WELDING OF POLYGONS AND THE TYPE OF RIEMANN SURFACES

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Introduction.

1. Let $f(x)$ be a continuous real-valued function which is defined on $0 \leqq x < \infty$, strictly monotone increasing, and is such that

$$f(0)=0 \quad \text{and} \quad \lim_{x\to+\infty} f(x)=+\infty.$$

Consider in the complex z-plane the half-strip

$$S=\{z;\ 0<\operatorname{Re} z<\infty,\ 0<\operatorname{Im} z<1\}$$

and put

$$l_1=\{z;\ 0<\operatorname{Re} z<\infty,\ \operatorname{Im} z=0\},\quad l_2=\{z;\ 0<\operatorname{Re} z<\infty,\ \operatorname{Im} z=1\},$$

and

$$\bar{S}=S\cup l_1\cup l_2.$$

On $\bar{S}$ with the natural relative topology, we identify

$$x\in l_1 \quad \text{with} \quad f(x)+i\in l_2$$

and get the space $\mathfrak{F}$ with the factor topology, which is a surface homeomorphic with a doubly-connected plane domain. Its boundary components will be denoted by B and B', where the former corresponds to $z=\infty$ in S. The identified $l_1=l_2$ is a simple open curve Λ in $\mathfrak{F}$ whose both ends tend to B and B' respectively. We shall denote by χ the projection of $\bar{S}$ onto the factor space $\mathfrak{F}$.

2. We say that *a Riemann surface $\mathfrak{R}$ is obtained from S by welding l_1 with l_2 by means of f* if we can introduce in $\mathfrak{F}$ a conformal structure to get a Riemann surface $\mathfrak{R}$ such that the mapping χ of S onto $\mathfrak{R}-\Lambda$ is conformal. As is seen from Example 1 given later, an f introduced in No. 1 does not necessarily permit us to do so.

Example 2 will show that, even though f permits the welding, the Riemann surface $\mathfrak{R}$ is not necessarily determined uniquely. Suppose, in general, we have $\mathfrak{R}_1$ and $\mathfrak{R}_2$ with projections χ_j $(j=1, 2)$ and seams Λ_j $(j=1, 2)$, respectively. The $\chi_2\circ\chi_1^{-1}$ is a homeomorphism of $\mathfrak{R}_1$ onto $\mathfrak{R}_2$ which is conformal on $\mathfrak{R}_1-\Lambda_1$. If $\chi_2\circ\chi_1^{-1}$ is conformal on $\mathfrak{R}_1$ for any pair of χ_1 and χ_2, we say that *a Riemann surface is obtained uniquely from S by welding l_1 with l_2 by means of f.*

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For instance, if f and f^{-1} are real analytic, then the welding is possible and, since Λ is an analytic curve in $\mathfrak{R}$, the $\mathfrak{R}$ is determined uniquely (see, e.g., Ahlfors-Sario [1], pp. 118–119).

If an $\mathfrak{R}$ is obtained by welding, it can be mapped conformally onto the annulus $0 \leqq q < |w| < 1$ so that B corresponds to $|w| = q$. We say that B is of *parabolic* type if $q = 0$, and *hyperbolic* otherwise.

The following problems, related closely to each other even though seem to be independent, have been discussed by Blanc [4, 5] and Volkoviskii [14, 15, 16], and the latter also by Nevanlinna [8, 9], Wirth [17], Sainouchi [12], and Jenkins [6]:*)

PROBLEM I. *Given $f(x)$ defined in No. 1, decide if the welding is possible and if it is possible uniquely.*

PROBLEM II. *Given $f(x)$ such that the welding is possible, determine the type of B.*

In the following lines, we shall use extremal length quite frequently. For its definition and elementary properties, the reader is referred to, e.g., Ahlfors-Sario [1], p. 220 ff.

§ 1. Possibility and uniqueness of welding.

3. We can see easily that an $\mathfrak{R}$ is obtained from S by welding l_1 with l_2 by means of f if and only if the following two requirements are fulfilled:

(i) Let ξ be an arbitrary point in $0 < \xi < \infty$. There exists $\varepsilon_\xi > 0$ such that the upper-half disc $D_{1\xi}$ with diameter $(\xi - \varepsilon_\xi, \xi + \varepsilon_\xi) \subset l_1$ and the lower-half disc $D_{2\xi}$ with diameter $(f(\xi - \varepsilon_\xi), f(\xi + \varepsilon_\xi)) \subset l_2$ are in S and mutually disjoint; there exists a simple arc c_ξ which is in $|w| < 1$ except for both endpoints on $|w| = 1$, and divides $|w| < 1$ into simply-connected domains $D'_{1\xi}$ and $D'_{2\xi}$; there exist conformal mappings $w = \varphi_{j\xi}(z)$ of $D_{j\xi}$ onto $D'_{j\xi}$ $(j = 1, 2)$ such that $\partial D_{j\xi} \cap l_j$ $(j = 1, 2)$ correspond to c_ξ, respectively, and that $\varphi_{2\xi}^{-1} \circ \varphi_{1\xi}(x) \equiv f(x)$ on $(\xi - \varepsilon_\xi, \xi + \varepsilon_\xi) \subset l_1$.

(ii) If $D_{1\xi} \cap D_{1\xi'} \neq \phi$, the conformal mappings $\varphi_{j\xi'} \circ \varphi_{j\xi}^{-1}$ of $\varphi_{j\xi}(D_{j\xi} \cap D_{j\xi'})$ onto $\varphi_{j\xi'}(D_{j\xi} \cap D_{j\xi'})$ $(j = 1, 2)$ are prolongable analytically to each other across $c_\xi \cap \varphi_{1\xi}(D_{1\xi} \cap D_{1\xi'})$.

Let $F(x)$ be a continuous strictly monotone increasing function defined on $a \leqq x \leqq b$. We shall simply say that $F(x)$ is *of class* $\boldsymbol{W}$ if the following C, Φ_1, and Φ_2 exist: C is a simple arc in $|w| < 1$ except for both end points on $|w| = 1$ which divides $|w| < 1$ into simply-connected domain D'_1 and D'_2; $\Phi_1(\Phi_2)$ is a conformal mapping of the upper- (lower-) half disc with diameter (a, b)

*) Added in proof. Prof. M. Ozawa notified the author that Problem I has been discussed also by Courant. See R. Courant, Dirichlet Principle. New York, 1950, p. 69, "Sewing Theorem".

