REMARKS TO "THE ADJOINT PROCESS OF A DIFFUSION WITH REFLECTING BARRIER"

BY MASAO NAGASAWA AND KENITI SATO

1. The adjoint process is an analytical characterization of the reversed process. In discussing the adjoint process of diffusions on a domain with boundary, a generalization of the Green's formula plays an essential role (cf. lemma 3.2 and lemma 5.1 in [3]). We shall note that a simliar formula is available for more general boundary conditions (cf. [5], [6]), especially, with the term of sojourn effect and killing and next give some remarks for the adjoint process of the diffusion with the boundary conditions.

2. We shall use the notations of the section 5 in [3]. Let

$$L=\frac{\partial}{\partial n}+B, \quad L^{\circ}=\frac{\partial}{\partial n}-b_n+B^{\circ}, \quad L^{*}=\frac{\partial}{\partial n}+B^{*},$$

and φ be the positive solution of the equation $A^{\circ}\varphi=0$ in D and $L^{\circ}\varphi=0$ on ∂D.

Let $\delta(x)$ be a non-positive continuous function on ∂D and define a measure μ by

$$\mu(\cdot)=\int_{\cdot}\varphi(x)dx-\int_{\cdot\cap\partial D}\delta(x)\varphi(x)d\tilde{x}. \tag{2.1}$$

Lemma 5.1 of [3] states that the following holds for any u and v in $C^2(\bar{D})$:

$$\int_D\{uAv-vA^*u\}\varphi dx=-\int_{\partial D}\{uLv-vL^*u\}\varphi d\tilde{x}. \tag{2.2}$$

Therefore we have

$$\begin{aligned}&\int_D\{uAv-vA^*u\}\varphi dx-\int_{\partial D}\{u\delta Av-v\delta A^*u\}\varphi d\tilde{x}\\ &=-\int_{\partial D}\{u(L+\delta A)v-v(L^*+\delta A^*)u\}\varphi d\tilde{x},\end{aligned} \tag{2.3}$$

which is written, using the measure μ, as

$$\int_{\bar{D}}\{uAv-vA^*u\}d\mu=-\int_{\partial D}\{u(L+\delta A)v-v(L^*+\delta A^*)u\}\varphi d\tilde{x}. \tag{2.4}$$

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Here the value of Av (resp. A^*u) on the boundary is understood as

$$\lim_{y\to x,\ y\in D} Av(y) \qquad (\text{resp. } A^*u(y)).$$

Moreover we may add killing term. Thus we obtain the following

PROPOSITION 2.1. *Let u and v belong to $C^2(\bar{D})$,*[1] *then we have*

$$\int_{\bar{D}} \{u(A+k)v - v(A^*+k)u\}d\mu = -\int_{\partial D} \{uL_1v - vL_1^*u\}\varphi d\tilde{x}, \tag{2.5}$$

$$L_1 = L + \gamma + \delta A, \quad \text{and} \quad L_1^* = L^* + \gamma + \delta A^*, \tag{2.6}$$

where k is a non-positive continuous function on $\bar{D}$ and γ is a non-positive continuous function on ∂D.

3. We now apply proposition 2.1 to the Markov processes with Wentzell's boundary condition L_1. In this section the domain of L_1 is understood to be $\mathcal{D}(L_1) = \{f\colon f \in C^H(\bar{D}),\ L_1 f \in C(\partial D)\}$.[2] The process is characterized to have the resolvent operator G_α of a strongly continuous semi-group T_t on $C(\bar{D})$ defined by

$$G_\alpha u = R_\alpha u - H_\alpha(\overline{L_1H_\alpha})^{-1}(\bar{L}_1 R_\alpha u), \qquad \text{for } u \in C(\bar{D}), \tag{3.1}$$

