A NOTE ON AN ABELIAN COVERING SURFACE, I

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§ 1. An abelian covering surface

1. We shall begin with a preliminary on a concept of a covering surface.[1] Throughout the present paper we assume that R is a closed or open Riemann surface, and $\tilde{R}$ is an unramified and unbounded covering surface of R.

LEMMA 1. *For a given subgroup* $\mathfrak{K}$ *of the fundamental group* $\mathfrak{F}$ *of* R, *there exists an unramified and unbounded covering surface* $\tilde{R}$ *of* R *such that the fundamental group* $\tilde{\mathfrak{F}}$ *of* $\tilde{R}$ *corresponds to* $\mathfrak{K}$.

LEMMA 2. *Let* $\tilde{R}$ *and* $\tilde{R}^*$ *be unramified and unbounded covering surfaces of* R. *If the fundamental groups* $\tilde{\mathfrak{F}}$ *and* $\tilde{\mathfrak{F}}^*$ *of* $\tilde{R}$ *and* $\tilde{R}^*$ *correspond to a common subgroup* $\mathfrak{K}$ *of the fundamental group* $\mathfrak{F}$ *of* R, *then* $\tilde{R}^*$ *coincides with* $\tilde{R}$.[2]

$\tilde{R}$ is called to be *normal,* if for any closed curve C on R all curves in $\tilde{R}$ lying over C are simultaneously closed or are simultaneously not closed. $\tilde{R}$ is normal if and only if the subgroup $\mathfrak{K}$ of $\mathfrak{F}$ corresponding to $\tilde{R}$ is a normal subgroup of $\mathfrak{F}$.

By a *covering transformation* of $\tilde{R}$ we mean a homeomorphism of $\tilde{R}$ onto itself which maps each point $\tilde{p} \in \tilde{R}$ over $p \in R$ into a point $\tilde{p}^*$ lying over the same point p. The covering transformation T which maps $\tilde{p}$ into $\tilde{p}^*$ is unique, provided it exists.

The set of covering transformations of $\tilde{R}$ clearly forms a group $\mathfrak{G}$ under the operation of composition of mappings which is called the *covering transformation group.*

There exists a transformation of the group $\mathfrak{G}$ which carries any point $\tilde{p}$ over p into any other prescribed point $\tilde{p}^*$ over p if and only if $\tilde{R}$ is normal. In this case, the covering transformation group $\mathfrak{G}$ is isomorphic to $\mathfrak{F}/\mathfrak{K}$, where $\mathfrak{K}$ is the normal subgroup of $\mathfrak{F}$ corresponding to $\tilde{R}$.

2. DEFINITION.[3] *A normal covering surface* $\tilde{R}$ *of* R *is called to be an (unramified) abelian covering surface of* R, *if its covering transformation group* $\mathfrak{G}$ *is abelian.*

Received August 15, 1962.

1) Cf. [2], [12], [15], etc.

2) We say that the two covering surfaces $\tilde{R}$ and $\tilde{R}^*$ of R coincide each other if there exists a one-to-one conformal mapping of $\tilde{R}^*$ onto $\tilde{R}$ such that two points corresponding each other by the mapping lie over the same point of R.

3) Cf. [11].

Let R have the fundamental group $\mathfrak{F}$ and $\mathfrak{C}$ be the commutator subgroup of $\mathfrak{F}$. $\mathfrak{C}$ determines a normal covering surface $\tilde{R}_{\mathfrak{H}}$ which we call the *homology covering surface.* $\tilde{R}_{\mathfrak{H}}$ is an abelian covering surface of R, for the quotient group $\mathfrak{F}/\mathfrak{C}$ is abelian. Further $\tilde{R}_{\mathfrak{H}}$ is the *strongest* abelian covering surface of R, for $\mathfrak{C}$ is the smallest subgroup whose quotient group is abelian. The quotient group $\mathfrak{F}/\mathfrak{C}$ is called the *homology group* of R. Let $\mathfrak{H}$ be the 1-dimensional singular homology group of R. Then $\mathfrak{H}$ is isomorphic with the quotient group $\mathfrak{F}/\mathfrak{C}$ and thus with the covering transformation group of $\tilde{R}_{\mathfrak{H}}$.

3. First let R be a closed Riemann surface of genus q. Then there exists a system of $2q$ cycles $\alpha_1, \ldots, \alpha_{2q}$ on R satisfying the following conditions:

(a) Any cycle C on R is homologous to a linear combination of $\alpha_1, \ldots, \alpha_{2q}$, that is

$$C \sim \prod_{j=1}^{2q} \alpha_j^{x_j},^{4)}$$

x_j being integers;

(b) The intersection number between them satisfies

$$\alpha_{2j-1} \times \alpha_{2k-1} = \alpha_{2j} \times \alpha_{2k} = 0, \quad \alpha_{2j-1} \times \alpha_{2k} = \delta_k^j \qquad (j=1, \ldots, q;\ \ k=1, \ldots, q),$$

δ_k^j being the Kronecker's symbol. Such a system is called a *canonical homology basis* of R.

Next let R be an open Riemann surface. A singular cycle is said to be a *dividing cycle,* or *homologous to* 0 *modulo the ideal boundary,* if it is homologous to a singular cycle which lies outside of any given compact set. The group formed by the homology classes of dividing singular cycles is denoted by $\mathfrak{H}_\beta$. The quotient group $\mathfrak{H}/\mathfrak{H}_\beta$ is the *homology group modulo dividing cycles.* It can also be called the *relative homology group with respect to the ideal boundary.*

It is known that there exists a system of cycles $\alpha_1, \alpha_2, \ldots$[5] on an open Riemann surface R satisfying the following conditions (cf. [1]):

(a′) Any cycle C on R is homologous to a linear combination of a finite number of the cycles $\alpha_1, \alpha_2, \ldots$, that is

$$C \sim \prod_{j=1}^{\kappa} \alpha_j^{x_j} \qquad (\mathrm{mod}\ \mathfrak{J}),$$

where x_j are integers and $\mathfrak{J}$ is the ideal boundary of R;

(b′) The intersection number between them satisfies

$$\alpha_{2j-1} \times \alpha_{2k-1} = \alpha_{2j} \times \alpha_{2k} = 0, \quad \alpha_{2j-1} \times \alpha_{2k} = \delta_k^j \qquad (j=1, 2, \ldots;\ k=1, 2, \ldots).$$

Such a system is called a *canonical homology basis of R modulo the ideal boundary.*

Now, let $\{R_n\}$ be an exhaustion of R. Then there exists a system of cycles satisfying moreover the following condition (cf. [1]):

4) In the present paper, we shall denote a sum of cycles by a product notation for convenience.

5) Here it may arise that the number of the cycles $\alpha_1, \alpha_2, \cdots$ is finite.

(c′) For every n there exists a number κ_n such that $\alpha_1, \ldots, \alpha_{2\kappa_n}$ form a relative homology basis of R_n mod ∂R_n, that is, any cycle $C \subset R_n$ is homologous to a linear combination of $\alpha_1, \ldots, \alpha_{2\kappa_n}$, i. e.

$$C \sim \prod_{j=1}^{2\kappa_n} \alpha_j^{x_j} \qquad (\bmod\ \partial R_n).$$

We shall call such a system a *canonical homology basis belonging to the exhaustion* $\{R_n\}$.

4. Let R be an open Riemann surface and K be a compact subregion of R. We shall call K a *canonical subregion*, if its boundary consists of a finite number of simple closed analytic curves, all components of $R-K$ are non-compact, and have a single contour.

Let $\{R_n\}_{n=1}^{\infty}$ be an exhaustion of R. We shall call $\{R_n\}_{n=1}^{\infty}$ a *canonical exhaustion* if all the R_n are canonical.

Let the contours of R_1 be denoted by $\beta_j, j=1, \ldots, s$. The orientation will be chosen such that $\partial R_1 = \Pi \beta_j$.

Consider the complement $R_2 - R_1$, always in the sense that the border is included in the complement. Because the exhaustion is canonical there will be exactly s components, and we denote by Q_j the component which has the contour β_j in common with R_1. The remaining contours of Q_j will be denoted by β_{jk}, $1 \leqq k \leqq s_j$, and we choose the orientation so that $\partial Q_j = \Pi_k \beta_{jk} \beta_j^{-1}$. Next, the complement $R_3 - R_2$ consists of components Q_{jk} where Q_j and Q_{jk} have the common contour β_{jk}.

When we continue in this way we obtain the symbol $\beta_{j_1 \cdots j_n}$ for each contour of R_n with the subscript j_n running from 1 to a number $s_{j_1 \cdots j_{n-1}}$. The components of $R_{n+1} - R_n$ have names $Q_{j_1 \cdots j_n}$ and further $\partial Q_{j_1 \cdots j_n} = \Pi_j \beta_{j_1 \cdots j_n j} \beta_{j_1 \cdots j_{n-1}}$. For the sake of conformity, Q will be another name for R_1, and β will be null.

LEMMA 3. *A* (*strong*) *homology basis for the open Riemann surface R is formed by the combined system of all cycles α_j of a canonical homology basis belonging to a canonical exhaustion $\{R_n\}$ and all cycles $\beta_{j_1 \cdots j_n}$ with $j_n > 1$.*

Such a system of cycles shall be called a (*strong*) *canonical homology basis of* R. Each cycle C satisfies a unique homology relation

$$C \sim \prod_{j} \alpha_j^{x_j} \prod_{j_n > 1} \beta_{j_1 \cdots j_n}^{y_{j_1 \cdots j_n}}$$

where x_j and $y_{j_1 \cdots j_n}$ are integers, the products are finite, and C is a dividing cycle if and only if all x_j are 0.

The system of cycles $\beta_{j_1 \cdots j_n}$, $j_n > 1$, determines a basis for $\mathfrak{H}_\beta$, which we shall call a *canonical homology basis of dividing cycles.*

5. Let Γ be a one-dimensional chain. We shall say that Γ is a *relative cycle* if and only if $\partial \Gamma = 0$. The group of relative cycles contains the subgroup of cycles (or finite cycles).

Consider a point on $\beta_{j_1 \cdots j_n}$, $j_n > 1$. It can be joined by a simple curve in $Q_{j_1 \cdots j_n}$ to a point on $\beta_{j_1 \cdots j_n 1}$. This point can be joined to a point on $\beta_{j_1 \cdots j_n 11}$ by a simple curve in $Q_{j_1\cdots j_n 1}$, and so on. In the opposite direction we can pass from the point on $\beta_{j_1\cdots j_n}$ through $Q_{j_1\cdots j_{n-1}}$ to a point on $\beta_{j_1\cdots j_{n-1}1}$, then through $Q_{j_1\cdots j_{n-1}1}$ to a point on $\beta_{j_1\cdots j_{n-1}11}$, and so on. Here we take the curves so that each of them does not intersect with any element α_j of canonical homology basis belonging to the exhaustion $\{R_n\}$, which is possible. Then the product of the curves between consecutive points is a relative cycle, which we denote by $\beta_{j_1\cdots j_n}{}^*$ and we call the *conjugate relative cycle of* $\beta_{j_1\cdots j_n}$. Its direction can be fixed so that $\beta_{j_1\cdots j_n}\times\beta_{j_1\cdots j_n}{}^*=1$. Then $\beta_{j_1\cdots j_n 1\cdots 1}\times\beta_{j_1\cdots j_n}{}^*=1$, $\beta_{j_1\cdots j_{n-1}1\cdots 1}\times\beta_{j_1\cdots j_n}{}^*=-1$, and for all others $\beta_{k_1\cdots k_{n'}}\times\beta_{j_1\cdots j_n}{}^*=0$, $\alpha_j\times\beta_{j_1\cdots j_n}{}^*=0$.

6. Let R be an arbitrary Riemann surface[6] and $\tilde{R}$ be an abelian covering surface of R with its covering transformation group $\mathfrak{G}$.

