A SUPPLEMENT TO "SZEGÖ KERNEL FUNCTION ON SOME DOMAINS OF INFINITE CONNECTIVITY"

BY MITSURU OZAWA

1. Let D and $\mathfrak{B}(D)$ be an n-ply connected analytic domain and the class of regular functions in D whose moduli are bounded by the value 1, respectively. In $\mathfrak{B}(D)$ there exists, up to a rotation, a unique extremal function by which the maximum $\max_{\mathfrak{B}(D)}|f'(z_0)|$ for a fixed point z_0 is attained. This extremal function $F(z, z_0)$ maps D onto the n times covered unit disc [1], [3], [4], [9], [11]. In $\mathfrak{B}(D)$ there exists an infinite number of essentially different functions which map D onto the n times covered unit disc [2], [5], [8]. Evidently the valence function $n(w)$ of these functions is equal to zero for any w lying in the exterior of the unit disc and n for any w lying in the interior of the unit disc. It is an important problem to define the meaning that the image domain of a mapping function covers a domain infinitely many times. Various attempts was made hitherto of this tendency. Among them Heins' notions of "*of type* Bl_1" and "*of type* Bl" give two important answers [6].

In the present paper we shall prove that the extremal function $F(z, z_0)$ is of type Bl_1 with at most one exceptional point at which the map is of type Bl in a case of an infinitely connected domain defined in later part. Is there a maximal planar domain for the class $\mathfrak{B}(D)$ on which the class of any extremals belonging to the class of type Bl is not empty while no extremal function of type Bl_1 exists? This problem is not yet settled, but it is very plausible to conjecture that there is such a domain. As far as we are aware, no systematic study has been made of this tendency for the class $\mathfrak{B}(D)$ in the case of infinitely connected domains.

2. Let D be a domain of finite type and of infinite island type defined in [10]. Then we have

$$\sum_{\nu=1}^{\infty} \frac{r_\nu}{a_\nu^2 - r_\nu^2} < \infty,$$

which is equivalent to a fact that the function $1/z$ is of class L^2. To proceed further we assume that the function $1/z^2$ belongs to the class L^2, which is equivalent to the condition (C) defined in [10]. Thus it is allowable to make free use of our earlier results.

Next we make some preparations on the positive or bounded harmonic functions in D. It is evident that there is no bounded harmonic function in D with vanishing boundary value on each C_n. Further, there is in D only one

Received April 20, 1961.

singular minimal positive harmonic function with vanishing boundary value on each C_n. For the definitions of these notions see [6]. This is not evident, since, in general, there is a domain for which this does not hold. In the first place we can map D onto a domain with an infinite number of slits lying on the real positive axis by the Riemann mapping theorem. Then we can apply the reasoning due to Kjellberg [7] sec. 23, since all situations needed are invariant by an arbitrary conformal mapping. For this minimal positive harmonic function $m(z)$ we have the following fact:

$$\lim_{z\to 0} m(z)=+\infty$$

along a path lying in an angular domain Θ: $\pi/2+\varepsilon<\arg z<3\pi/2-\varepsilon$ defined in [10].

Let $F(z, z_0)$ be the extremal function in the class $\mathfrak{B}(D)$. Then we have the representation

$$F(z, z_0)=\frac{k(z, z_0)}{l(z, z_0)}$$

and the uniqueness of the extremal function up to the rotation. And further we have the analyticity of $F(z, z_0)$ and $|F(z, z_0)|=1$ on each C_n and the limiting behaviors

$$k(z, z_0)-\overline{k_1(z_0)}=O(|z|) \quad \text{and} \quad l(z, z_0)+l_1(z_0)=O(|z|)$$

in the angular domain Θ. Thus we have

$$F(z, z_0)+\frac{\overline{k_1(z_0)}}{l_1(z_0)}=O(|z|)$$

there. The function $k_1(z_0)/l_1(z_0)$, as a function of z_0, is also bounded:

$$\left|\frac{k_1(z_0)}{l_1(z_0)}\right|\leqq 1$$

and is regular in D. If its modulus is equal to 1 at an inner point z_0, then it is a constant of modulus 1. However it was also proved already that $k_1(z_0)$ tends to a positive constant $\|k_1(z_0)\|^2$, when z_0 tends to the origin in Θ and $l_1(z_0)$ tends to the infinity simultaneously. Therefore we arrive at a contradiction. Thus we have a fact that

$$0\leqq\left|\frac{k_1(z_0)}{l_1(z_0)}\right|<1$$

at any point z_0 in D.

Let a be any constant lying in the unit disc $\{|w|<1\}$. Then we have

$$-\log\left|\frac{F(z, z_0)-a}{1-\bar{a}F(z, z_0)}\right|=\sum_{j=0}^{\infty} g(z, z_j(z_0))+\mu(z_0)\, m(z),$$

where the summation is taken over all the a-points $z_j(z_0)$ of $F(z, z_0)$ and $g(z, p)$ is the Green function of D with its singularity p and $\mu(z_0)$ is a non-negative constant. If a is not equal to $-\overline{k_1(z_0)}/l_1(z_0)$, then $\mu(z_0)$ must be equal to zero.

Indeed, suppose that $\mu(z_0)$ is not equal to zero, then the right hand side term tends to $+\infty$ but the left hand side term is bounded, when z tends to the origin in Θ. This is absurd. This leads to a relation

$$-\log\left|\frac{F(z, z_0)-a}{1-\bar{a}F(z, z_0)}\right| = \sum_{j=0}^{\infty} g(z, z_j(z_0))$$

and

$$\lim_{z\to 0}\sum_{j=0}^{\infty} g(z, z_j(z_0)) < \infty$$

in Θ. If a is equal to $-\overline{k_1(z_0)}/l_1(z_0)$, then we cannot yet say that $\mu(z_0)$ reduces to zero. However it must satisfy a system of simultaneous equations

$$\sum_{j=0}^{\infty}\omega_\nu(z_j(z_0)) + \mu(z_0)\frac{1}{2n}\int_{C_\nu}\frac{\partial}{\partial n}m(z)\,ds \equiv 0 \pmod 1,\ \nu = 1, 2, \cdots,$$

where $\omega_\nu(z)$ is the harmonic measure $\omega(z, C_\nu, D)$.

THEOREM. *Any extremal function $F(z, z_0)$ is of type Bl_1 for the unit disc $|w|<1$ with at most one exceptional point $w=-\overline{k_1(z_0)}/l_1(z_0)$, at which the map is locally of type Bl.*

3. It should be here remarked that the system of equations reduces to a simpler one in our case. We shall state only our result:

The image curve lying on the unit circle $|w|=1$ of each boundary circle C_ν by the function $F(z, z_0)$ circumscribes just once the origin, and further the system of equations

$$\sum_{j=0}^{\infty}\omega_\nu(z_j(z_0)) + \mu(z_0)\frac{1}{2\pi}\int_{C_\nu}\frac{\partial}{\partial n}m(z)\,ds = 1, \qquad \nu = 1, 2, \cdots$$

for $a=-\overline{k_1(z_0)}/l_1(z_0)$ is valid. These equations have a system of solutions $z_j(z_0)$ for any given z_0 and $-\overline{k_1(z_0)}/l_1(z_0)$.

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DEPARTMENT OF MATHEMATICS,
TOKYO INSTITUTE OF TECHNOLOGY.