$((F(a), F(b)))$ on the real axis onto D_1' (D_2') under which the diameter of D_1 (D_2) on the real axis corresponds to C; $\Phi_2^{-1}\circ\Phi_1(x)\equiv F(x)$ on $a\leqq x\leqq b$. An $F(x)$ of class W will be called *of class* W^* if, for any other C^*, Φ_1^*, and Φ_2^*, there exists a linear transformation L of $|w|<1$ onto itself such that $\Phi_j = L\circ\Phi_j^*$ on D_j $(j=1, 2)$. These notations have been introduced by Volkoviskii [14].

The previous statement (i) means that, for any ξ, the restriction of f on $[\xi-\varepsilon_\xi,\ \xi+\varepsilon_\xi]$ is of class W. Of course (i) does not imply (ii) (cf. Example 2). We note, however, that $\mathfrak{R}$ is determined uniquely by the welding if and only if the restriction of f on $[\xi-\varepsilon_\xi,\ \xi+\varepsilon_\xi]$ is of class W^* for every ξ.

We show here by an example that an $f(x)$ introduced in No. 1 does not necessarily permit us to get a Riemann surface $\mathfrak{R}$ from S by the welding:

EXAMPLE 1. *If $\alpha,\beta>0$ and $\alpha\neq\beta$, then the function*

$$F(x)=\begin{cases} x^\alpha & 0\leqq x\leqq 1,\\ -|x|^\beta & -1\leqq x\leqq 0\end{cases}$$

is not of class W on $-1\leqq x\leqq 1$.

Proof. Suppose that $F(x)$ is of class W and retain the previous notations. Let w_0 be the point in $|w|<1$ corresponding to $x=0$. $C-\{w_0\}$ consists of two analytic curves. Without loss of generality we assume that $\alpha>\beta$. Consider, for every r with $2^{-\alpha/\beta}<r<2^{-1}$, the smooth curve γ_r in $(|w|<1)-\{w_0\}$ defined as follows:

$$\gamma_r=\bigcup_{n=0}^{\infty}[\{r^{(\alpha/\beta)^n}e^{i\theta};\ -\pi\leqq\theta\leqq 0\}\cup\{r^{(\alpha/\beta)^n\beta^{-1}}e^{i\theta};\ 0\leqq\theta\leqq\pi\}],$$

where each arc has clockwise direction. Since every γ_r connects the point w_0 with the set in $(|w|<1)-\{w_0\}$ which corresponds to the set $\{z;\ 2^{-\alpha/\beta}<\operatorname{Re} z<2^{-1},\ \operatorname{Im} z=0\}$ of boundary points of D_2, the extremal length $\lambda(\Gamma)$ of the family $\Gamma=\{\gamma_r;\ 2^{-\alpha/\beta}<r<2^{-1}\}$ is infinite (cf. [1; p. 224]). We shall show $\lambda(\Gamma)<\infty$ to disprove that $F(x)$ is of class W.

Let I_n and I_n' be the intervals $2^{-(\alpha/\beta)^{n+1}}\leqq x\leqq 2^{-(\alpha/\beta)^n}$ and $2^{-(\alpha/\beta)^n\beta^{-1}}\leqq x\leqq 2^{-(\alpha/\beta)^{n-1}\beta^{-1}}$, respectively. Consider also the quadrilaterals $A_n=\{z;\ |z|\in I_n,\ -\pi<\arg z<0\}$ and $A_n'=\{z;\ |z|\in I_n',\ 0<\arg z<\pi\}$. Let $\rho^*|dw|$ be the auxiliary density on $|w|<1$ defined as follows: $\rho^*\equiv 0$ on C; in $(|w|<1)-C$, it is defined, in terms of $z\in D_1\cup D_2$, by

$$\rho^*(z)|dz|=\begin{cases}(\alpha/\beta)^{-n}|z|^{-1}|dz| & \text{for } z\in A_n\subset D_1\quad (n=0,1,2,\cdots),\\ (\alpha/\beta)^{-n}\beta^{-1}|z|^{-1}|dz| & \text{for } z\in A_n'\subset D_2\quad (n=0,1,2,\cdots),\\ 0 & \text{otherwise.}\end{cases}$$

On using the notations in [1; p. 220 ff], we have, with respect to any admissible density $\rho|dw|$,

$$L(\Gamma, \rho)^2 \leqq \left(\int_{\gamma_r} \rho|dw|\right)^2 \leqq \left(\int_{\gamma_r} \rho^*|dw|\right)\left(\int_{\gamma_r} (\rho^2/\rho^*)|dw|\right)$$
$$= \frac{\alpha(1+\beta)\pi}{\beta(\alpha-\beta)} \cdot \sum_{=0}^{\infty}\left\{\int_0^{\pi} \rho(r^{(\alpha/\beta)^n} e^{i\theta})^2 \left(\frac{\alpha}{\beta}\right)^n r^{2(\alpha/\beta)^n} d\theta\right.$$
$$\left. + \int_{-\pi}^{0} \rho(r^{(\alpha/\beta)^n\beta-1} e^{i\theta})^2 \left(\frac{\alpha}{\beta}\right)^n \frac{1}{\beta} r^{2(\alpha/\beta)^n\beta-1} d\theta\right\}.$$

Divide it by r and integrate with respect to r over $(2^{-\alpha/\beta}, 2^{-1})$:

$$L(\Gamma, \rho)^2 \frac{\alpha-\beta}{\beta} \log 2 \leqq \frac{\alpha(1+\beta)\pi}{\beta(\alpha-\beta)} \sum_{n=0}^{\infty} \iint_{A_n \cup A'_n} \rho^2 r\, dr d\theta \leqq \frac{\alpha(1+\beta)\pi}{\beta(\alpha-\beta)} \iint_{|w|<1} \rho^2 du dv,$$

therefore $\lambda(\Gamma)<\infty$. Consequently $F(x)$ is not of class $\boldsymbol{W}$.