Let $\tilde{p}$ be an arbitrarily fixed point on $\tilde{R}$. For each $j=1, 2, \ldots$, we denote by $\alpha_j(\tilde{p})$ the end point of the curve on $\tilde{R}$ starting from $\tilde{p}$, whose projection on R is closed and homotopic with the curve α_j. Similarly, we denote by $\beta_{j_1\cdots j_n}(\tilde{p})$ $(j_n>1)$ the end point of a curve on $\tilde{R}$ starting from $\tilde{p}$, whose projection on R is closed and homotopic to the curve $\beta_{j_1\cdots j_n}$. Since $\tilde{R}$ is normal by the assumption, there exists a unique covering transformation of $\tilde{R}$, which carries $\tilde{p}$ into $\alpha_j(\tilde{p})$ $(j=1, 2, \ldots)$ or $\beta_{j_1\cdots j_n}(\tilde{p})$ $(j_n>1)$, respectively. Further, since $\tilde{R}$ is an abelian covering surface of R, by the consequence in **2** and the lemma 3, these transformations form a *system of generators* of the group $\mathfrak{G}$, which shall be denoted by the same letters α_j and $\beta_{j_1\cdots j_n}$, respectively.[7]

Let $\mathfrak{G}_n$ be the subgroup of $\mathfrak{G}$ generated by the strong canonical homology basis α_j $(j=1, \ldots, 2\kappa_n)$, $\beta_{j_1\cdots j_\nu}$ $(\nu=1, \ldots, n;\ j_\nu>1)$ of R_n. For simplicity of notation, we shall agree to denote the canonical homology basis $\beta_{j_1\cdots j_\nu}$ $(\nu=1, \ldots, n;\ j_\nu>1)$ of dividing cycles of R_n by $\beta_1, \ldots, \beta_{\iota_n}$ with the changed subindices.

Now, there exists a finite number of *defining relations* between the elements of $\mathfrak{G}_n$:

$$\prod_{j=1}^{2\kappa_n}\alpha_j{}^{a_{kj}}\prod_{j=1}^{\iota_n}\beta_j{}^{a_{k,\,2\kappa_n+j}}=I \tag{1}$$

$$(k=1, \ldots, \lambda_n;\ 0\leqq\lambda_n\leqq 2\kappa_n+\iota_n),$$

with integral exponents a_{kj} such that each relation

$$\prod_{j=1}^{2\kappa_n}\alpha_j{}^{x_j}\prod_{j=1}^{\iota_n}\beta_j{}^{x_{2\kappa_n+j}}=I$$

is generated by the system (1), where I is the identical transformation.[8] Here, if the $\lambda_n\times(2\kappa_n+\iota_n)$ matrix

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6) Here the result will be stated for the case that R is most general, but it will be done more easily for the other cases (of closed one or finite genus, etc.). Cf. [11] for the case of closed one.

7) In general, these transformations depend also on the choice of the point $\tilde{p}$. In the case of abelian $\mathfrak{G}$, however, they are uniquely determined by the curves α_j or $\beta_{j_1\cdots j_n}$.

8) Cf. [14].

$$(a_{kj})_{k=1,\dots,\lambda_n,\, j=1,\dots,2\kappa_n+\iota_n}$$

is of rank $\rho_n(\leqq\lambda_n)$, then the number $2\kappa_n+\iota_n-\rho_n=\sigma_n$ is the rank (Betti number) of the abelian group $\mathfrak{G}_n$.

Conversely, when we give the defining relations (1) for each n such that the relations (1) for n contains all ones for $n-1$, an abelian group $\mathfrak{G}$ with the system of generators α_j and β_j $(j=1, 2, \dots)$ satisfying (1) is uniquely determined and further a subgroup $\mathfrak{K}$ of the fundamental group $\mathfrak{F}$ of R uniquely corresponds to it. Then, by the lemmas 1 and 2, there exists one and only one abelian covering surface $\tilde{R}$ of R whose fundamental group $\tilde{\mathfrak{F}}$ corresponds to $\mathfrak{K}$ and whose covering transformation group is $\mathfrak{G}$.

Now we shall show one of the methods of constructing the abelian covering surface $\tilde{R}$ of R with the covering transformation group $\mathfrak{G}$ when a Riemann surface R and a group $\mathfrak{G}$ with the system of generators α_j and β_j $(j=1, 2, \dots)$ satisfying (1) are given.

We shall represent the elements $\Pi_{j=1}^{2\kappa_n}\alpha_j^{x_j}\Pi_{j=1}^{\iota_n}\beta_j^{x_{2\kappa_n+j}}$ of $\mathfrak{G}_n$, where x_j $(j=1, \dots, 2\kappa_n+\iota_n)$ are integers, by the lattice points $(x_1, \dots, x_{2\kappa_n+\iota_n})$ of a $2\kappa_n+\iota_n$-dimensional euclidean space $\mathfrak{E}^{2\kappa_n+\iota_n}$. Let $\mathfrak{Z}_n$ be the group of transformations of $\mathfrak{E}^{2\kappa_n+\iota_n}$ generated by the $2\kappa_n+\iota_n$ translations carrying the origin into $(1, 0, \dots, 0), \dots, (0, \dots, 0, 1)$ respectively, and $\mathfrak{Z}_n^*$ be its subgroup generated by the λ_n translations carrying the origin into $(a_{k1}, \dots, a_{k,2\kappa_n+\iota_n})$, $(k=1, \dots, \lambda_n)$, respectively. Obviously, two lattice points of $\mathfrak{E}^{2\kappa_n+\iota_n}$ represent one and the same element of $\mathfrak{G}_n$ if and only if they are equivalent with respect to $\mathfrak{Z}_n^*$, and $\mathfrak{G}_n$ is isomorphic with the factor group $\mathfrak{Z}_n/\mathfrak{Z}_n^*$.

Let the two shores of each of the curves $\alpha_{2j-1}, \alpha_{2j}, j=1, 2, \dots$, be denoted by $\alpha_{2j-1}^+, \alpha_{2j-1}^-, \alpha_{2j}^+, \alpha_{2j}^-$ respectively, in such a manner that the oriented curve α_{2j} intersects α_{2j-1} from the shore α_{2j-1}^+ to the other shore α_{2j-1}^-, and that α_{2j-1} intersect α_{2j} from α_{2j}^+ to α_{2j}^-.[9] Similarly, let the two shores of each of the curves β_j^* $(j=1, 2, \dots)$, be denoted by $\beta_j^{*+}, \beta_j^{*-}$, respectively, in such a manner that the oriented curve β_j intersects β_j^* from the shore β_j^{*+} to the other shore β_j^{*-}. We cut R along the all curves α_j $(j=1, 2, \dots)$ and β_j^* $(j=1, 2, \dots)$ to obtain a surface F of planar character having an (infinite) number of boundary components, one of which consists of all $\beta_j^{*+}, \beta_j^{*-}$ $(j=1, 2, \dots)$ and the ideal boundary $\mathfrak{I}$, and other ones of which consist of four sides $\alpha_{2j-1}^+, \alpha_{2j}^+, \alpha_{2j-1}^-, \alpha_{2j}^-$. To each residue class $(x_1, \dots, x_{2\kappa_n+\iota_n})$ mod $\mathfrak{Z}_n^*$, we associate a replica $F(x_1, \dots, x_{2\kappa_n+\iota_n})$ of F. Next, we identify the side $\alpha_{2j}^+(j=1, \dots, \kappa_n)$ of each $F(x_1, \dots, x_{2j-1}, x_{2j}, \dots, x_{2\kappa_n+\iota_n})$ with the side α_{2j}^- of $F(x_1, \dots, x_{2j-1}+1, x_{2j}, \dots, x_{2\kappa_n+\iota_n})$, α_{2j-1}^+ $(j=1, \dots, \kappa_n)$ of $F(x_1, \dots, x_{2j-1}, x_{2j}, \dots, x_{2\kappa_n+\iota_n})$ with α_{2j-1}^- of $F(x_1, \dots, x_{2j-1}, x_{2j}+1, x_{2\kappa_n+\iota_n})$, and β_j^{*+} $(j=1, \dots, \iota_n)$ of $F(x_1, \dots, x_{2\kappa_n+j}, \dots, x_{2\kappa_n+\iota_n})$ with β_j^{*-} of $F(x_1, \dots, x_{2\kappa_n+j}+1, \cdots, x_{2\kappa_n+\iota_n})$, where each point on α_j^+ and β_j^{*+} must be identified with the corresponding point on α_j^- and β_j^{*-}, respectively.

By these procedures, the sides α_j $(j=1, \dots, 2\kappa_n)$, β_j^* $(j=1, \dots, \iota_n)$ of each F is identified with the unique side of some other (or the same) F, and at each vertex

9) We should note that in each homology class (α_j) there exists a simple closed curve and thus so α_j may be assumed itself to be.

of each F, there meet four F's; $F(x_1, \ldots, x_{2j-1}, x_{2j}, \ldots, x_{2\kappa_n+\iota_n})$, $F(x_1, \ldots, x_{2j-1}+1, x_{2j}, \ldots, x_{2\kappa_n+\iota_n})$, $F(x_1, \ldots, x_{2j-1}, x_{2j}+1, \ldots, x_{2\kappa_n+\iota_n})$ and $F(x_1, \ldots, x_{2j-1}+1, x_{2j}+1, \ldots, x_{2\kappa_n+\iota_n})$ $(j=1, \ldots, \kappa_n)$ (some of these four may be mutually identical). Let $\tilde{F}_n$ be a surface obtained by such a procedure. Here we should note that $\tilde{F}_{n+1}$ is obtained by the identification process of a number of replicas of $\tilde{F}_n$ along the sides $\alpha_{2\kappa_n+1}{}^+$, $\alpha_{2\kappa_n+1}{}^-$, $\ldots$, $\alpha_{2\kappa_{n+1}}{}^+$, $\alpha_{2\kappa_{n+1}}{}^-$, $\beta_{\iota_n+1}{}^{*+}$, $\beta_{\iota_n+1}{}^{*-}$, $\ldots$, $\beta_{\iota_{n+1}}{}^{*+}$, $\beta_{\iota_{n+1}}{}^{*-}$ according to (1) replaced n by $n+1$. According to the present procedure, in the first step we construct $\tilde{F}_1$ of a number of replicas of F, in the second step $\tilde{F}_2$ of a number of replicas of $\tilde{F}_1$, and so on. By these infinite identification process, the unramified and unbounded covering surface $\tilde{R}$ of R with the covering transformation group $\mathfrak{G}$ is constructed

§ 2. A free abelian covering surface

7. Let R be an open Riemann surface. A non-compact subregion Ω of R whose relative boundary consists of a finite number of disjoint closed analytic curves will be called an *end.*

We introduce the following definition of an *ideal boundary component*:

An ideal boundary component is a non-void collection γ of ends Ω which satisfies the following conditions:

(i) If $\Omega_0 \in \gamma$ and $\Omega \supset \Omega_0$, then $\Omega \in \gamma$;

(ii) If $\Omega_1, \Omega_2 \in \gamma$, there exists an $\Omega_3 \subset \Omega_1 \frown \Omega_2$ which belongs to γ;

(iii) The intersection of all closures $\bar{\Omega}$, $\Omega \in \gamma$, is empty.

Let $\{p_j\}_{j=1}^{\infty}$ be a sequence of points of R. If for any $\Omega \in \gamma$ there exists a number j_0 such that $p_j \in \Omega$ for all $j \geqq j_0$, we say that the *point sequence* $\{p_j\}_{j=1}^{\infty}$ *tends to the ideal boundary component* γ.

