# AFFINE CONNEXIONS IN AN ALMOST PRODUCT SPACE

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Let us consider an almost complex space $M$ of class $C^\infty$ and denote by $F_\lambda{}^\kappa$ its structure tensor satisfying $F_\mu{}^\lambda F_\lambda{}^\kappa = -A_\mu^\kappa$. As is well known now, in order that the tensor $F_\lambda{}^\kappa$ define a complex structure, it is necessary and sufficient that the Nijenhuis tensor

$$N_{\mu\lambda}{}^\kappa = F_\mu{}^\rho(\partial_\rho F_\lambda{}^\kappa - \partial_\lambda F_\rho{}^\kappa) - F_\lambda{}^\rho(\partial_\rho F_\mu{}^\kappa - \partial_\mu F_\rho{}^\kappa)$$

constructed from $F_\lambda{}^\kappa$ vanish. (Eckmann [1], Eckmann and Frölicher [1], Ehresmann [1], Frölicher [1], Libermann [1], [2], [3], Newlander and Nirenberg [1], Nijenhuis [1], de Rham (unpublished), Yano [2], [3].)

If there exists a symmetric affine connexion $\Gamma_{\mu\lambda}^\kappa$ in $M$, then denoting by $\nabla_\mu$ the covariant derivative with respect to this affine connexion, we have

$$N_{\mu\lambda}{}^\kappa = F_\mu{}^\rho(\nabla_\rho F_\lambda{}^\kappa - \nabla_\lambda F_\rho{}^\kappa) - F_\lambda{}^\rho(\nabla_\rho F_\mu{}^\kappa - \nabla_\mu F_\rho{}^\kappa)$$

and consequently we can see that if there exists a symmetric affine connexion such that $\nabla_\mu F_\lambda{}^\kappa = 0$, then $N_{\mu\lambda}{}^\kappa = 0$ and consequently the almost complex structure is a complex structure.

Now the eigenvalues of the matrix $(F_\lambda{}^\kappa)$ are $+i$ and $-i$ and the eigenvectors corresponding to the eigenvalue $+i$ span a distribution $B$ of complex dimension $n$ and those corresponding to the eigenvalue $-i$ span a distribution $\bar{B}$ which is complex conjugate to $B$. The condition $\nabla_\mu F_\lambda{}^\kappa = 0$ means then that these two complex conjugate distributions are parallel with respect to the symmetric affine connexion. (Yano [2].)

Now the following converse problem arises. We assume that $N_{\mu\lambda}{}^\kappa = 0$. Then does there exist a symmetric affine connexion $\Gamma_{\mu\lambda}^\kappa$ such that the covariant derivative $\nabla_\mu F_\lambda{}^\kappa$ of the structure tensor $F_\lambda{}^\kappa$ vanishes? This problem was studied by Eckmann [1] and Frölicher [1] and answered affirmatively. (Cf. Yano [3].)

Now problems quite analogous to this arise in a space which we call here an almost product space. Suppose that there are given two complementary distributions $B$ and $C$ of respective dimensions $p$ and $q$ ($p \geqq 1$, $q \geqq 1$, $p+q=n$), then denoting by $B_\lambda^\kappa$ and $C_\lambda^\kappa$ the projection tensors on these distributions, we have

$$B_\lambda^\kappa + C_\lambda^\kappa = A_\lambda^\kappa.$$

It is easy to verify that if we put

$$B_\lambda^\kappa - C_\lambda^\kappa = F_\lambda{}^\kappa,$$

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then the tensor field $F_\lambda{}^\kappa$ satisfies

$$F_\mu{}^\lambda F_\lambda{}^\kappa = +A_\mu^\kappa,$$

and conversely if we have a tensor field $F_\lambda{}^\kappa$ satisfying $F_\mu{}^\lambda F_\lambda{}^\kappa = +A_\mu^\kappa$, then the two tensors $B_\lambda^\kappa = \frac{1}{2}(A_\lambda^\kappa + F_\lambda{}^\kappa)$ and $C_\lambda^\kappa = \frac{1}{2}(A_\lambda^\kappa - F_\lambda{}^\kappa)$ are projection tensors on two complementary distribution $B$ and $C$ respectively. The eigenvectors corresponding to the eigenvalue $+1$ span the distribution $B$ and those corresponding to $-1$ span the distribution $C$.