4. LEMMA 1 (Blanc [5], Volkoviskii [14, 15]). *$F(x)$ defined on $a \leqq x \leqq b$ is of class $\boldsymbol{W}$ if and only if it is of class "quasi-$\boldsymbol{W}$", which means the validity of the requirements for $F(x)$ to be of class $\boldsymbol{W}$ (No. 3) with quasi-conformal Φ_1 and Φ_2.*[1)]

Proof. Let Φ_j $(j=1, 2)$ be quasi-conformal mappings of D_j onto D'_j $(j=1, 2)$, respectively, such that $\Phi_2^{-1} \circ \Phi_1 \equiv F$ on $[a, b]$. The quasi-conformal mapping Φ_j^{-1} satisfies the Beltrami equation

$$\frac{\partial \Phi_j^{-1}}{\partial \bar{w}} = \mu_j \frac{\partial \Phi_j^{-1}}{\partial w}$$

on D'_j, where $\mu_j(w)$ is a measurable function with $|\mu_j| \leqq k < 1$ on D'_j $(j=1, 2)$.[1)] Consider the $\mu(w)$ defined on $|w|<1$ by

$$\mu(w) = \begin{cases} \mu_1(w) & \text{for } w \in D'_1, \\ \mu_2(w) & \text{for } w \in D'_2, \\ 0 & \text{otherwise,} \end{cases}$$

and let $\omega = \Phi(w)$ be a quasi-conformal mapping of $|\omega|<1$ onto $|w|<1$ satisfying the Beltrami equation $\partial\Phi/\partial\bar{w} = \mu \cdot \partial\Phi/\partial w$, the existence of which is known.[1)] Then the curve $\Phi(C)$ and the conformal mappings $\Phi \circ \Phi_j$ of D_j into $|\omega|<1$ $(j=1, 2)$ guarantee that the $F(x)$ is of class $\boldsymbol{W}$.

LEMMA 2. *Let $F(x)$ be of class $\boldsymbol{W}$ on $a \leqq x \leqq b$.*

(a) *If $F(x)$ is of class $\boldsymbol{W}^*$, then the area of C is zero.*

(b) *Suppose there exists a set E which is in a domain Δ in the ζ-plane, is closed relative to Δ, and is the union of at most a countable number of sets with finite 1-dimensional outer measure. Suppose further there is a quasi-conformal mapping $w=\psi(\zeta)$ of Δ into $|w|<1$ such that $\psi(E) = C \cap (|w|$*

1) We mean by a quasi-conformal mapping the one in Pfluger-Ahlfors' sense. For its definition as well as its generalized derivatives and relation with Beltrami differential equations, the reader is referred to, e.g., Bers [2].

$<1)$. *Then* $F(x)$ *is of class* W^*.

Proof. (a) If C has positive area, we can construct a measurable $\mu(w)$ in $|w|<1$ such that $|\mu|\leqq k<1$, $\mu\equiv 0$ in $(|w|<1)-C$, and $\mu\neq 0$ on a set of positive area. The quasi-conformal mapping $\omega=\Psi(w)$ of $|w|<1$ onto $|\omega|<1$ satisfying $\partial\Psi/\partial\bar{w}=\mu\cdot\partial\Psi/\partial w$ is conformal in $(|w|<1)-C$ and is not conformal in $|w|<1$. The curve $\Psi(C)$ and the conformal mappings $\Psi\circ\Phi_j$ $(j=1, 2)$ determine another welding which violates the requirement for F to be of class W^*.

(b) Let Φ_1^* and Φ_2^* determine another welding. It is sufficient to show that the mapping

$$T(w)=\begin{cases}\Phi_1^*\circ\Phi_1^{-1} & \text{in } \bar{D}_1'\cap(|w|<1),\\ \Phi_2^*\circ\Phi_2^{-1} & \text{in } \bar{D}_2'\cap(|w|<1),\end{cases}$$

a homeomorphism of $|w|<1$ onto itself being conformal in $(|w|<1)-C$, is conformal in $|w|<1$.

The composite mapping $T\circ\psi$ is homeomorphic on Δ and quasi-conformal in $\Delta-E$. By a result of Strebel [13; p. 906] and Mori [7; p. 66], $T\circ\psi$ is quasi-conformal in Δ, therefore, so is T on $|w|<1$. Since a quasi-conformal mapping is measurable (see Bers [2; p. 18]), the area of C vanishes. We conclude that T is conformal in $|w|<1$ since a quasi-conformal mapping which is conformal almost everywhere is conformal.[2]

REMARK. As is seen from the proof immediately, the assumption for C in (b) came from a condition for "quasi-conformal removability of topological mappings" due to Strebel [13] and Mori [7]. Amelioration of the latter would narrow the gap between (a) and (b).

The reasoning in the proof of (a) leads us to the following:

EXAMPLE 2. *There is a function* $F(x)$ *on* $-1\leqq x\leqq 1$ *which is of class* W *but is not of class* W^*.

Proof. Consider a simple arc C in $|w|<1$ except for both endpoints on $|w|=1$ which has positive area[3] and divides $|w|<1$ into simply-connected domains D_1' and D_2'. Let Φ_1 (Φ_2) map D_1' (D_2') conformally onto the upper- (lower-) unit disc in such a way that the diameter corresponds to C. The desired is $F(x)=\Phi_2^{-1}\circ\Phi_1(x)$ $(-1\leqq x\leqq 1)$.

We have not succeeded in constructing an explicit example of $F(x)$ with this property.

2) It is an immediate consequence of Weyl's lemma when we start from the analytic definition of quasi-conformality (see, Bers [2]). Strebel [13; p. 909] proved it directly from the geometric definition.

3) An explicit example of a Jordan curve with positive area is found in Osgood [10].

5. A sufficient condition for $F(x)$ to be of class W has been obtained by Blanc [5] and Volkoviskii [14, 15]. Our Lemma 2 shows that their condition even implies that $F(x)$ is of class W^*.

LEMMA 3. *A function $F(x)$ on $a \leqq x \leqq b$ with the following property is of class W^*: There exists a quasi-conformal mapping $\zeta = \Psi(z)$ which maps $D_1 = \{z;\ |z-(a+b)/2| < (b-a)/2,\ \operatorname{Im} z > 0\}$ onto a simply-connected domain Δ in the upper-half ζ-plane with $[F(a), F(b)] \subset \partial\Delta$ and satisfies $\Psi(x) \equiv F(x)$ for $a \leqq x \leqq b$.*

Proof. Map the simply-connected domain $\Delta \cup (F(a), F(b)) \cup \{\zeta;\ |\zeta - (F(a) + F(b))/2| < (F(b)-F(a))/2,\ \operatorname{Im}\zeta < 0\}$ onto $|w|<1$ conformally by $w = \Phi(\zeta)$. Then the curve $\Phi([F(a), F(b)])$ and quasi-conformal mappings $\Phi_1^*(z) \equiv \Phi\circ\Psi(z)$, $\Phi_2^*(z) \equiv \Phi(z)$ determine a "quasi-conformalw elding". By Lemma 1 we find C, $\Phi_j(z)$ $(j=1, 2)$ showing that the $F(x)$ is of class W. Up to here we merely repeated the reasoning of Blanc [5] and Volkoviskii [14, 15]. Here the C obtained by the method in the proof of Lemma 1 is a quasi-conformal image of the line segment $[F(a), F(b)]$, therefore, by Lemma 2, (b), $F(x)$ is of class W^*.