8. In the present section we shall state a *topological character* of a Riemann surface $\tilde{R}$ admitting a group $\mathfrak{G}$ of one-to-one conformal transformations onto itself which is *free abelian,* finitely generated and properly discontinuous.[10] Here we assume that no transformation of $\mathfrak{G}$ other than the identity has a fixed point. Let R be a Riemann surface constructed from $\tilde{R}$ by identifying equivalent points by $\mathfrak{G}$, denoted by $R \equiv \tilde{R} \pmod{\mathfrak{G}}$. Then, $\tilde{R}$ is an abelian covering surface of R with its covering transformation group $\mathfrak{G}$.

We distinguish several cases by a number of elements of basis of $\mathfrak{G}$ in the following.

I. *The case where $\mathfrak{G}$ is generated by only one element T.*

LEMMA 4. *Let $\tilde{p}$ be a fixed point on $\tilde{R}$. Then the point sequence $\{T^m(\tilde{p})\}_{m=1}^{\infty}$ (and also $\{T^{-m}(\tilde{p})\}_{m=1}^{\infty}$) tends to an ideal boundary component of $\tilde{R}$.*

Proof. Let $\tilde{C}_0$ be a curve from $\tilde{p}$ to $T(\tilde{p})$ on $\tilde{R}$, $\tilde{C}_m = T^m(\tilde{C}_0)$ $(m=0, 1, \ldots)$ and K be any compact region of $\tilde{R}$. Then there exists a number m_0 such that $\tilde{C}_m \subset \tilde{R}$

10) For any abelian group $\mathfrak{G}$, let $\mathfrak{T}$ be the torsion group of $\mathfrak{G}$, then the quotient group $\mathfrak{G}/\mathfrak{T}$ is a free abelian group without torsion. Thus, in the present problem there is an essential interest for only the case that $\mathfrak{G}$ is free abelian.

$-K$ for all $m \geqq m_0$ and thus all $T^m(\tilde{p})$ $(m \geqq m_0)$ lie on a common connected component of $\tilde{R}-K$. Thus $\{T^m(\tilde{p})\}_{m=1}^{\infty}$ tends to an ideal boundary component of $\tilde{R}$.

LEMMA 5. *Let $\tilde{p}_1$ and $\tilde{p}_2$ be any two points on $\tilde{R}$. Then two point sequences $\{T^m(\tilde{p}_1)\}_{m=1}^{\infty}$ and $\{T^m(\tilde{p}_2)\}_{m=1}^{\infty}$ (or $\{T^{-m}(\tilde{p}_1)\}_{m=1}^{\infty}$ and $\{T^{-m}(\tilde{p}_2)\}_{m=1}^{\infty}$) tend to a common ideal boundary component of $\tilde{R}$.*

Proof. Let $\tilde{C}_0$ be a curve from $\tilde{p}_1$ to $\tilde{p}_2$ on $\tilde{R}$ and $\tilde{C}_m = T^m(\tilde{C}_0)$ $(m=0, 1, \ldots)$. Then we may apply a similar argument as in the lemma 4.

In the case I., two subcases can be distinguished.

(H) *The case where $\{T^{-m}(\tilde{p})\}$ and $\{T^m(\tilde{p})\}$ $(\tilde{p} \in \tilde{R};\ m=1, 2, \ldots)$ tend to distinct ideal boundary components γ_1 and γ_2 of $\tilde{R}$, respectively. Then $\tilde{R}$ will be called the hyperbolic type.*

(P) *The case where both sequences of points $\{T^{-m}(\tilde{p})\}$ and $\{T^m(\tilde{p})\}$ $(\tilde{p} \in \tilde{R};\ m=1, 2, \ldots)$ tend to a common ideal boundary component γ of $\tilde{R}$. Then $\tilde{R}$ will be called the parabolic type.*

II. *The case where $\mathfrak{G}$ is generated by two elements T_1, T_2.*

LEMMA 6. *Let $\tilde{p}$ be a fixed point on $\tilde{R}$. Then the four point sequences $\{T_1^{-m}(\tilde{p})\}_{m=1}^{\infty}$, $\{T_1^m(\tilde{p})\}_{m=1}^{\infty}$, $\{T_2^{-m}(\tilde{p})\}_{m=1}^{\infty}$ and $\{T_2^m(\tilde{p})\}_{m=1}^{\infty}$ tend to a common ideal boundary component of $\tilde{R}$.*

Proof. Let $\tilde{C}_1(0, 0)$ and $\tilde{C}_2(0, 0)$ be curves from $\tilde{p}$ to $T_1(\tilde{p})$ and $T_2(\tilde{p})$, respectively, and $T_1^{m_1} \circ T_2^{m_2}(\tilde{C}_1(0, 0)) = \tilde{C}(m_1, m_2)$ and $T_1^{m_1} \circ T_2^{m_2}(\tilde{C}_2(0, 0)) = \tilde{C}_2(m_1, m_2)$ $(m_1, m_2 = 0, \pm 1, \ldots)$. Then for any compact subregion K of $\tilde{R}$ there exists a number m_0 such that $\tilde{C}_1(m_1, m_2) \subset \tilde{R}-K$ and $\tilde{C}_2(m_1, m_2) \subset \tilde{R}-K$ for all pairs (m_1, m_2) except for $|m_1| < m_0$, $|m_2| < m_0$, simultaneously. Then $T_1^{-m}(\tilde{p})$, $T_1^m(\tilde{p})$, $T_2^{-m}(\tilde{p})$ and $T_2^m(\tilde{p})$ $(m \geqq m_0)$ can be connected each other by a curve on $\tilde{R}-K$ (e.g. so can be $T_1^m(\tilde{p})$ and $T_2^m(\tilde{p})$ $(m \geqq m_0)$ by the curve $\tilde{C}_2(m, 0) \ldots \tilde{C}_2(m, m-1) \tilde{C}_1^{-1}(m-1, m) \ldots \tilde{C}_1^{-1}(0, m)$), and thus they all lie on a common connected component of $\tilde{R}-K$.

III. *The case where $\mathfrak{G}$ is generated by three or more elements $T_1, \ldots, T_N$ $(N \geqq 3)$.*

In this case also each point sequence $\{T_1^{-m}(\tilde{p})\}$, $\{T_1^m(\tilde{p})\}$, ..., $\{T_N^{-m}(\tilde{p})\}$, $\{T_N^m(\tilde{p})\}$ $(\tilde{p} \in \tilde{R};\ m=1, 2, \ldots)$ always tends to a common ideal boundary component of $\tilde{R}$.

In the following we concern ourselves with only the case I. (H). In the next paper we will concern ourselves with the other cases I. (P), II. and III.

9. The strong homology basis of R determines a system of generators of the covering transformation group $\mathfrak{G}$ of $\tilde{R}$ as was discussed in § **1**.

LEMMA 7. *In the case* I. (H), *no dividing cycle on R can be a non-trivial generator of $\mathfrak{G}$.*[11]

Proof. (i) We would assume that an infinite number of elements in a canonical homology basis of dividing cycles are non-trivial generators of $\mathfrak{G}$, and let $\mathfrak{B} = \{\beta_1,$

11) Here by a non-trivial generator of $\mathfrak{G}$ we mean the element of $\mathfrak{G}$ which is not the identical transformation.

$\beta_2, \ldots\}$ be the system of such ones.

Since $T^{-m}(\tilde{p}) \to \gamma_1$, $T^m(\tilde{p}) \to \gamma_2$ $(m \to \infty;\ \tilde{p} \in \tilde{R};\ \gamma_1 \neq \gamma_2)$, there exists a dividing cycle $\tilde{C}$ of $\tilde{R}$ such that $\tilde{C}$ divides $\tilde{R}$ into two ends Ω_1 and Ω_2 being $\Omega_1 \in \gamma_1$ and $\Omega_2 \in \gamma_2$.

Let p_j be a point on β_j $(j=1, 2, \ldots)$ and $\tilde{p}_j$ be one of the points on $\tilde{R}$ lying over p_j. We can select a subsequence of the point sequence $\tilde{p}_j$ $(j=1, 2, \ldots)$ that tends to an ideal boundary component γ of $\tilde{R}$. Without loss of generality, we may assume that $\tilde{p}_j \to \gamma$ $(j \to \infty)$. Then, $\beta_j{}^m(\tilde{p}_j) \to \gamma$ $(j \to \infty$; for each $m = \pm 1, \ldots)$. For, there exists a number j_0, for any compact region $K \subset \tilde{R}$, such that two points $\beta_j{}^{m-1}(\tilde{p}_j)$ and $\beta_j{}^m(\tilde{p}_j)$ can be connected by one of the curves $\tilde{\beta}_j$ on $\tilde{R}-K$ lying over β_j for all $j \geqq j_0$. Now, either $\Omega_1 \in \gamma$ or $\Omega_2 \in \gamma$, e. g. let $\Omega_1 \in \gamma$. Then there exists a number j_1 such that $\beta_j \subset R - C$ and $\tilde{p}_j \in \Omega_1$ for all $j \geqq j_1$ and thus

$$(2) \qquad \beta_j{}^m(\tilde{p}_j) \in \Omega_1 \quad (j \geqq j_1; \text{ for each } m=0, \pm 1, \ldots),$$

where $C \equiv \tilde{C}$ (mod $\mathfrak{G}$). On the other hand, for an arbitrarily positive integer j there exists a positive integer m_j such that

$$(3) \qquad \beta_j{}^m(\tilde{p}_j) \in \Omega_2 \ (\text{or } \beta_j{}^{-m}(\tilde{p}_j) \in \Omega_2) \quad \text{for any } m \geqq m_j.$$

For, β_j $(j=1, 2, \ldots)$ is a non-trivial generator of $\mathfrak{G}$ by the assumption and $T^m(\tilde{p}) \to \gamma_2$ $(m \to \infty;\ \tilde{p} \in \tilde{R})$. (3) contradicts to (2).

(ii) We would assume that only a (non-zero) finite number of elements in a canonical homology basis of dividing cycles are non-trivial generators of $\mathfrak{G}$, and let $\mathfrak{B}$ be a system of such ones. There exists an element $\beta_{j_1^0 \cdots j_n^0}$ $(j_n^0 > 1)$ in $\mathfrak{B}$ such that at least one of ends of the conjugate relative cycle $\beta_{j_1^0 \cdots j_n^0}{}^*$ of $\beta_{j_1^0 \cdots j_n^0}$ does not tend to an ideal boundary component in common with any end of a conjugate one of any other element of $\mathfrak{B}$. On the end $\beta_{j_1^0 \cdots j_n^0}{}^*$ intersects with an infinite number of dividing cycles $\beta_{j_1^0 \cdots j_n^0 1}, \beta_{j_1^0 \cdots j_n^0 11}, \ldots$ or $\beta_{j_1^0 \cdots j_{n-1}^0 1}, \beta_{j_1^0 \cdots j_{n-1}^0 11}, \ldots$. For instance, we shall assume that the first case occurs. Then

$$\beta_{j_1^0 \cdots j_n^0 1} \sim \beta_{j_1^0 \cdots j_n^0} \prod_{j_{n+1}=2}^{s_{j_1^0 \cdots j_n^0}} (\beta_{j_1^0 \cdots j_n^0 j_{n+1}})^{-1}$$

and

$$\beta_{j_1^0 \cdots j_n^0 j_{n+1}} = I \qquad (j_{n+1} = 2, \ldots, s_{j_1^0 \cdots j_n^0}).$$

For, if $\beta_{j_1^0 \cdots j_n^0 j_{n+1}} \neq I$, $\beta_{j_1^0 \cdots j_n^0 j_{n+1}} \in \mathfrak{B}$ and an end of $\beta_{j_1^0 \cdots j_n^0 j_{n+1}}{}^*$ would tend to the ideal boundary component in common with *the* end of $\beta_{j_1^0 \cdots j_n^0}{}^*$. Thus $\beta_{j_1^0 \cdots j_n^0 1} = \beta_{j_1^0 \cdots j_n^0}$. By the similar procedure, we have that

$$\beta_{j_1^0 \cdots j_n^0} = \beta_{j_1^0 \cdots j_n^0 1} = \beta_{j_1^0 \cdots j_n^0 11} = \cdots$$

and thus each $\beta_{j_1^0 \cdots j_n^0 1}, \beta_{j_1^0 \cdots j_n^0 11}, \ldots$ is a non-trivial generator. In the second case, we can also show by the similar argument that each $\beta_{j_1^0 \cdots j_{n-1}^0 1}, \beta_{j_1^0 \cdots j_{n-1}^0 11}, \ldots$ is a non-trivial generator. Then we may apply the argument in (i) for a system of such ones and deduce a contradiction.