satisfying

$$(\alpha - \bar{A}_1)G_\alpha u = u, \qquad A_1 = A + k, \tag{3.2}$$

$$\bar{L}_1 G_\alpha u = 0, \tag{3.3}$$

where $R_\alpha u$ is the solution of $(\alpha - A_1)f = u$ in D with $f=0$ on ∂D, $H_\alpha u$ is the solution of $(\alpha - A_1)f = 0$ in D with $f = u$ on ∂D, $\overline{L_1H_\alpha}$ is the closure of L_1H_α on $\mathcal{D}(L_1H_\alpha) = \{f\colon f \in C^H(\partial D),\ H_\alpha f \in \mathcal{D}(L_1)\}$, $\bar{A}_1$ is the closure of A_1 on $\mathcal{D}(A_1) = \{f\colon f \in C(\bar{D}) \cap C^2(D),\ A_1 f \in C(\bar{D})\}$, and $\bar{L}_1$ is the extension of L_1 defined by Ueno. For detailed definitions, we refer to Ueno [5] (for a sufficient condition of the existence of G_α, see also [5]). We shall call this process (A_1, L_1)-process.

Now let G be the generator with the domain $\mathcal{D}(G) = \{G_\alpha u\colon u \in C(\bar{D})\}$. For any $f = G_\alpha u \in \mathcal{D}(G)$, there exists such a sequence $f_n \in \mathcal{D}(L_1) \cap \mathcal{D}(A_1)$ that $\lim_{n\to\infty} f_n = f$ and $\lim_{n\to\infty} A_1 f_n$ exist. For example, we can take $f_n = R_\alpha u_n - H_\alpha(\overline{L_1H_\alpha})^{-1} v_n$ where $\lim_{n\to\infty} u_n = u$ with $u_n \in C^H(\bar{D})$ and $\lim_{n\to\infty} v_n = \bar{L}_1 R_\alpha u$ such that $(\overline{L_1H_\alpha})^{-1} v_n \in \mathcal{D}(L_1H_\alpha)$, then, in fact, $f_n \in \mathcal{D}(A_1) \cap \mathcal{D}(L_1)$ and $\lim_{n\to\infty} A_1 f_n = \alpha G_\alpha u - u \in C(\bar{D})$.

Consider now $(A+k, L_1)$-process and (A^*+k, L_1^*)-process, and let G and G^* be their generators, respectively. For any $f \in \mathcal{D}(G)$ and $g \in \mathcal{D}(G^*)$, take $f_n \to f$ and $g_n \to g$ as above, then we have by proposition 2.1

1) It is sufficient that u and v belong to $\mathcal{D}(A+k) \cap \mathcal{D}(L_1)$ and $\mathcal{D}(A^*+k) \cap \mathcal{D}(L_1^*)$, respectively (cf. § 3).

2) $C^H(\bar{D})$ and $C^H(\partial D)$ are the spaces of uniformly Hölder continuous functions on $\bar{D}$ and ∂D, respectively.

$$(3.4)\qquad \int_{\bar{D}}\{g_n(A+k)f_n-f_n(A^*+k)g_n\}d\mu=-\int_{\partial D}\{g_nL_1f_n-f_nL_1^*g_n\}\varphi d\tilde{x}.$$

Letting $n\to\infty$, it follows that

$$(3.5)\qquad \int_{\bar{D}}\{gGf-fG^*g\}d\mu=-\int_{\partial D}\{g\bar{L}_1f-f\bar{L}_1^*g\}\varphi d\tilde{x}=0.$$

Next, letting $u\equiv 1$ (resp. $v\equiv 1$), $v=f_n$ (resp. $u=g_n$) in (2.5) and $n\to\infty$, we have

$$(3.6)\qquad \int_{\bar{D}}Gfd\mu=\int_{\bar{D}}kfd\mu+\int_{\partial D}\gamma f\varphi d\tilde{x}\leqq 0,\qquad \text{for non-negative } f\in\mathcal{D}(G),$$

$$(3.7)\qquad \int_{\bar{D}}G^*gd\mu=\int_{\bar{D}}kgd\mu+\int_{\partial D}\gamma g\varphi d\tilde{x}\leqq 0,\qquad \text{for non-negative } g\in\mathcal{D}(G^*).$$