This lemma reminds us of the "ρ-condition" due to Beurling and Ahlfors [3]. In fact, on localizing it, we have

THEOREM 1. *Let $f(x)$ be a function on $0 \leqq x < \infty$ introduced in No. 1. If it satisfies the following "local ρ-condition", a Riemann surface $\mathfrak{R}$ is obtained uniquely from S by welding l_1 with l_2 by means of f: At every ξ $(0<\xi<\infty)$ there exist ε_ξ $(0<\varepsilon_\xi<\xi)$ and ρ_ξ $(0<\rho_\xi<\infty)$ such that*

$$\frac{1}{\rho_\xi} \leqq \frac{f(x+t)-f(x)}{f(x)-f(x-t)} \leqq \rho_\xi$$

holds for all x and t with $\xi - \varepsilon_\xi < x-t < x+t < \xi + \varepsilon_\xi$.

The equivalence of Theorem 1 with Lemma 3 is apparent from the following Lemmas:

LEMMA 4. *Let $h(x)$ be a continuous strictly monotone increasing function on $-1 \leqq x \leqq 1$ such that $h(-1) = -1$ and $h(1) = 1$. Suppose there exists a quasi-conformal mapping $w = T(z)$ which maps the "unit triangle" $\Delta = \{z;\ \operatorname{Im} z > 0,\ |\operatorname{Re} z| + \operatorname{Im} z < 1\}$ onto a simply-connected domain Δ' in the upper-half w-plane with $[-1, 1] \subset \partial\Delta'$ and satisfies $T(x) = h(x)$ for $-1 \leqq x \leqq 1$. Then there exists a positive constant ρ such that*

$$\frac{1}{\rho} \leqq \frac{h(x+t)-h(x)}{h(x)-h(x-t)} \leqq \rho \tag{1}$$

for all x and t with $-1 < x-2t < x+2t < 1$.

LEMMA 5. *Let $h(x)$ be as above on $-1 \leqq x \leqq 1$. If there exists a positive constant ρ such that* (1) *holds for all x and t with $-1 < x-t < x+t < 1$, then*

$$(2) \qquad T(z)=\int_0^1\left\{\frac{1+i}{2}h(x+ty)+\frac{1-i}{2}h(x-ty)\right\}dt \qquad (z=x+iy)$$

maps quasi-conformally the "unit triangle" $\varDelta$ *onto a* $\varDelta'$ *described above in such a way that* $T(x)=h(x)$ *holds for* $-1\leqq x\leqq 1$.

These lemmas will be proved by repeating the qualitative part of the proof in Beurling and Ahlfors [3] with minor technical modifications, which for the sake of completeness we shall indicate in the following lines.

Proof of Lemma 4. For x and t with $-1<x-2t<x+2t<1$, let λ be the extremal length of the family consisting of all the curves in $\varDelta$ connecting the interval $[x-t,\ x]$ with $[x+t,\ 1]$. Let λ' be that of the family consisting of all the curves in $\varDelta'$ connecting $[h(x-t),\ h(x)]$ with $[h(x+t),\ 1]$. We have $\lambda'\leqq K\lambda$, where K is the maximal dilatation of T. Let λ_0 be the extremal length of the family consisting of all the curves in $\{\zeta;\ \operatorname{Im}\zeta>0,\ |\operatorname{Re}\zeta-x|+\operatorname{Im}\zeta<2t\}$ connecting $[x-t,\ x]$ with $[x+t,\ x+2t]$. Let λ'' be that of the family consisting of all the curves in the upper-half plane connecting $[h(x-t),\ h(x)]$ with $[h(x+t),\ \infty)$. Evidently $\lambda''\leqq\lambda'$ and $\lambda\leqq\lambda_0$, hence $\lambda''\leqq K\lambda_0$. Here $K\lambda_0$ is independent of x and t while λ'' is given by cross-ratio as in [3; p. 130]. We thus get the right inequality in (1). The left one is proved analogously on considering the intervals $[-1,\ x-t]$ and $[x,\ x+t]$.

Proof of Lemma 5. Extend the given $h(x)$ by

$$h^*(x)=\begin{cases}h(x) & \text{for } |x|\leqq 1,\\ x & \text{for } |x|>1\end{cases}$$

on $-\infty<x<\infty$ and define $T^*(z)$ by the integral (2) with respect to h^*. $T(z)=T^*(z)$ holds on $\varDelta$. Even though h^* does not necessarily satisfy the ρ-condition throughout $-\infty<x<\infty$, the reasoning in [3; pp. 135–136] is applicable to show that T^* maps topologically the upper-half plane onto itself with the boundary correspondence h^*. The estimation of the dilatation of T in $\varDelta$ is performed in completely the same way as in [3; pp. 136–138].

A continuously differentiable function $F(x)$ on $-1\leqq x\leqq 1$ with positive $F'(x)$ is of class $\boldsymbol{W}^*$ by Theorem 1. Volkoviskii [16; p. 42] showed that, if $F(x)$ further satisfies

$$\int_{-1}^1\left|\frac{F'(x)-F'(t)}{x-t}\right|dx\leqq K$$

for every $t\in[-1,\ 1]$ with K independent of t, the $\varPhi_j'(x)$ $(j=1,\ 2)$ exist and are uniformly bounded and bounded away from zero on each closed interval in $(-1,\ 1)$.

6. THEOREM 1′. *Let* $f(x)$ *be a function on* $0\leqq x<\infty$ *introduced in No.* 1. *Suppose there exists an at most countable set* $E\subset[0,\ \infty]$ *which has no finite positive accumulation point and satisfies the following conditions*:

(i) *At every* $\xi \notin E$, $\xi > 0$, $f(x)$ *satisfies the "local* ρ*-condition" in Theorem* 1;

(ii) *For every* $\xi \in E$, *there exists a* $\delta > 0$ *such that*

$$\int_0^\delta \frac{\min\left(\frac{1}{t}, \frac{f'(\xi+t)}{f(\xi+t)-f(\xi)}, \frac{f'(\xi-t)}{f(\xi)-f(\xi-t)}\right)}{\pi^2 + \left(\log \frac{f(\xi+t)-f(\xi)}{f(\xi)-f(\xi-t)}\right)^2} dt = \infty$$

(note that f' exists almost everywhere). *Then the Riemann surface* $\mathfrak{R}$ *is obtained uniquely from* S *by welding* l_1 *with* l_2 *by means of* f.

Theorem 1′ is not necessarily a generalization of Theorem 1. However, the function

$$f(x) = \begin{cases} 1 + \dfrac{x-1}{\exp\sqrt{-\log(1-x)}} & \text{for } 0 \leqq x < 1, \\ x & \text{for } 1 \leqq x < \infty \end{cases}$$

satisfies the assumption of Theorem 1′ for $E = \{1\}$, while does not that of Theorem 1 at $x = 1$.