LEMMA 8. *In the case* I. (H), *only a finite number of* α_j *can be non-trivial generators of* $\mathfrak{G}$.

We can prove the lemma by the argument similar to the lemma 7. We shall omit its proof.

LEMMA 9. *In the case* I., *if only a finite number of elements* $\alpha_1, \cdots, \alpha_{2\kappa}$ *of canonical homology basis modulo the ideal boundary* $\mathfrak{J}$ *of* R *form a system of generators of* $\mathfrak{G}$, *then* $\tilde{R}$ *is the type* (H) *and a suitable cycle* $\tilde{C}$ *on* $\tilde{R}$ *whose projection on* R *is homologous to the cycle* $C=\Pi_{j=1}^{\kappa}\alpha_{2j-1}{}^{-m_{2j}}\alpha_{2j}{}^{m_{2j-1}}$, *forms a dividing cycle of* $\tilde{R}$ *separating* γ_2 *from* γ_1, *where* m_j *are the integers such that* $\alpha_j=T^{m_j}$ $(j=1, \ldots, 2\kappa)$.

We shall prove the present lemma in **11**.

By the lemmas 7, 8 and 9, we have the following theorem.

THEOREM 1. *In the case* I., $\tilde{R}$ *is of hyperbolic type* (H) *if and only if a finite number of elements* $\alpha_1, \ldots, \alpha_{2\kappa}$ *of canonical homology basis modulo the ideal boundary* $\mathfrak{J}$ *of* R *form a system of generators of* $\mathfrak{G}$.

By the theorem, if $\tilde{R}$ is the type I. (H) then it must be

$$(4)\qquad \begin{cases} \alpha_j=T^{m_j} \qquad (j=1, \ldots, 2\kappa), \\ \quad (m_1, \ldots, m_{2\kappa})=1, \\ \alpha_j=I \qquad (j=2\kappa+1, \ldots), \end{cases}$$

for some $\kappa\geqq 1$ since $\{T\}$ is a base of $\mathfrak{G}$, where m_j $(j=1, \ldots, 2\kappa)$ are integers and $(m_1, \ldots, m_{2\kappa})$ denotes the greatest common measure of the integers $m_1, \ldots, m_{2\kappa}$.

By the theorem we have immediately the following corollary.

COROLLARY. *In the case* I., *if* R *is closed, then* $\tilde{R}$ *is of hyperbolic type* (H).

10. LEMMA 10. *In the case* I. (H), *we can always select a canonical homology basis* $\bar{\alpha}_1, \bar{\alpha}_2, \ldots$ *on* R, *called regular for* $\tilde{R}$ *which satisfies the condition*

$$(5)\qquad \begin{cases} \bar{\alpha}_{2j-1}=I, \\ \qquad\qquad (j=1, 2, \ldots), \\ \bar{\alpha}_{2j}=T^{\bar{m}_j}, \end{cases}$$

where $\bar{m}_j=0$ *if* $m_{2j-1}=m_{2j}=0$, *otherwise* $\bar{m}_j=(m_{2j-1}, m_{2j})>0$.

Conversely, we can always select a canonical homology basis $\alpha_1, \alpha_2, \ldots$ *on* R *from a given regular canonical homology basis* (5) *for* $\tilde{R}$ *which satisfies the condition*

$$\begin{cases} \alpha_{2j-1}=T^{m_{2j-1}}, \\ \qquad\qquad (j=1, 2, \ldots), \\ \alpha_{2j}=T^{m_{2j}}, \end{cases}$$

where m_{2j-1}, m_{2j} $(j=1, 2, \ldots)$ *are arbitrarily given integers such that* $(m_{2j-1}, m_{2j})=\bar{m}_j$ *if* $\bar{m}_j>0$ *and* $m_{2j-1}=m_{2j}=0$ *if* $\bar{m}_j=0$.

Proof. If $m_{2j-1}=m_{2j}=0$ we put $\bar{\alpha}_{2j-1}=\alpha_{2j-1}$ and $\bar{\alpha}_{2j}=\alpha_{2j}$. Otherwise we put

$$\bar{\alpha}_{2j-1}=\alpha_{2j-1}{}^{m_{2j}/(m_{2j-1},m_{2j})}\alpha_{2j}{}^{-m_{2j-1}/(m_{2j-1},m_{2j})}. \tag{6}$$

Then $\bar{\alpha}_{2j-1}=I$. Now there always exists a unique pair of integers x_{2j-1} and x_{2j} such that

$$x_{2j-1}m_{2j-1}+x_{2j}m_{2j}=(m_{2j-1},\ m_{2j}). \tag{7}$$

For such x_{2j-1} and x_{2j} we put

$$\bar{\alpha}_{2j}=\alpha_{2j-1}{}^{x_{2j-1}}\,\alpha_{2j}{}^{x_{2j}}. \tag{8}$$

Then by (7)

$$\bar{\alpha}_{2j}=T^{(m_{2j-1},m_{2j})}.$$

Conversely, by (6), (7) and (8) we have

$$\begin{cases} \alpha_{2j-1}=\bar{\alpha}_{2j-1}{}^{x_{2j}}\,\bar{\alpha}_{2j}{}^{m_{2j-1}/(m_{2j-1},m_{2j})}, \\ \alpha_{2j}=\bar{\alpha}_{2j-1}{}^{-x_{2j-1}}\,\bar{\alpha}_{2j}{}^{m_{2j}/(m_{2j-1},m_{2j})}. \end{cases}$$

Thus the condition (a) or (a′) in **3** is satisfied by the new basis $\bar{\alpha}_1, \bar{\alpha}_2, \ldots$. And also it is immediately shown that the condition (b) or (b′) for the intersection number in **3** is satisfied.

The converse statement is also easily shown.

11. *The proof of the lemma* 9. By using the proof of the lemma 10, the present lemma is reduced to the case where α_{2j} ($\alpha_{2j}=T^{m_j}$, $m_j>0$; $j=1, \ldots, \kappa$) form a system of generators of $\mathfrak{G}$. Then we need only to show that a suitable cycle $\tilde{C}$ on $\tilde{R}$ the projection on R of which is homologous to $C=\Pi_{j=1}^{\kappa}\alpha_{2j-1}{}^{-m_j}$, forms a dividing cycle on $\tilde{R}$ separating γ_2 from γ_1.

Let $\tilde{p}$ be a fixed point on $\tilde{R}$. We can connect $\tilde{p}$ to $\alpha_{2j}(\tilde{p})$ ($j=1, \ldots, \kappa$) by simple curves, say $\tilde{\alpha}_{2j}$, on $\tilde{R}$ the projections on R of which are homologous to α_{2j}, respectively. At the same time we can select simple closed curves, say $\tilde{\alpha}_{2j-1}$ ($j=1, \ldots, \kappa$), on $\tilde{R}$ which intersect to $\tilde{\alpha}_{2j}$ just one time and the projections on R of which are homologous to α_{2j-1}, respectively.

Let

$$\tilde{C}=\prod_{j=1}^{\kappa}\prod_{k=1}^{m_j}T^{-k+1}(\tilde{\alpha}_{2j-1}{}^{-1}).$$

We shall first prove that if $\tilde{C}^*$ is an arbitrary curve connecting $\tilde{p}$ to $T(\tilde{p})$ on $\tilde{R}$ then $\tilde{C}^*$ intersects to $\tilde{C}$ and thus $\tilde{C}$ separates $T(\tilde{p})$ from $\tilde{p}$. Without loss of generality we may assume that $\tilde{C}^*$ have the form

$$\tilde{C}^*=\prod_{\nu=1}^{\iota}T^{k_\nu}(\tilde{\alpha}_{2j_\nu}{}^{\delta_\nu}),$$

because only the homotopy classes containing α_{2j} ($j=1, \ldots, \kappa$) among all homotopy classes of R can be non-trivial generators of $\mathfrak{G}$. Here $1\leqq j_\nu\leqq\kappa$, $\delta_\nu=\pm1$, k_ν are integers ($\nu=1, \ldots, \iota$), and the initial points $T^{\bar{k}_\nu}(\tilde{p})$ and the terminal points $T^{\bar{k}_\nu'}(\tilde{p})$ of T^{k_ν}

$(\tilde{\alpha}_{2j_\nu}{}^{\delta_\nu})$ $(\nu=1, \ldots, \iota)$ satisfy that $T^{\bar{k}_1}(\tilde{p})=\tilde{p}$, $T^{\bar{k}_{\nu+1}}(\tilde{p})=T^{\bar{k}_\nu'}(\tilde{p})$ $(\nu=1, \ldots, \iota-1)$, $T^{\bar{k}_\iota'}(\tilde{p})=T(\tilde{p})$, respectively. There exists the minimum ν, say ν_0, such that $\bar{k}_\nu \leqq 0, \bar{k}_\nu' > 0$. Then, $\delta_{\nu_0}=1$, $\bar{k}_{\nu_0}=k_{\nu_0}$, $\bar{k}_{\nu_0}+\bar{m}_{j\nu_0}=\bar{k}_{\nu_0}'$ and thus $-m_{j\nu_0}+1 \leqq k_{\nu_0} \leqq 0$. Therefore, noting that $T^{k_{\nu_0}}(\tilde{\alpha}_{2j\nu_0})$ intersects to $T^{k_{\nu_0}}(\tilde{\alpha}_{2j\nu_0-1})$, we see that $\tilde{C}^*$ intersects to $\tilde{C}$.

Now it is immediately seen that $\tilde{C}$ separates γ_2 from γ_1. For there exists a relative cycle from $\tilde{p}$ to γ_1, e. g. $\Pi_{k=1}^{\infty} T^{-km_1}(\tilde{\alpha}_2)$ which does not intersect to $\tilde{C}$ and similarly one from $T(\tilde{p})$ to γ_2.

§ 3. Abelian covering surface with finite spherical area

12. Let R be an arbitrary Riemann surface and f be a meromorphic function on R. We introduce a quantity defined by

$$I(f)=\iint_R \frac{|df/d\zeta|^2}{(1+|f|^2)^2}\, d\xi d\eta\,,$$

where $\zeta=\xi+i\eta$ is a local uniformizing parameter at a point on R. It expresses the spherical area of the covering surface over the Riemann sphere which is formed by the image of R by f. We denote by O_{MD} the class of Riemann surfaces R which do not admit any non-constant meromorphic function f with $I(f)<\infty$ (cf. [17]). We say briefly that R has *finite spherical area* if $R \notin O_{MD}$.

By the *valence* $\mathfrak{v}_f$ of f we mean the function on the w-sphere S defined by

$$\mathfrak{v}_f(w)=\sum_{f(p)=w} \mu(p;\, f)\,, \qquad w \in S\,,$$

where $\mu(p\,;\, f)$ is the multiplicity of f at p. Let $\mathfrak{B}(R)$ be the class of non-constant meromorphic functions of bounded valence on R. We denote by O_V the class of Riemann surfaces R with $\mathfrak{B}(R)=\phi$.