The equality holds if and only if $k=0$ and $\gamma=0$. Thus μ is the sub-invariant measure of $(A+k, L_1)$-process and (A^*+k, L_1^*)-process, in particular, the invariant measure when $k=0$ and $\gamma=0$. Therefore (3.5) implies that they are adjoint to one another with respect to μ. Thus we have

THEOREM 3.1. *$(A+k, L_1)$-process and (A^*+k, L_1^*)-process are, if exist, adjoint to one another with respect to the measure μ given in* (2.1) *which is the sub-invariant measure, in particular, the invariant measure when $k=0$ and $\gamma=0$.*

4. In the following let

$$Lu=\frac{\partial u}{\partial n}+\delta Au \quad\text{and}\quad L^*u=\frac{\partial u}{\partial n}+\delta A^*u,$$

where $\delta\in C^H(\partial D)$ and $\delta\leqq 0$. Let $\varphi\in C^2(\bar{D})$ be the positive solution of $A^\circ\varphi=0$ in D with $\partial\varphi/\partial n-b_n\varphi=0$ on ∂D (cf. Proposition 3.1 in [3]). Then (A, L)-diffusion $\boldsymbol{M}_\delta=(W_c, B(W_c), \bar{P}_x, x\in\bar{D})$ exists (cf. [1], [5]). Also we can prove the existence of (A^*, L^*)-diffusion $\boldsymbol{M}_\delta^*=(W_c, B(W_c), \bar{P}_x^*, x\in\bar{D})$.[3] Moreover it is known that they are obtained by a time change from reflecting A-(resp. A^*)-diffusion $\boldsymbol{M}$ (resp. $\boldsymbol{M}^*$) (cf. [4]). Consequently, $\boldsymbol{M}_\delta$ (resp. $\boldsymbol{M}_\delta^*$) has the same hitting probability as $\boldsymbol{M}$ (resp. $\boldsymbol{M}^*$). Further, it is easily verified that $\boldsymbol{M}_\delta$ (resp. $\boldsymbol{M}_\delta^*$) is recurrent. Therefore they have the unique common invariant measure (cf. appendix in [3]). Thus we have

THEOREM 4.1. *(A, L)-diffusion $\boldsymbol{M}_\delta$ and (A^*, L^*)-diffusion $\boldsymbol{M}_\delta^*$ have the common unique invariant measure μ defined by* (2.1) *with φ. They are adjoint to one another.*

3) For sufficiently large $\beta>0$, let $K_\beta f$ be the solution of the equation $(\alpha-A)v=0$ and $(\beta+b_n-\alpha\delta)v-\partial v/\partial n=f$, for $f\in C^H(\partial D)$ (cf. [2]). Then the solution of the equation $(\alpha-A^*)u=0$ and $(\beta-L^*)u=f$, for $f\in C^H(\partial D)$, is given by $u=\varphi^{-1}K_\beta(\varphi f)$. This is the sufficient condition of the existence of M_δ^*-diffusion (cf. [5]).

$(A+k, L+\gamma)$-diffusion and $(A^*+k, L^*+\gamma)$-diffusion are obtained by killings from $\boldsymbol{M}_\delta$ and $\boldsymbol{M}_\delta^*$, respectively. Hence we have

COROLLARY 4.1. *$(A+k, L+\gamma)$-diffusion and $(A^*+k, L^*+\gamma)$-diffusion are adjoint to one another with respect to μ.*

These are direct extensions of Theorem 3.1 in [3].

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DEPARTMENT OF MATHEMATICS, TOKYO INSTITUTE OF TECHNOLOGY, AND
DEPARTMENT OF MATHEMATICS, TOKYO METROPOLITAN UNIVERSITY.