Proof. By Theorem 1 we obtain the Riemann surface $\mathfrak{R}^*$ determined uniquely from S by welding $l_1 - E$ with $l_2 - f(E)$ by means of f. To each $\xi \in E$ the boundary component B_ξ of $\mathfrak{R}^*$ corresponds. The image Λ^* of $l_1 - E$ consists of at most a countable number of simple curves connecting different ideal boundary components of $\mathfrak{R}^*$. The "union" of Λ^* and all the B_ξ ($\xi \in E$) "connects" B with B' of $\mathfrak{R}^*$. It is not difficult to see that the proof is complete if we show that each B_ξ is parabolic.

We map conformally the upper-half disc with diameter $(\xi - \delta, \xi + \delta)$ by $w = -\log(z - \xi)$ into the strip S_1: $0 < \operatorname{Im} w < \pi$. Similarly we map the lower-half disc with diameter $(f(\xi - \delta), f(\xi + \delta))$ by $w = -\log(z - f(\xi))$ into S_2: $\pi < \operatorname{Im} w < 2\pi$. Then the decision of the type of B_ξ is reduced to the type problem concerning the welding of two strips, which has also been discussed by Volkoviskii [14, 15]. In our case we have two functions $g(u)$ and $h(u)$ transformed from $f(x)$, by means of which we weld S_1 and S_2 as follows: The point u on the lower edge of S_1 is identified with $g(u) + 2\pi i$ on the upper edge of S_2; the point $u + \pi i$ on the upper edge of S_1 is identified with $h(u) + \pi i$ on the lower edge of S_2. Evidently B_ξ is parabolic if and only if the boundary component corresponding to $w = +\infty$ of thus obtained doubly-connected Riemann surface is parabolic.

With respect to this welding, Volkoviskii [14; p. 193] gave the following criterion: The relevant boundary component is of parabolic type if

$$\int^\infty \frac{du}{\{\pi^2 + (u-s)^2\}\left\{\max\left(1, \frac{du}{ds}\right) + \max\left(\frac{du}{ds^*}, \frac{du}{du^*}\right)\right\}} = \infty,$$

where s is the function of u defined by $s - u = g(u) - h(s)$, $s^* = h(s)$, and

$u^* = g(u)$. Since the implicit relation involved is unpleasant, we shall avoid to use it.

To get another criterion for parabolicity, we estimate the extremal length of the family $\{\gamma_u;\ u_0 < u < \infty\}$ (u_0: sufficiently large), where γ_u is defined as the union of line segments connecting u with $u+\pi i$ in S_1 and $h(u)+\pi i$ with $g(u)+2\pi i$ in S_2, respectively. On performing completely the same process as in Theorem 2 in the next section, we have that

$$\int^{\infty} \frac{\min(1, g', h')}{\pi^2+(g-h)^2} du = \infty$$

implies the parabolicity. Consequently, on expressing g and h in terms of f given, we get the condition (ii) of our theorem.

§ 2. Types of surfaces.

7. For completeness we begin by proving the following theorem due to Volkoviskii [14, 15] and Nevanlinna [8, 9]; the proof is essentially a mere repetition of the former's:

THEOREM 2. *Suppose that $f(x)$ on $0 \leqq x < \infty$ introduced in No. 1 determines an $\mathfrak{R}$ by welding.* (Note that $f'(x)$ exists almost everywhere.) *If it satisfies*

$$\int^{\infty} \frac{\min(1, f'(x))}{1+(f(x)-x)^2} dx = \infty,$$

then the boundary component B of every $\mathfrak{R}$ is parabolic.

Proof. For every x $(0 < x < \infty)$, let γ_x be the closed curve in $\mathfrak{R}$ which corresponds in $\bar{S}$ to the line segment connecting x with $f(x)+i$. γ_x separates boundary components of $\mathfrak{R}$. Every subarc of γ_x which does not contain the point corresponding to x is rectifiable. As is well known (see, e.g., [1; pp. 224–227]) B is of parabolic type if the extremal length $\lambda(\Gamma)$ of $\Gamma = \{\gamma_x;\ 0 < x < \infty\}$ vanishes. Using the notations in [1; p. 220 ff] we have, for every admissible ρ,

$$\begin{aligned} L(\Gamma, \rho)^2 &\leqq \left(\int_{\gamma_x} \rho |dw|\right)^2 \leqq \int_{\gamma_x} |dw| \int_{\gamma_x} \rho^2 |dw| \\ &= (1+(f(x)-x)^2) \int_0^1 \rho(x+y(f(x)-x)+iy)^2 dy \\ &\leqq \frac{1+(f(x)-x)^2}{\min(1, f'(x))} \int_0^1 \rho(x+y(f(x)-x)+iy)^2 (1+y(f'(x)-1))\, dy, \end{aligned}$$

so that

$$L(\Gamma, \rho)^2 \int^{\infty} \frac{\min(1, f'(x))\, dx}{1+(f(x)-x)^2} \leqq \int_0^1 \int^{\infty} \rho(x+y(f(x)-x)+iy)^2 (1+y(f'(x)-1)) dx\, dy.$$

The topological mapping $u+iv = x+y(f(x)-x)+iy$ of S onto itself is partially differentiable almost everywhere with the Jacobian $1+y(f'(x)-1)$. On changing the variables (x, y) into (u, v) and omitting the singular part (see, e.g., [11], p. 199; cf. also pp. 271, 413), we have

$$L(\Gamma, \rho)^2\int^{\infty}\frac{\min(1, f'(x))\,dx}{1+(f(x)-x)^2} \leqq \iint_{\mathfrak{R}}\rho^2 du dv.$$

The left integral is infinite by assumption, so that $\lambda(\Gamma)=0$.

A sufficient condition for hyperbolicity has been given by Volkoviskii [14, 15], which was recently improved by Jenkins [6] by using a different method. Making use of the former's method has two advantages that we can avoid a rather complicated topological consideration required in the proof of the latter and that we need not assume the absolute continuity of f. In fact, we can state Volkoviskii-Jenkins' theorem in the following way:

THEOREM 3. *Suppose that $f(x)$ on $0 \leqq x < \infty$ introduced in No. 1 determines an $\mathfrak{R}$ by welding. Let $f_0(x) = x$ and $f_n(x) = f \circ f_{n-1}(x)$ $(n = 1, 2, \cdots)$. If there exist a number $a > 0$ with $f(a) > a$, $\lim_{n\to\infty} f_n(a) = \infty$ and a measurable set $E \subset [a, f(a))$ with positive measure on which*

$$\sum_{n=0}^{\infty}\frac{1}{f_n'(x)} < \infty$$

holds, then the boundary component B of every $\mathfrak{R}$ is hyperbolic.