It is known (cf. [5], [10]) that if $R \in O_G$, O_G being the class of Riemann surfaces not admitting Green's function, two alternative cases can occur; namely

(i) $\mathfrak{v}_f(w) \equiv \text{const} < \infty$ except for a set of w of capacity zero, and

(ii) $\mathfrak{v}_f(w) \equiv \infty$ except for a set of w of capacity zero.

Thus we can easily see that if $R \in O_G$, either R belongs to O_V and O_{MD} simultaneously or not.

In the present section we shall state a *function-theoritic character* of abelian covering surfaces with finite spherical area of the class O_G. For some special cases, related problems have already been discussed by Ozawa [13], M. Tsuji [18] and the author [9].

In the present paper we concern ourselves with the problem for only the case I. (H) in 8. In the next paper we will concern ourselves with the similar problem for the other cases I. (P), II. and III. in 8.

13. Let R be an open Riemann surface of the class O_G and Ω be an end of R. Throughout the following, by Ω we shall mean an end (with subscripts if necessary). We shall need several lemmas (the lemmas 11, 12 and 13) for an analytic function on an end Ω, which have been proved in [4] for the case where the end has only one ideal boundary component (see [4] and also cf. [7]). Since in our case they can also be proved analogously, we omit their proofs. (Cf. the theorems 4.1, 4.2 and 5.1 in [4] for the proof of the following lemmas 11, 12 and 13, respectively.)

LEMMA 11. *Any analytic function bounded on Ω possesses a limit at each ideal boundary component of Ω.*

Let φ be an analytic function on an end Ω_0. In analogy to [4], we define the *local degree $d(\varphi, \gamma)$ of φ at γ ($\Omega_0 \in \gamma$)* as follows: Let Ω be a generic end such that $\Omega \subset \Omega_0$ ($\Omega \in \gamma$) and $\mathfrak{n}(\varphi, \Omega) = \max_{w=\varphi(p), p \in \Omega} \mathfrak{v}_\varphi(w)$. We then put $d(\varphi, \gamma) = \min_{\Omega \in \gamma} \mathfrak{n}(\varphi, \Omega)$.

LEMMA 12. *If φ ($\not\equiv$ const) is meromorphic on Ω and $\Omega \in \gamma$, then either* (i) *φ possesses a limit at γ and is then $(1, d(\varphi, \gamma))$ on some subend $\Omega_1 (\subset \Omega$ and $\Omega_1 \in \gamma)$ with the exception of a set of capacity zero, on which φ is of valency less than $d(\varphi, \gamma)$, or else* (ii) *the set of limiting values at γ is the extended plane and φ assumes every value infinitely often except for a set of capacity zero.*

We suppose that φ_0 is a preferred analytic function bounded on Ω which has the minimal local degree d_0 at γ ($\Omega \in \gamma$) and that (i) φ_0 possesses a zero limit at γ and a non-zero limit at every other ideal boundary component of Ω, (ii) φ_0 does not assume zero on Ω and is $(1, d_0)$ on the closure of Ω with the exception of a set of capacity zero, and (iii) $|\varphi_0| = 1$ on the frontier of Ω.

LEMMA 13. *If there exists the above-mentioned analytic function φ_0 on Ω, then for each bounded analytic function φ ($|\varphi| < 1$) on Ω there exists a unique analytic function ψ ($|\psi| < 1$) on $|z| < 1$ such that $\varphi = \psi \circ \varphi_0$ holds on Ω.*

14. In the following we shall assume that $\tilde{R}$ is a Riemann surface of the class O_G which admits a conformal transformation group $\mathfrak{G} = \{T\}$ of the type I. (H) in 8. Then we have

THEOREM 2. *If $\tilde{R}$ has finite spherical area there exists a function $f_0 \in \mathfrak{B}(\tilde{R})$ uniquely determined except a multiplicative constant which satisfies the conditions*

$$f_0 \circ T(\tilde{p}) = t f_0(\tilde{p}), \qquad \tilde{p} \in \tilde{R}$$

and

$$f = g \circ f_0 \qquad \text{for each } f \in \mathfrak{B}(\tilde{R}),$$

where t is a complex constant $|t| > 1$ uniquely determined by $\tilde{R}$ and $\mathfrak{G}$, and g is a rational function.

The theorem can be proved by a similar argument with somewhat difference to the proposition 1 in [13]. For completeness, we shall state the proof in detail (**15**—**20**).

15. If $\tilde{R}$ has finite spherical area, then there exists a function $f \in \mathfrak{B}(\tilde{R})$ and $\mathfrak{v}_f(w) \equiv \text{const} < \infty$ except for a set of w of capacity zero. *We can regard that $\tilde{R}$ is a finite covering surface over the w-sphere S such that f is the projection map of*

$\tilde{R}$ *onto* S. By the lemma 12, f has a limit at every ideal boundary component of $\tilde{R}$. Especially there exist the limits

$$\lim_{m\to\infty} f\circ T^{-m}(\tilde{p}) \text{ and } \lim_{m\to\infty} f\circ T^{m}(\tilde{p}).$$

Without loss of generality, *we may assume that* $\lim_{m\to\infty} f\circ T^{-m}(\tilde{p})=0$. We shall fix such a function f and make constructively the function f_0 in the theorem from it.

16. *We can find a suitable end* Ω_0 $(\Omega_0\in\gamma_1)$ *of* $\tilde{R}$ *and a preferred analytic function* φ_0 *on* Ω_0 *which has the minimal local degree* d_0 *at* γ_1 *and such that* (i) φ_0 *and* f *possess a zero limit at* γ_1 *and a non-zero limit at every other ideal boundary component of* Ω_0, (ii) φ_0 *and* f *do not assume zero on* Ω_0 *and* φ_0 *is* $(1, d_0)$ *on the closure of* Ω_0 *except for a set of capacity zero, and* (iii) $|\varphi_0|=1$ *on the boundary of* Ω_0 *and* $\sup_{\Omega_0}|f|<1$.

Proof. Since f has bounded valence, it has only a finite number of zero points and only a finite number of ideal boundary components such that $\lim_{\tilde{p}\to\gamma} f(\tilde{p})=0$ $(\gamma\neq\gamma_1)$. Then we can take an end $\Omega^*(\in\gamma_1)$ which does not contain any zero point and which separates all those ideal boundary components from γ_1. We can select a constant r_1 such that $0<r_1\leqq\min(\min_{\partial\Omega^*}|f|, 1)$ and any point on $|w|=r_1$ is not a defect value of f. Then, $\Omega_1=\{\tilde{p}|\,|f(\tilde{p})|<r_1\}(\subset\Omega^*)$ is an end of γ_1 and f/r_1 on Ω_1 satisfies the conditions (i), (ii) and (iii) imposed for φ_0 on Ω_0. Let $\varphi_0{}^*$ be one of the functions analytic, bounded on Ω_1 such that $\lim_{\tilde{p}\to\gamma_1}\varphi_0{}^*(\tilde{p})=0$ and which have the minimal local degree at γ_1. By the similar procedure as for f, we can find a subend $\Omega_0(\in\gamma_1)$ of Ω_1 such that $\varphi_0\equiv r_0\varphi_0{}^*$ (r_0 being a suitable positive constant) satisfies all the conditions imposed for φ_0 on Ω_0. Of course, f also satisfies all the conditions imposed for f on Ω_0. Here, owing to taking the subend $\Omega_0(\subset\Omega_1)$, we can no more guarantee in general that $\varphi_0{}^*$ provides the minimal local degree at γ_1 among a family of the functions on Ω_0. However, by taking r_1 sufficiently small previously, we can guarantee that the minimal local degree at γ_1 is invariant even if we take any subend $\Omega_0(\in\gamma_1)$ of Ω_1.

17. By **16** and the lemma 13, there holds $f=\psi_0\circ\varphi_0$ on Ω_0, where ψ_0 is a unique analytic function on $|w|<1$, $|\psi_0|<1$, $\psi_0(0)=0$ and $\psi_0(w)\neq 0$ for $w\neq 0$. We may suppose that ψ_0 has no multiple points except possibly at the origin, by reselecting Ω_0 and φ_0 if necessary. Let ψ_0 have the multiplicity $\mu(\geqq 1)$ at the origin. Then $w^{1/\mu}\circ\psi_0$ is one-valued, univalent on $|w|<1$, and thus $\varphi=w^{1/\mu}\circ f=w^{1/\mu}\circ\psi_0\circ\varphi_0$ is a one-valued analytic function defined on Ω_0 which has the minimal local degree d_0 at γ_1, possesses a zero limit at γ_1 and is $(1, d_0)$ on the closure of Ω_0 except for a set of capacity zero.

Let $\Omega=\{\tilde{p}|\,|\varphi(\tilde{p})|<r\}$ be a subend of $\Omega_0\frown T^{-1}(\Omega_0)$ such that $\partial\Omega=\{\tilde{p}|\,|\varphi(\tilde{p})|=r\}$ forms the compact boundary of Ω and does not contain multiple points of f, where the real constant r should be selected suitably small. Then φ maps Ω onto an r-disk $G=\{|z|<r\}$ with some defect set of capacity zero, and $\varphi\circ T$ is bounded, $|\varphi\circ T|$

<1, on $\Omega(\subset T^{-1}(\Omega_0))$. Further, φ and $\varphi\circ T$ possess a zero limit at γ_1, are $(1, d_0)$ on the closure of Ω except for a set of capacity zero, and have the minimal local degree d_0 at γ_1. Therefore, by the lemma 13, *we have a functional relation*

$$(9) \qquad \varphi\circ T=\psi\circ\varphi \qquad \text{on } \Omega,$$

where ψ is an analytic function of z on G which is one-valued, univalent, bounded, $|\psi|<1$, and $\psi(0)=0$.

18. $\varphi=w^{1/\mu}\circ f$ maps Ω onto the r-disk with the defect set of capacity zero. Let G' be a set of points of $G=\{|z|<r\}$ which are not defect values of φ and also which are not images of multiple points of φ. Then, $G-G'$ is a set of capacity zero and thus $\Omega-\Omega'$ is possibly a Green null set on Ω, being $\Omega'=\varphi^{-1}(G')$.

Let z_0 be an arbitrarily point of G', $\tilde{p}_1^0, \dots, \tilde{p}_{d_0}^0$ be all points with the φ-image z_0, i.e. $z_0=\varphi(\tilde{p}_\nu^0)$ $(\nu=1, \dots, d_0)$. Then we have that

$$f\circ T^m(\tilde{p}_1^0)=\cdots=f\circ T^m(\tilde{p}_{d_0}^0) \quad (m=0, 1, \dots).$$

Proof. First, obviously

$$f(\tilde{p}_1^0)=\cdots=f(\tilde{p}_{d_0}^0)=w_0(=z_0{}^\mu).$$

Let $\tilde{p}_1^1=T(\tilde{p}_1^0), \dots, \tilde{p}_{d_0}^1=T(\tilde{p}_{d_0}^0)$. Then by (9) we have

$$f(\tilde{p}_\nu^1)=f\circ T(\tilde{p}_\nu^0)=z^\mu\circ\varphi\circ T(\tilde{p}_\nu^0)=z^\mu\circ\psi\circ\varphi(\tilde{p}_\nu^0)=z^\mu\circ\psi(z_0)=w_1 \qquad (\nu=1, \dots, d_0),$$

that means, f takes the common value w_1 at all $\tilde{p}_\nu^1$ $(\nu=1, \dots, d_0)$.

If we take $w=f(\tilde{p})$ as local uniformizers at the points $\tilde{p}_1^0, \dots, \tilde{p}_{d_0}^0$, then by (9) we have

$$(10) \qquad f\circ T(\tilde{p}_\nu)=z^\mu\circ\psi\circ\varphi(\tilde{p}_\nu)=z^\mu\circ\psi\circ w^{1/\mu}\circ f(\tilde{p}_\nu)=z^\mu\circ\psi\circ w^{1/\mu}(w)$$
$$\text{in } \tilde{p}_\nu\in U(\tilde{p}_\nu^0) \qquad (\nu=1, \dots, d_0)$$

on taking a branch of $w^{1/\mu}$ such that $w^{1/\mu}(w_0)=z_0$, where $U(\tilde{p}_\nu^0)$ $(\nu=1, \dots, d_0)$ are the definition regions of the local uniformizers $w=f(\tilde{p}_\nu)$ at $\tilde{p}_\nu^0$ such that $f(U(\tilde{p}_\nu^0))=\{w||w-w_0|<r(w_0)\}$ and $U(\tilde{p}_\nu^0)\subset\Omega$. It means that $f\circ T(\tilde{p}_\nu)$ $(\nu=1, \dots, d_0)$ are the common analytic functions of local uniformizers $w=f(\tilde{p}_\nu)$ on $U(\tilde{p}_\nu^0)$.

We can find an analytic curve $\Lambda(w_0, w_1)$ on S from w_0 to w_1 such that each image curve $C(\tilde{p}_\nu^0)$ running from $\tilde{p}_\nu^0$ $(\nu=1, \dots, d_0)$ of $\Lambda(w_0, w_1)$ by f^{-1} is a simple curve on $\Omega\cup T(\Omega)$ not passing through the multiple points of f. Then each $C(\tilde{p}_\nu^0)$ $(\nu=1, \dots, d_0)$ ends one-to-one corresponding to one of $\tilde{p}_1^1, \dots, \tilde{p}_{d_0}^1$. If we continue analytically the functions $f\circ T(\tilde{p}_\nu)$ along $C(\tilde{p}_\nu^0)$ $(\nu=1, \dots, d_0)$, then by (10) we see that $f\circ T(\tilde{p}_\nu^1)$ $(\nu=1, \dots, d_0)$ takes a common value independent of ν, that is,

$$f\circ T(\tilde{p}_1^1)=\cdots=f\circ T(\tilde{p}_{d_0}^1)=w_2.$$

By the similar successive process, we see that

$$f\circ T^m(\tilde{p}_1^0)=\cdots=f\circ T^m(\tilde{p}_{d_0}^0) \qquad (m=0, 1, \dots).$$

We identify the points $T^m(\tilde{p}_1^0), \dots, T^m(\tilde{p}_{d_0}^0)$ on $\tilde{R}$ for each $z_0\in G'$ and each m $(m=0, 1, \dots)$. Then any point on $\tilde{R}$ possibly with the exception of a Green null set takes a part in this identification process. For there exists a positive integer m_0 for any $\tilde{p}\in\tilde{R}$ such that $T^{-m}(\tilde{p})\in\Omega$ for $m\geqq m_0$. By this successive identification process, we obtain a finitely-sheeted covering surface W' over S over which $\tilde{R}'$ is a d_0-sheeted covering surface, where $\tilde{R}'=\cup_{m=1}^{\infty}T^m(\Omega')$.

19. Let $\mathfrak{f}$ be a projection map of R' onto W'. Then the covering transformation T of $\tilde{R}$ onto itself induces the one-to-one conformal transformation $\mathfrak{g}$ of W' onto itself which is uniquely determined by the functional relation

$$\mathfrak{f}\circ T=\mathfrak{g}\circ\mathfrak{f}. \tag{11}$$

Now let Δ_m' $(m=0, 1, \ldots)$ be the images on W' of $T^m(\Omega')$ by $\mathfrak{f}$, respectively. Then, $\mathfrak{g}^{-m}$ $(m=1, 2, \ldots)$ map one-to-one conformally Δ_m' onto Δ_0', respectively. Since $\varphi\circ\mathfrak{f}^{-1}$ is a one-valued function on Δ_0' independent of the choice of branches of $\mathfrak{f}^{-1}$ and defines a one-to-one conformal mapping of Δ_0' onto the r-disk G' with the exception of a set of capacity zero, we see that Δ_0' can be uniquely continued to a region Δ_0 conformally equivalent to the whole r-disk G, preserving the original conformal structure of Δ_0', and thus Δ_m' $(m=1, 2, \ldots)$ can also be continued to regions Δ_m conformally equivalent to G, respectively. Then, $W'=\cup_{m=0}^{\infty}\Delta_m'$ can be uniquely continued to the covering surface $\mathring{W}=\cup_{m=0}^{\infty}\Delta_m$ on S, preserving the original conformal structure of W', and the mapping $\mathfrak{f}$ of $\tilde{R}'$ onto W' can be analytically continued to the projection map (again denoted by $\mathfrak{f}$) of $\tilde{R}$ into $\mathring{W}$. Further, $\mathfrak{g}$ can be extended to the one-to-one conformal transformation of $\mathring{W}$ onto $\mathring{W}$.

$\mathring{W}$ is a simply-connected covering surface on S with only one ideal boundary component $\mathfrak{p}_2$, corresponding to the ideal boundary component γ_2 of $\tilde{R}$ by $\mathfrak{f}$, and thus $\mathring{W}$ can be continued to a simply-connected compact Riemann surface W, preserving the original conformal structure.

Proof. Since $\partial\Delta_\nu$ $(\nu=0, 1, \ldots)$ are analytic curves on W', $W_m=\cup_{\nu=0}^{m}\Delta_\nu$ $(\subset\mathring{W})$ $(m=0, 1, \ldots)$ are bordered Riemann surfaces. Further W_m $(m=0, 1, \ldots)$ are planar, for there exists a number m' $(>m)$ for each m such that W_m is one-to-one mapped into Δ_0 by $\mathfrak{g}^{-m'}$. If we note that there exists a number m_1 $(>m_0)$ for any number m_0 such that $\overline{W}_{m_0}\subset W_m$ for all $m\geqq m_1$ and $\mathfrak{g}^m(\mathfrak{p})$ for any $\mathfrak{p}\in\mathring{W}$ tends to the ideal boundary component $\mathfrak{p}_2$ of $\mathring{W}$, corresponding to the ideal boundary component γ_2 of $\tilde{R}$ by $\mathfrak{f}$, as $m\to\infty$, then we see that ∂W_m tends to $\mathfrak{p}_2$ as $m\to\infty$. Thus we see that $\mathring{W}$ is planar and has only one ideal boundary component $\mathfrak{p}_2$.

Now let $\mathfrak{f}$ be again the projection map of $\tilde{R}$ into W and $\mathfrak{g}$ be again a one-to-one conformal mapping of W onto itself. It is easily seen that $\mathfrak{g}$ has two fixed points $\mathfrak{p}_1$ and $\mathfrak{p}_2$ on W corresponding to the ideal boundary components γ_1 and γ_2 of $\tilde{R}$ by $\mathfrak{f}$, respectively. Then, we have the functional relation (11) on $\tilde{R}$.

The Riemann surface W can be mapped one-to-one conformally onto S. Let χ be one of such mapping functions satisfying

$$\chi(\mathfrak{p}_1)=0 \text{ and } \chi(\mathfrak{p}_2)=\infty. \tag{12}$$

Then $\chi\circ\mathfrak{g}\circ\chi^{-1}$ is a one-to-one conformal mapping of S onto itself with the only fixed points 0 and ∞. Thus it must be

$$\chi \circ \mathfrak{g} \circ \chi^{-1}(w) = tw,$$

where t is a constant with $|t| \neq 1$. Thus, by (11) we have

$$\chi \circ \mathfrak{f} \circ T = t\chi \circ \mathfrak{f}. \tag{13}$$

If we define a meromorphic function $f_0 \in \mathfrak{V}(\tilde{R})$ on $\tilde{R}$ by $\chi \circ \mathfrak{f}$, then by (13) we have

$$f_0 \circ T = tf_0 \quad \text{on } \tilde{R}. \tag{14}$$

Obviously, $\max_w \mathfrak{v}_{f_0}(w) = d_0$. By (12) and the definition of f_0 we have $\lim_{m\to\infty} f_0 \circ T^m(\tilde{p}) = \infty$ ($\tilde{p} \in \tilde{R}$) and thus by (14) it must be $|t| > 1$.

If we put $g = \mathfrak{w} \circ \chi^{-1}$, $\mathfrak{w}$ being the projection map of W onto S, then g is obviously a rational function on S and there holds

$$f = \mathfrak{w} \circ \mathfrak{f} = \mathfrak{w} \circ \chi^{-1} \circ f_0 = g \circ f_0.$$

20. *The proof of uniqueness.* Let f_1 be another function of $\mathfrak{V}(\tilde{R})$ and let f_{10} be the function of $\mathfrak{V}(\tilde{R})$ constructed from f_1 by the above procedure which satisfies the conditions

$$f_{10} \circ T = t_1 f_{10}$$

and

$$f_1 = g_1 \circ f_{10}, \tag{15}$$

where t_1 is a complex constant $|t_1| > 1$ and g_1 is a rational function on S. Then, by the construction method of f_0 and f_{10}, we see that there exists a function g_{10} which maps one-to-one conformally S onto itself and which satisfies the condition

$$f_{10} = g_{10} \circ f_0.$$

Since g_{10} has two fixed points $w=0$ and $w=\infty$, it must be of the form

$$g_{10}(w) = c_1 w,$$

c_1 being a constant. Then we have

$$f_{10} = c_1 f_0. \tag{16}$$

Substituting (16) to (15), we have

$$f_1 = g_2 \circ f_0,$$

where $g_2(w) \equiv g_1(c_1 w)$ is a rational function.

Finally, let f_0^* be an arbitrary function of $\mathfrak{V}(\tilde{R})$ satisfying the conditions

$$f_0^* \circ T = t^* f_0^* \quad \text{on } \tilde{R} \tag{17}$$

and

$$f = g^* \circ f_0^* \quad \text{for each } f \in \mathfrak{V}(\tilde{R}),$$

where t^* is a complex constant $|t^*| > 1$ and g^* is a rational function. Then there exist rational functions g_0, g_0^* on S such that

$$f_0^* = g_0 \circ f_0, \qquad f_0 = g_0^* \circ f_0^*.$$

Then we have

$$g_0^{-1}=g_0^*.$$

Thus we see that g_0 maps one-to-one conformally S onto itself with two fixed points $w=0$ and $w=\infty$ and thus it has the form

$$g_0(w)=cw,$$

c being constant. Then we have

$$f_0^*=cf_0. \tag{18}$$

Substituting (18) to (17), we have

$$cf_0\circ T=t^*cf_0$$

or

$$f_0\circ T=t^*f_0. \tag{19}$$

Comparing (19) with (14), we see that it must be

$$t^*=t.$$

21. In the theorem 2, f_0 is a function which has the minimal local degree d_0 at two ideal boundary components γ_1 and γ_2 of $\tilde{R}$, and $\max_w \mathfrak{v}_{f_0}(w)=d_0$. Thus f_0 takes all values on S d_0-times, except for a set of capacity zero, and never takes 0 and ∞ on $\tilde{R}$. Thus we can find a subregion Ω of $\tilde{R}$ defined by $|f_0|<r$ $(r>0)$ and such that no point on $|w|=r$ is an exceptional point of f_0 or the image of a multiple point of f_0. Then $\tilde{C}=\partial\Omega$ consists of a finite number of simple closed analytic curves $\tilde{C}_1, \ldots, \tilde{C}_{\kappa_0}$ $(\kappa_0\leqq d_0)$, and $|f_0|=r$ holds on $\tilde{C}$. By the theorem 2 the subregion F of $\tilde{R}$ defined by $r\leqq|f_0|<r|t|$, gives a fundamental region for the group $\mathfrak{G}$ and $R\equiv\tilde{R}\ (\mathrm{mod}\ \mathfrak{G})$ is obtained from F by identifying $\tilde{C}_j$ to $T(\tilde{C}_j)$ for each $j=1, \ldots, \kappa_0$. Thus $\tilde{R}$ must be conformally equivalent to such a d_0-sheeted covering surface $\tilde{R}^*$ on S that $\tilde{R}^*$ is mapped onto itself by the transformation $w \mid tw$.