Proof. Let E^* be the set in $\mathfrak{R}$ corresponding to $\{z; \operatorname{Re} z \in E, \operatorname{Im} z = 1\} \subset \bar{S}$. If B is parabolic then $\lambda(\Gamma) = \infty$ holds for any family Γ of curves in $\mathfrak{R}$ connecting E^* with B (cf. [1; p. 224]). This proposition is valid under the assumption that each member of Γ satisfies the following condition: In its parameter representation $p = p(t)$ $(0 < t < 1)$, there are t_n $(n = 0, \pm 1, \pm 2, \cdots)$ such that $0 < t_{n-1} < t_n < 1$, $\lim_{n\to\infty} t_n = 1$, $\lim_{n\to\infty} t_{-n} = 0$, and that each subarc of the curve corresponding to $t_{n-1} < t < t_n$ is rectifiable.

It is possible to find a measurable $E_0 \subset E$ with positive measure on which $\sum_{n=0}^{\infty} 1/f_n'(x) \leqq M < \infty$. We note that, for each $x \in E_0$, $f_n(x) < f_{n+1}(x)$ and $\lim_{n\to\infty} f_n(x) = \infty$ hold. Let γ_x be the curve in $\mathfrak{R}$ which is represented in $\bar{S}$ by

$$\gamma_x = \bigcup_{n=0}^{\infty}\{z; \operatorname{Re} z = f_n(x), 0 \leqq \operatorname{Im} z \leqq 1\}$$

with the downward direction. Each γ_x $(x \in E_0)$ connects E^* with B in $\mathfrak{R}$ and satisfies the above requirement for piecewise rectifiability. Therefore, to prove the hyperbolicity of B, it suffices to show that the extremal length of $\Gamma = \{\gamma_x; x \in E_0\}$ is finite.

We remark that $E_n = f_n(E_0)$ $(n = 1, 2, \cdots)$ are mutually disjoint. Define a density $\rho^*|dw|$ on $\mathfrak{R}$ as follows: $\rho^* \equiv 0$ on Λ; in $\mathfrak{R} - \Lambda$ it is defined, in terms of $z \in S$, as follows:

$$\rho^*|dz| = \begin{cases} \dfrac{|dz|}{f_n'(\xi)} & \text{if } \operatorname{Re} z \in E_n, \\ 0 & \text{otherwise,} \end{cases}$$

where $\xi \in E_0$ is such that $f_n(\xi) = \mathrm{Re}\, z$. For any admissible ρ on $\mathfrak{R}$, we have

$$L(\Gamma, \rho)^2 \leqq \left(\int_{\gamma_x} \rho |dw|\right)^2 \leqq \left(\int_{\gamma_x} \rho^* |dw|\right)\left(\int_{\gamma_x} (\rho^2/\rho^*) |dw|\right)$$
$$= \left(\sum_{n=0}^{\infty} \frac{1}{f_n'(x)}\right)\left(\sum_{n=0}^{\infty} \int_0^1 \rho(f_n(x) + iy)^2 f_n'(x)\, dy\right)$$
$$\leqq M \sum_{n=0}^{\infty} \int_0^1 \rho(f_n(x) + iy)^2 f_n'(x)\, dy.$$

We integrate it with respect to x over E_0. Concerning the change of variables we know

$$\int \rho^2 f_n' dx \leqq \int \rho^2 du \quad (u = f_n),$$

so that

$$L(\Gamma, \rho)^2 mE_0 \leqq M \sum_{n=0}^{\infty} \int_0^1 \int_{E_0} \rho(f_n(x) + iy)^2 f_n'(x)\, dx dy$$
$$\leqq M \sum_{n=0}^{\infty} \int_0^1 \int_{E_n} \rho(x + iy)^2 dx dy \leqq M \iint_{\mathfrak{R}} \rho^2 du dv,$$

consequently $\lambda(\Gamma) \leqq M/mE_0 < \infty$.

REMARK. The following comparison theorem due to Blanc [5; p. 142] would enlarge the range of applicability of above criteria:

Let $f_1(x)$ and $f_2(x)$ on $0 \leqq x < \infty$ be those introduced in No. 1 *and absolutely continuous on every finite closed interval in $0 < x < \infty$. Suppose that $f_1(x)$ satisfies the assumption of Theorem* 1 *or* 1′ *and that $f_2(x)$ determines an $\mathfrak{R}_2$ by welding. If*

$$\left|\frac{f_1'(x)}{f_2'(x)} - 1\right| \leqq \frac{M}{x + f_1(x)} \quad a.\ e.,$$
$$|f_1(x) - f_2(x)| \leqq N \quad a.\ e.,$$

then the type of B_1 of the $\mathfrak{R}_1$ determined uniquely by f_1 and that of B_2 of every $\mathfrak{R}_2$ determined by f_2 are the same.

8. Our plan of reclaiming the space beyond the range of Theorems 2 and 3 begins by looking over the first requirement

$$(3) \qquad f(a) > a, \qquad \lim_{n\to\infty} f_n(a) = \infty$$

in Theorem 3. This condition for the inverse f^{-1} is expressed as

$$(4) \qquad f(a) < a, \qquad \lim_{n\to\infty} f_{-n}(a) = \infty,$$

where $f_{-n}=(f^{-1})_n$. If $f(a)>a$ and $\lim f_n(a)=b<\infty$, then $a<b$ and $f(b)=b$; conversely if $f(a)>a$ and $f(b)=b$ for some $b>a$, then $\lim f_n(a)<\infty$. Concerning (4) we see the similar. Therefore, *an* $f(x)$ *introduced in No.* 1 *satisfies neither* (3) *nor* (4) *for any* a $(0<a<\infty)$ *if and only if there exists a sequence* $0<b_1<b_2<\cdots$ *such that* $\lim_{n\to\infty}b_n=\infty$ *and* $f(b_n)=b_n$ $(n=1,2,\cdots)$; Theorem 3 is applicable neither to f nor f^{-1} of this property.