22. THEOREM 3. *Let m_j $(j=1, \ldots, 2\kappa)$ be arbitrarily given integers with $(m_1, \ldots, m_{2\kappa})=1$. Then there exists a marked Riemann surface $\langle R\rangle$*[12] *of finite or infinite genus q $(\kappa\leqq q\leqq\infty)$ such that an abelian covering surface $\tilde{R}$ of $\langle R\rangle$ with a covering transformation group $\mathfrak{G}=\{T;\ \alpha_j=T^{m_j}\,(j=1, \ldots, 2\kappa), \alpha_j=I\,(j\geqq 2\kappa+1)\}$ has finite spherical area.*

Proof. By the lemma 10, it is sufficient to prove the theorem for only the case $m_{2j-1}=0$ and $m_{2j}\geqq 0$ $(j=1, \ldots, \kappa)$. Further it is not an essential restriction to assume that $m_{2j}>0$ $(j=1, \ldots, \kappa)$.

(i) *The case where $m_1=0$, $m_2=1$, and $\kappa=q=1$.*

Let $\tilde{R}=\{0<|z|<\infty\}$, $\mathfrak{G}=\{T;\ T=tz\ (t\text{: real, } t>1)\}$, $\tilde{\alpha}_1=\{|z|=1\}$, $\tilde{\alpha}_2=\{1\leqq\Re z<t, \Im z=0\}$ and $F=\{1\leqq|z|<t\}$. Let R be a Riemann surface (torus) obtained from F

12) By a marked Riemann surface $\langle R\rangle$ we mean the Riemann surface R preassigned a canonical homology basis. cf. ([3])

by identifying $\tilde{\alpha}_1$ to $T(\tilde{\alpha}_1)$, and α_1, α_2 be the images on R of $\tilde{\alpha}_1$, $\tilde{\alpha}_2$, respectively. Then the marked Riemann surface $\langle R\rangle$ obtained by the suitable selection of the orientation of α_1, α_2, provides a required one for the present case.

(ii) *The case where* $m_1=0$, $m_2=1$, $\kappa=1$ *and* $q>1$.

Let

$$T'=t^{1/2}z \qquad (t\text{: real, } t>1),$$

$$l_\nu^{1\prime}=\left\{|z|=t^{1/4},\ \frac{\pi}{2^{2\nu+1}}\leqq \arg z\leqq \frac{\pi}{2^{2\nu}}\right\},$$

$$l_\nu^{2\prime}=\left\{|z|=t^{1/4},\ \left(1+\frac{1}{2^{2\nu+1}}\right)\pi\leqq \arg z\leqq \left(1+\frac{1}{2^{2\nu}}\right)\pi\right\}$$

$$(\nu=1,\ \dots,\ q-1,\ \text{if } q<\infty;\ \ \nu=1,2,\ \dots,\ \text{if } q=\infty),$$

$$\tilde{F}'=\{0<|z|<\infty\}-\bigcup_{m=-\infty}^{\infty} T'^m\left(\bigcup_\nu l_\nu^{1\prime}\cup\bigcup_\nu l_\nu^{2\prime}\right),$$

$$\tilde{\alpha}_1{}'=\{|z|=1\},$$

$$\tilde{\alpha}_2{}'=\{1\leqq \Re z<t^{1/2},\ \Im z=0\}.$$

Let $\tilde{F}$ be a two-sheeted covering surface over w-plane obtained as an image of $\tilde{F}'$ by the transformation $w=z^2$. Then, we denote the images of $l_\nu^{1\prime}$, $l_\nu^{2\prime}$ $(\nu=1,2,\dots)$, $\tilde{\alpha}_1'$ and $\tilde{\alpha}_2'$ by l_ν^1, l_ν^2, $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$, respectively, and $T=tw$. Let $\tilde{R}$ be a Riemann surface obtained from $\tilde{F}$ by connecting crosswise each other along $T^m(l_\nu^1)$ and $T^m(l_\nu^2)$ for each ν and each m, respectively, and F be a subregion of $\tilde{R}$ surrounded by $\tilde{\alpha}_1$ and $T(\tilde{\alpha}_1)$. Let R be a Riemann surface obtained from F by identifying $\tilde{\alpha}_1$ with $T(\tilde{\alpha}_1)$, and α_1 and α_2 be the images on R of $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$. Then, by the suitable selection of the orientation of α_1, α_2 and of the remaining basis α_3, α_4, ... on R, we shall obtain one of the desired marked Riemann surfaces $\langle R\rangle$ for the present case.

(iii) *The case except the cases* (i) *and* (ii).

Then, necessarily $\kappa>1$. Let

$$\mathfrak{G}=\{T;\ T=tz\ (t\text{: real, } t>1)\},$$

$$l_j=\left\{|z|=t^{1/2},\ \frac{2j-1}{2\kappa}\pi\leqq \arg z\leqq \frac{j}{\kappa}\pi\right\} \qquad (j=1,\ \dots,\ \kappa-1),$$

$$l_k'=\left\{|z|=t^{1/2}\left(1+\frac{1}{2^{2k}}\right)\pi\leqq \arg z\leqq \left(1+\frac{1}{2^{2k-1}}\right)\pi\right\}$$

$$(k=1,\ \dots,\ q-\kappa,\ \text{if } q<\infty;\ k=1,2,\ \dots,\ \text{if } q=\infty),$$

$$\tilde{F}_1^1=\{0<|z|<\infty\}-\bigcup_{m=-\infty}^{+\infty} T^{m_2m}\left(l_1\cup\bigcup_k l_k'\right.,$$

$$\tilde{F}_j^1=\{0<|z|<\infty\}-\bigcup_{m=-\infty}^{+\infty} T^{m_{2j}m}(l_{j-1}\cup l_j) \qquad (j=2,\ \dots,\ \kappa-1),$$

$$\tilde{F}^1_\kappa=\{0<|z|<\infty\}-\bigcup_{m=-\infty}^{+\infty}T^{m_{2\kappa}m}{}'l_{\kappa-1}\smile\bigcup_k l_k{}',$$

$$\tilde{F}^\nu_j=T^{\nu-1}(\tilde{F}^1_j)\qquad(\nu=1,\ \dots,\ m_{2j}\,;\ j=1,\ \dots,\ \kappa),$$

where $\cup_k l_k'=\phi$ if $q=\kappa$. Let $\tilde{\alpha}_{2j-1}=\{|z|=1\}$ and $\tilde{\alpha}_{2j}=\{1\leqq\Re z<t^{m_{2j}},\ \Im z=0\}$ be curves on $\tilde{F}^1_j$ $(j=1, \dots, \kappa)$, and F^1_j be a subregion of $\tilde{F}^1_j$ surrounded by $\tilde{\alpha}_{2j-1}$ and $T^{m_{2j}}(\tilde{\alpha}_{2j-1})$. Let $\tilde{R}$ be a Riemann surface obtained from $\tilde{F}^1_1, \dots, \tilde{F}^{m_2}_1, \dots, \tilde{F}^1_\kappa, \dots, \tilde{F}^{m_{2\kappa}}_\kappa$ by connecting crosswise along the each slit corresponding up and down each other, where each slit corresponds obviously to one and only one slit. Since $(m_1, \dots, m_{2\kappa})=1$, $\tilde{R}$ is obviously connected. Further $\tilde{R}$ admits obviously the covering transformation group $\mathfrak{G}$. Let F^* be a subregion of $\tilde{R}$ surrounded by $\tilde{\alpha}_1$, $T^{m_2}(\tilde{\alpha}_1), \dots, \tilde{\alpha}_{2\kappa-1}$ and $T^{m_{2\kappa}}(\tilde{\alpha}_{2\kappa-1})$ and thus F^* consists of the portions $F^1_1, \dots, F^1_\kappa$ connected along the corresponding slits. Let R be a Riemann surface obtained from F^* by identifying $\tilde{\alpha}_{2j-1}$ to $T^{m_{2j}}(\tilde{\alpha}_{2j-1})$ $(j=1, \dots, \kappa)$, and α_j $(j=1, \dots, 2\kappa)$ be images on R of $\tilde{\alpha}_j$, respectively. Then it is easily shown that $R\equiv\tilde{R}$ (mod $\mathfrak{G}$). Then, by the suitable selection of the orientation of $\alpha_1, \dots, \alpha_{2\kappa}$ and of the remaining basis $\alpha_{2\kappa+1}, \dots$ on R, we shall obtain one of the desired marked Riemann surfaces $\langle R\rangle$ for the present case. We shall omit the detailed argument.

23. Let R be an open Riemann surface. We now consider a system of curves $\mathfrak{L}=\{L\}$ such that $L\subset R-K_0$, K_0 being a simply connected compact subregion of R, and $L\sim\partial K_0$. Suppose further that L consists a finite number of disjoint analytic dividing cycles of R. Let $E=\{R_n\}_{n=1}^{\infty}$ be an exhaustion of R such that $\partial R_n\in\mathfrak{L}$ and $\mathfrak{L}_E$ be the union of $\mathfrak{L}_n$ for all n where $\mathfrak{L}_n$ is a set of curves of $\mathfrak{L}$ contained in annuli including ∂R_n.[13] We shall denote by O'' a class of Riemann surfaces on which an exhaustion E exists such that $\lambda\{\mathfrak{L}_E\}=0$, where $\lambda\{\mathfrak{L}_E\}$ denote the extremal length of $\mathfrak{L}_E$. It is known that $O''\subset O_G$ (see [8]).

24. Let $\tilde{R}$ be a Riemann surface with finite spherical area in **14** and let $R\equiv\tilde{R}$ (mod $\mathfrak{G}$). Let $\alpha_1, \alpha_2, \dots$ be a canonical homology basis belonging to an exhaustion $E=\{R_n\}$ of R. Here if R is a closed Riemann surface of genus q, we may take an arbitrary canonical homology basis $\alpha_1, \dots, \alpha_{2q}$. We assume that $\mathfrak{G}=\{T\}$ and $\alpha_j=T^{m_j}$ $(j=1, 2, \dots)$ where $m_j=0$ for $j>2\kappa$ and $(m_1, \dots, m_{2\kappa})=1$.

Let $\zeta=\zeta(\tilde{p})$ be a local uniformizing parameter at $\tilde{p}\in\tilde{R}$. By the theorem 2 the differential $(f_0'/f_0)\,d\zeta$ is invariant under the group $\mathfrak{G}$, and thus it is an abelian differential of the first kind on the Riemann surface R. We can easily verify that it has a finite Dirichlet integral over R.

It is known (see [1]) that there exists a system of abelian differentials $d\omega_j$ of the first kind with finite Dirichlet integrals on R such that

$$\int_{\alpha_{2k-1}}d\omega_j=\delta^j_k\ ,\ \text{and}\ \int_{\alpha_{2k}}d\omega_j=\tau^j_k,\ \ \tau^j_k=\tau^k_j.$$

13) By an annulus including $L\in\mathfrak{L}$ we mean the union of doubly connected domains each of which includes a component of L.

We shall assume that R is a Riemann surface of the class O'' or is a closed Riemann surface of genus q.[14] Then, if $\alpha_1, \alpha_2, \ldots$ is a canonical homology basis belonging to the exhaustion $E=\{R_n\}$ of R satisfiying $\lambda\{\mathfrak{L}_E\}=0$, we have

$$\frac{f_0'}{f_0}d\zeta=2\pi i\sum_{j=1}^{\infty}c_j d\omega_j, \tag{20}$$

where

$$c_j=\frac{1}{2\pi i}\int_{\alpha_{2j-1}}\frac{f_0'}{f_0}d\zeta.$$

In the case of R of infinite genus, (20) can be verified as follows. Putting

$$\sum_{j=1}^{\nu}c_j d\omega_j=du_\nu+i\,dv_\nu, \qquad \frac{f_0'}{f_0}d\zeta=du+i\,dv;$$

$$du_\nu,\ dv_\nu,\ du,\ dv:\ \text{real differentials},$$

and using the theorem 6 in [8], we obtain

$$D_R(u-u_\nu)=\sum_{k=\nu+1}^{\infty}\left(\int_{\alpha_{2k-1}}du\int_{\alpha_{2k}}dv-\int_{\alpha_{2k}}du\int_{\alpha_{2k-1}}dv\right).$$

Since the right hand side tends to zero as $\nu\to\infty$ (see [9]), we obtain

$$D_R(u-u_\nu)\to 0 \qquad (\nu\to\infty),$$

and the desired equality.

By calculating the periodicity moduli of (20) along $C\equiv\Pi_{j=1}^{k}\alpha_{2j-1}{}^{-m_{2j}}\alpha_{2j}{}^{m_{2j-1}}$ and each homology base α_j ($j=1, 2, \ldots$), we have the following system of equations

$$\tag{21}\left\{\begin{aligned}
2\pi i d_0&=\int_C\frac{f_0'}{f_0}d\zeta=2\pi i\sum_{j=1}^{\infty}c_j\int_C d\omega_j\\
&=2\pi i\sum_{k=1}^{\kappa}\left(-m_{2k}c_k+m_{2k-1}\sum_{j=1}^{\infty}c_j\tau_k^j\right),\\
m_{2k-1}\log t+2\pi i\,m_{2k-1}{}^*&=\int_{\alpha_{2k-1}}\frac{f_0'}{f_0}d\zeta=2\pi i\sum_{j=1}^{\infty}c_j\int_{\alpha_{2k-1}}d\omega_j\\
&=2\pi i\,c_k \qquad (k=1, 2, \ldots),\\
m_{2k}\log t+2\pi i\,m_{2k}{}^*&=\int_{\alpha_{2k}}\frac{f_0'}{f_0}d\zeta=2\pi i\sum_{j=1}^{\infty}c_j\int_{\alpha_{2k}}d\omega_j\\
&=2\pi i\sum_{j=1}^{\infty}c_j\tau_k^j \qquad (k=1, 2, \ldots).
\end{aligned}\right.$$

Here $m_{2k-1}{}^*$, $m_{2k}{}^*$ ($k=1, 2, \ldots$) are integers. However we have

14) In the following, the argument shall be done for only the case where R is of infinite genus because it is done similarly and more easily for the other cases.

LEMMA 14. *There exists a number κ^* such that $m_k{}^*=0$ for all $k>2\kappa^*$.*

Proof. We would assume that $m_k{}^*\neq 0$ for an infinite number of k, e. g. $m_{k_\nu}{}^*\neq 0$ ($\nu=1, 2, \ldots$). Then for any θ ($0\leqq\theta<2\pi$) there exists at least one point $p_\nu(\theta)$ on α_{k_ν} such that $\arg f_0(\tilde{p}_\nu(\theta))\equiv\theta$ (mod 2π; $\nu=1, 2, \ldots$), where $\tilde{p}_\nu(\theta)$ ($\nu=1, 2, \ldots$) is the point on $F=\{\tilde{p}|\ r\leqq|f_0(\tilde{p})|<r|t|\}$ lying over $p_\nu(\theta)$. We may assume that two point sequences $\{p_\nu(0)\}_{\nu=1}^\infty$ and $\{p_\nu(\pi)\}_{\nu=1}^\infty$ tend to ideal boundary components γ' and γ'' of R, respectively, if necessary by taking subsequences with a set of common subindices of them. Then, two number sequences $\{f_0(\tilde{p}_\nu(0))\}_{\nu=1}^\infty$ and $\{f_0(\tilde{p}_\nu(\pi))\}_{\nu=1}^\infty$ converge, and $r<\lim_{\nu\to\infty}f_0(\tilde{p}_\nu(0))<r|t|$ and $-r|t|<\lim_{\nu\to\infty}f_0(\tilde{p}_\nu(\pi))<-r$. Thus, by the lemma 11, $\gamma'\neq\gamma''$. On the other hand, for any compact region K on R there exists a number ν_0 such that $\alpha_{k_\nu}\subset R-K$ for all $\nu\geqq\nu_0$, which shows that $\gamma'=\gamma''$. Contradiction.

Therefore by (21) and the lemma 14 we have a system of algebraic equations:

$$\frac{\sum_{j=1}^{\kappa^*} m_{2j-1}^*\tau_j^1-m_2{}^*}{\sum_{j=1}^{\kappa} m_{2j-1}\,\tau_j^1-m_2}=\frac{\sum_{j=1}^{\kappa^*} m_{2j-1}^*\tau_j^2-m_4{}^*}{\sum_{j=1}^{\kappa} m_{2j-1}\,\tau_j^2-m_4}=\cdots \tag{22}$$

$$(m_{2j}{}^*=0 \quad \text{for } j>\kappa^*),$$

and a relation

$$\sum_{k=1}^{\kappa}(m_{2k-1}\,m_{2k}^*-m_{2k}\,m_{2k-1}{}^*)=d_0\,, \tag{23}$$

where $\sum_{j=1}^{\kappa^*}m_{2j-1}^*\tau_j^k-m_{2k}^*=0$ if and only if $\sum_{j=1}^{\kappa}m_{2j-1}\,\tau_j^k-m_{2k}=0$.

25. Now we shall proceed to the converse problem. If (22) has a solution $(m_1{}^*, \ldots, m_{2\kappa^*}^*)$ for some $\kappa^*\geqq 1$ whose menbers are all integers and at least one of them is not zero, then $(m_1{}^*, \ldots, m_{2\kappa^*}^*)$ is called to be an *admissible integral system* for (22). With this terminology we see that $(m_1{}^*/m, \ldots, m_{2\kappa^*}^*/m)$ for a common measure m of $m_1{}^*, \ldots, m_{2\kappa^*}^*$ and $(m'm_1{}^*, \ldots, m'm_{2\kappa^*}^*)$ for any non-zero integer m' are also admissible for (22), if so is $(m_1{}^*, \ldots, m_{2\kappa^*}^*)$. Among all the admissible integral system for (22) there exists an unique system $(m_1{}^{*0}, \ldots, m_{2\kappa^*}{}^{*0})$ for which the minimum d_0 in (23) is provided,

Now we shall assume that (22) has an admissible integral system $(m_1{}^{*0}, \ldots, m_{2\kappa^*}{}^{*0})$ with the minimum d_0 in (23). Then, putting

$$Z=-\frac{\sum_{j=1}^{\kappa^*} m_{2j-1}^{*0}\,\tau_j^1-m_2^{*0}}{\sum_{j=1}^{\kappa} m_{2j-1}\,\tau_j^1-m_2}=-\frac{\sum_{j=1}^{\kappa^*} m_{2j-1}^{*0}\tau_j^2-m_4^{*0}}{\sum_{j=1}^{\kappa} m_{2j-1}\tau_j^2-m_4}=\cdots,$$

the differential

$$dh=\sum_{j=1}^{\kappa^{**}}(m_{2j-1}\,Z+m_{2j-1}{}^{*0})d\omega_j$$

satisfies the period relations

$$\begin{cases} \displaystyle\int_{\alpha_{2k-1}} dh = m_{2k-1}Z + m_{2k-1}^{*0} & (k=1, 2, \ldots), \\ \displaystyle\int_{\alpha_{2k}} dh = m_{2k}Z + m_{2k}^{*0} & (k=1, 2, \ldots), \\ \displaystyle\int_{C} dh = d_0, \end{cases}$$

where $\kappa^{**}=\max(\kappa, \kappa^*)$ and $m_{2k-1}^{*0}=m_{2k}^{*0}=0$ for all $k>\kappa^*$. Here the imaginary part of Z does not reduce to zero, because no non-zero abelian differential of the first kind on $R\in O''$ with finite Dirichlet integral has a real periodicity modulus along every cycle. Thus the potential function

$$f(\tilde{p})=\exp\left(2\pi i\int^{\tilde{p}} dh\right)$$

has the periods t^{m_k} ($|t|\neq 1$) along α_k ($k=1, 2, \ldots$), where $t=\exp(2\pi iZ)$. Thus f is an analytic function one-valued and regular on $\tilde{R}$, satisfies the functional relation

$$f\circ T(\tilde{p})=tf(\tilde{p})$$

and has the minimum local degree d_0 at two ideal boundary components γ_1 and γ_2 of $\tilde{R}$. Then the function $f_0\equiv f$ or $f_0\equiv 1/f$ corresponding to $|t|>1$ or $|t|<1$, respectively, provides the property in the theorem 2.

26. By the argument throughout **24** and **25**, we obtain the following consequence.

THEOREM 4. *Let R be a Riemann surface of O'', $\alpha_1, \alpha_2, \ldots$ be a canonical homology basis belonging to the exhaustion E satisfying $\lambda\{\mathfrak{L}_E\}=0$, and $(\tau_k^j)_{j,k=1,2,\ldots}$ be the period matrix corresponding to the canonical homology basis $\alpha_1, \alpha_2, \ldots$. Let $\tilde{R}$ ($\in O_G$) be an abelian covering surface of R, and have a covering transformation group $\mathfrak{G}=\{T\}$ and a system of generators $\alpha_1, \ldots, \alpha_{2\kappa}$ with $\alpha_j=T^{m_j}$ ($j=1, \ldots, 2\kappa$). Then, $\tilde{R}$ has finite spherical area if and only if there exists an admissible integral system $(m_1^*, \ldots, m_{2\kappa^*}^*)$ for some κ^* such that the period matrix $(\tau_k^j)_{j,k=1,2,\ldots}$ satisfies the system of algebraic equations*

$$(22) \qquad \frac{\sum_{j=1}^{\kappa^*} m_{2j-1}^*\tau_j^1 - m_2^*}{\sum_{j=1}^{\kappa} m_{2j-1}\tau_j^1 - m_2} = \frac{\sum_{j=1}^{\kappa^*} m_{2j-1}^*\tau_j^2 - m_4^*}{\sum_{j=1}^{\kappa} m_{2j-1}\tau_j^2 - m_4} = \cdots,$$

where $m_{2j}=0$ for $j>\kappa$ and $m_{2j}^*=0$ for $j>\kappa^*$.

If the canonical homology basis $\alpha_1, \alpha_2, \ldots$ is *regular* for $\tilde{R}$ which can be taken by the lemma 10, the condition (22) is replaced by the simpler system

$$(24) \qquad \frac{1}{m_2}\left(\sum_{j=1}^{\kappa^*} m_{2j-1}^*\tau_j^1 - m_2^*\right) = \frac{1}{m_4}\left(\sum_{j=1}^{\kappa^*} m_{2j-1}^*\tau_j^2 - m_4^*\right) = \cdots.$$

(Cf. [9] for the case where $m_{2j}=1$ ($j=1, \ldots, \kappa$) and $m_j=0$ for other j.)

If R is *of finite genus* q, then the condition (22) has the following form

$$\frac{\sum_{j=1}^{q} m_{2j-1}{}^* \tau_j^1 - m_2{}^*}{\sum_{j=1}^{q} m_{2j-1} \tau_j^1 - m_2} = \cdots = \frac{\sum_{j=1}^{q} m_{2j-1}{}^* \tau_j^q - m_{2q}{}^*}{\sum_{j=1}^{q} m_{2j-1} \tau_j^q - m_{2q}}. \tag{25}$$

(Cf. [13] for the case where $m_2=1$ and $m_j=0$ for other j.)

The author should like to express his heartiest thanks to Professor Yûsaku Komatu and Professor Kôtaro Oikawa for their many valuable advices.

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