We shall give a sufficient condition for hyperbolicity which is essentially to be applied to such functions. Let $f(x)$ be what is introduced in No. 1 and such that $f(b_n)=b_n$ for a sequence $0=b_0<b_1<b_2<\cdots\to\infty$. On introducing auxiliarly transformations

$$\sigma_n(x)=b_n+b_{n+1}-x \qquad \text{on } b_n\leqq x\leqq b_{n+1}$$

and

$$\tau_n(x)=2b_{n+1}-x \qquad \text{on } b_n\leqq x\leqq b_{n+1}$$

for $n=0,1,2,\cdots$, we consider the sequences $\{g_n(x)\}_{n=0}^{\infty}$ and $\{h_n(x)\}_{n=0}^{\infty}$ defined on $0\leqq x\leqq b_1$ as follows:

$$g_0(x)=x,$$

$$g_n(x)=\tau_{n-1}\circ f^{-1}\circ\sigma_{n-1}\circ f\circ g_{n-1}(x),$$

$$h_n(x)=f\circ g_n(x).$$

THEOREM 4. *Suppose that* $f(x)$ *on* $0\leqq x<\infty$ *introduced in No.* 1 *determines an* $\mathfrak{R}$ *by welding and that*

$$f(b_n)=b_n \qquad (n=1,2,\cdots)$$

for a sequence

$$0=b_0<b_1<b_2<\cdots\to\infty.$$

Let $g_n(x)$ *and* $h_n(x)$ $(0\leqq x\leqq b_1;\ n=0,1,2,\cdots)$ *be as above. If there exists a measurable set* $E\subset(0,b_1/4)$ *with positive measure such that*

$$g_n(E)\subset\left(b_n,\frac{3b_n+b_{n+1}}{4}\right),\qquad h_n(E)\subset\left(\frac{3b_n+b_{n+1}}{4},\frac{b_n+b_{n+1}}{2}\right)$$
$$(n=0,1,2,\cdots)$$

and

$$\sum_{n=0}^{\infty}\left\{\frac{g_n(x)-b_n}{xg_n'(x)}+\frac{\dfrac{b_n+b_{n+1}}{2}-h_n(x)}{xh_n'(x)}\right\}<\infty \tag{5}$$

on E, *then the boundary component* B *of every* $\mathfrak{R}$ *is hyperbolic.*

Proof. It is possible to find a measurable $E_0\subset E$ with positive measure on which the left-hand side of (5) is bounded by $M<\infty$. For every $x\in E_0$, let γ_x be the curve in $\mathfrak{R}$ which is realized in $\bar{S}$ as

$$\gamma_x=(\bigcup_{n=1}^{\infty}\gamma_x^{(n)})\cup(\bigcup_{n=0}^{\infty}\gamma_x'^{(n)}),$$

where

$$\gamma_x^{(n)} = \{z;\ |z - b_n| = g_n(x) - b_n,\ 0 \leqq \arg(z - b_n) \leqq \pi\}$$

with clockwise direction and

$$\gamma_x'^{(n)} = \left\{z;\ \left|z - \frac{b_n + b_{n+1}}{2} - i\right| = \frac{b_n + b_{n+1}}{2} - h_n(x),\right.$$
$$\left. -\pi \leqq \arg\left(z - \frac{b_n + b_{n+1}}{2} - i\right) \leqq 0\right\}$$

with counter-clockwise direction. As in the proof of Theorem 3, the hyperbolicity of B is seen from the finiteness of the extremal length of the family $\Gamma = \{\gamma_x;\ x \in E_0\}$.

To estimate $\lambda(\Gamma)$, consider

$$A_n = \left\{z;\ |z - b_n| < \frac{1}{2}\min(1, b_{n+1} - b_n, b_n - b_{n-1}),\ \mathrm{Im}\, z > 0\right\} \qquad (n = 1, 2, \cdots)$$

and

$$B_n = \left\{z;\ \left|z - \frac{b_n + b_{n+1}}{2} - i\right| < \frac{1}{2}\min(1, b_{n+1} - b_n),\ \mathrm{Im}\, z < 0\right\} \qquad (n = 0, 1, 2, \cdots)$$

which are in S and mutually disjoint. Let $\rho^*|dw|$ be the density on $\mathfrak{R}$ defined as follows: $\rho^* \equiv 0$ on Λ; in $\mathfrak{R} - \Lambda$, in terms of $z \in S$,

$$\rho^*(z) = \begin{cases} \dfrac{1}{\xi g_n'(\xi)} & \text{if } z \in A_n\ (n = 1, 2, \cdots), \\ \dfrac{1}{\tilde{\xi} h_n'(\tilde{\xi})} & \text{if } z \in B_n\ (n = 1, 2, \cdots), \\ 0 & \text{otherwise,} \end{cases}$$

where $\xi \in E_0$ is such that $g_n(\xi) = b_n + |z - b_n|$ and $\tilde{\xi} \in E_0$ is such that $h_n(\tilde{\xi}) = ((b_n + b_{n+1})/2) - |z - ((b_n + b_{n+1})/2) - i|$. Then, for any admissible ρ on $\mathfrak{R}$,

$$L(\Gamma, \rho)^2 \leqq \left(\int_{\gamma_x} \rho^*|dw|\right)\left(\int_{\gamma_x} (\rho^2/\rho^*)|dw|\right)$$
$$\leqq \pi M \sum_{n=1}^{\infty} \int_0^{\pi} \rho((g_n(x) - b_n)e^{i\theta})^2 (g_n(x) - b_n) x g_n'(x)\, d\theta$$
$$+ \pi M \sum_{n=0}^{\infty} \int_{-\pi}^{0} \rho\left(\left(\frac{b_n + b_{n+1}}{2} - h_n(x)\right)e^{i\theta}\right)^2 \left(\frac{b_n + b_{n+1}}{2} - h_n(x)\right) x h_n'(x)\, d\theta.$$

On dividing it by x and integrating over E_0, we have

$$L(\Gamma, \rho)^2 \int_{E_0} \frac{dx}{x} \leqq \pi M \left(\sum_{n=1}^{\infty} \iint_{A_n} \rho^2 dxdy + \sum_{n=0}^{\infty} \iint_{B_n} \rho^2 dxdy\right) \leqq \pi M \iint_{\mathfrak{R}} \rho^2 dudv,$$

and, therefore, $\lambda(\Gamma) \leqq \pi M / 4mE_0 < \infty$.

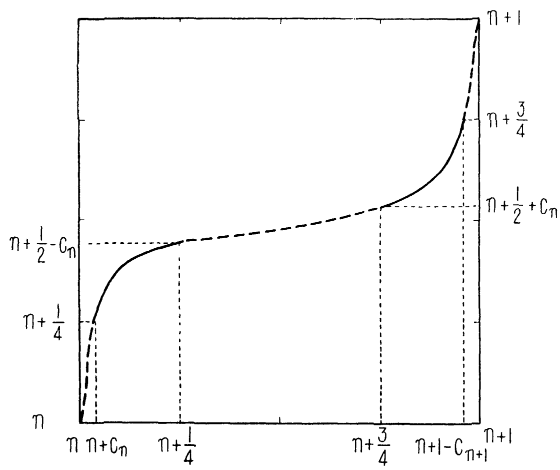
Since the theorem is rather complicated, we give here an illustrative example. Consider a sequence $\{c_n\}_{n=0}^{\infty}$ of real numbers such that $0 < c_n < 1/4$. Define $f(x)$ as follows:

$$f(x) = n + \frac{1}{2} - \frac{c_n}{4(x-n)} \qquad \text{for } n + c_n \leqq x \leqq n + \frac{1}{4},$$

$$f(x) = n + \frac{1}{2} + c_n \left(\frac{1}{4(n+1-x)} \right)^{\log 4c_n / \log 4c_{n+1}}$$

$$\text{for } n + \frac{3}{4} \leqq x \leqq n + 1 - c_{n+1},$$

$n = 0, 1, 2, \cdots$, and for remaining x define suitably so that the resulting $f(x)$ on $0 \leqq x < \infty$ is strictly monotone increasing, continuously differentiable with non-vanishing $f'(x)$, and is such that $f(n) = n$ $(n=0, 1, 2, \cdots)$. By Theorem 1, f determines an $\mathfrak{R}$ uniquely. We infer the following:



EXAMPLE 3. *With respect to the $f(x)$ defined above, the B of the $\mathfrak{R}$ is parabolic if*

$$\sum_{n=1}^{\infty} \sqrt{c_n} = \infty,$$

and is hyperbolic if

$$\sum_{n=1}^{\infty} \frac{1}{\log(1/c_n)} < \infty.$$

Proof. $f'(x) \geqq 1$ on $I_n = [n + c_n, n + (\sqrt{c_n}/2)]$ and $|f(x) - x| \leqq 1$ on $0 \leqq x < \infty$. We thus have

$$\int^{\infty} \frac{\min(1, f'(x))}{1 + (f(x) - x)^2} dx \geqq \frac{1}{2} \sum^{\infty} \int_{I_n} dx \geqq \frac{1}{8} \sum^{\infty} \sqrt{c_n}$$

and see, by Theorem 2, that the divergence of the last term implies the parabolicity.

To apply Theorem 4 for hyperbolicity, we take $E = (1/8, 1/4)$. Clearly $g_n(E) \subset (n, n + (1/4))$ and $h_n(E) \subset (n + (1/4), n + (1/2))$ $(n = 0, 1, 2, \cdots)$. Since

$$\tau_n \circ f^{-1} \circ \sigma_n \circ f(x) = n + 1 + \frac{1}{4}(4x - 4n)^{\log 4c_{n+1}/\log 4c_n} \qquad \left(n < x < n + \frac{1}{4}\right)$$

for $x \in E$, we have successively,

$$\begin{cases} g_n(x) - n = \dfrac{1}{4}(4(g_{n-1}(x) - n + 1))^{\log 4c_n/\log 4c_{n-1}}, \\ n + \dfrac{1}{2} - h_n(x) = \dfrac{c_n}{4(g_n(x) - n)}, \end{cases}$$

$$\begin{cases} \dfrac{g_n'(x)}{g_n(x) - n} = \dfrac{\log 4c_n}{\log 4c_{n-1}} \dfrac{g_{n-1}'(x)}{g_{n-1}(x) - (n-1)} = \cdots = \dfrac{\log 4c_n}{\log 4c_0} \dfrac{1}{1+x}, \\ \dfrac{h_n'(x)}{n + \dfrac{1}{2} - h_n(x)} = \dfrac{g_n'(x)}{g_n(x) - n}, \end{cases}$$

and

$$\sum_{n=1}^{\infty} \frac{g_n(x) - n}{x g_n'(x)} = \sum_{n=1}^{\infty} \frac{n + 1/2 - h_n(x)}{x h_n'(x)} \leqq 9 \log \frac{1}{4c_0} \sum_{n=1}^{\infty} \frac{1}{\log(1/c_n)}$$

for $x \in E$. Consequently, by Theorem 4, the convergence of the last term implies the hyperbolicity.

REFERENCES

[1] AHLFORS, L.V., AND L. SARIO, Riemann surfaces. Princeton Univ. Press, 1960.

[2] BERS, L., On a theorem of Mori and on the definition of quasiconformality. Trans. Amer. Math. Soc. 84 (1957), 78–84.

[3] BEURLING, A., AND L.V. AHLFORS, The boundary correspondence under quasiconformal mappings. Acta Math. **96** (1956), 125–141.

[4] BLANC, C., Les surfaces de Riemann des fonctions méromorphes. Comment. Math. Helv. **9** (1937), 193–216, 335–368.

[5] ——, Les demi-surfaces de Riemann. Ibid. **11** (1939), 130–150.

[6] JENKINS, J. A., On a type problem. Canad. J. Math. **11** (1959), 427–431.

[7] MORI, A., On quasi-conformality and pseudo-analyticity. Trans. Amer. Math. Soc. 84 (1957), 56–77.

[8] NEVANLINNA, R., Über die Polygonaldarstellung einer Riemannschen Fläche. Ann. Acad. Sci. Fenn. Ser. I-A, 122 (1952), 9 pp.

[9] ——, Polygonal representations of Riemann surfaces. Lectures on functions of a complex variable, Univ. Michigan Press, 1955, pp. 65–70.

[10] OSGOOD, W. F., A Jordan curve of positive area. Trans. Amer. Math. Soc. 4 (1903), 107–112.

[11] RADÓ, T., AND P. V. REICHELDERFER, Continuous transformations in analysis, with an introduction to algebraic topology. Springer, 1955.

[12] SAINOUCHI, Y., A remark on the polygonal representation of Riemann surfaces. Bull. Univ. Osaka Prefect. Ser A, **6** (1958), 1–9.

[13] STREBEL, K., On the maximal dilation of quasiconformal mappings. Proc. Amer. Math. Soc. **6** (1955), 903–909.

[14] VOLKOVISKIĬ, L. I., On the problem of type of simply-connected Riemann surfaces. Mat. Sbor. 18 (60) (1946), 185–211. (Russian)

[15] ——, Studies in the problem of type of simply-connected Riemann surfaces. Trudi Mat. Inst. B. A. Steklova, 34 (1950). (Russian)

[16] ——, Quasiconformal mapping and the problem of conformal pasting. Ukrain. Mat. Z. 3 (1951), 39–51. (Russian)

[17] WIRTH, E. M., Über die Bestimmung des Typus einer Riemannschen Fläche. Comment. Math. Helv. 31 (1956), 90–107.

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