We shall call a space in which the structure tensor $F_\lambda{}^\kappa$ satisfying $F_\mu{}^\lambda F_\lambda{}^\kappa = +A_\mu^\kappa$ $(F_\lambda{}^\kappa \neq A_\lambda^\kappa)$ is given an almost product space. When $p=q$, the space is called a space with paracomplex structure. (Libermann [2], [3].)

Now a necessary and sufficient condition for $B$ $(C)$ to be completely integrable is given by

$$N_{\mu\lambda}{}^\kappa - N_{\mu\lambda}{}^\rho F_\rho{}^\kappa = 0 \qquad (N_{\mu\lambda}{}^\kappa + N_{\mu\lambda}{}^\rho F_\rho{}^\kappa = 0).$$

It is easy to see that if there exists a symmetric affine connexion with respect to which the distribution $B$ $(C)$ is parallel, then the distribution $B$ $(C)$ becomes integrable.

Conversely suppose that one of the distributions, say, $B$ is integrable. In the case of almost complex spaces the integrability of $B$ implies that of $\bar{B}$, but in the case of almost product space the integrability of a distribution does not imply that of the other. Under the assumption that $B$ is integrable, does there exist a symmetric affine connexion with respect to which the distribution $B$ is parallel? This problem and more general problems were studied from a global point of view by Walker [1], [2] and Willmore [1], [2] and were answered affirmatively.

Now what we call here distributions are nothing but the non-holonomic subspaces studied by Dienes [1], Schouten [1], Vranceanu [1], the present author and Petrescu [1] and others some more than twenty years ago, but the authors who studied the almost complex or product spaces did not use the existing classical theory of non-holonomic subspaces.

The present author [2] has shown already how to use the existing theory of non-holonomic subspaces to study the almost complex spaces. The purpose of the present paper is to study the problem of the existence of affine connexions which satisfy certain conditions imposed on the distributions $B$ and $C$ in almost product space and which are symmetric whenever the distribution $B$ or $C$ is integrable, the problem which was already studied by Walker and Willmore in great details. But we shall study here the problem utilizing fully the existing theory of non-holonomic subspaces and we shall try to express the results in terms of the structure tensor $F_\lambda{}^\kappa$ only.

## §1. Almost product spaces.

Let $M$ be an $n$-dimensional manifold of differentiability class $C^\infty$ and there be given globally two complementary distributions $B$ and $C$ of dimensions $p$ and $q$ respectively, where $p+q=n$ and $p \geqq 1$, $q \geqq 1$.

When there is given only one distribution $B$ in $M$, we can construct globally a complementary distribution $C$ in the following way. Since the manifold $M$ is of differentiability class $C^\infty$, we can introduce a global Riemannian metric of class $C^\infty$ in $M$. Then we have only to define $C$ as a distribution which is always orthogonal to the distribution $B$ with respect to the introduced Riemannian metric. The distribution $C$ is thus globally defined.

We take $p$ linearly independent contravariant vectors $B_b{}^\kappa$ ($\kappa, \lambda, \mu, \cdots = 1, 2, \cdots, n$; $a, b, c, \cdots = 1, 2, \cdots, p$; $h, i, j, \cdots = p+1, \cdots, n$) in $B$ and $q$ linearly independent contravariant vectors $C_i{}^\kappa$ in $C$. Then $n$ vectors $B_b{}^\kappa$ and $C_i{}^\kappa$ being linearly independent, we construct the inverse of the matrix $(B_b{}^\kappa, C_i{}^\kappa)$ which we denote by $(B^a{}_\lambda, C^h{}_\lambda)$. Then we have the identities:

(1.1) $$B_b{}^\kappa B^a{}_\kappa = \delta_b^a, \quad B_b{}^\kappa C^h{}_\kappa = 0, \quad C_i{}^\kappa B^a{}_\kappa = 0, \quad C_i{}^\kappa C^h{}_\kappa = \delta_i^h,$$

where $\delta$ is the Kronecker symbol and

(1.2) $$B_a{}^\kappa B^a{}_\lambda + C_i{}^\kappa C^i{}_\lambda = A_\lambda^\kappa,$$

$A_\lambda^\kappa$ being the unit tensor.

We use also the notations

(1.3) $$A_\beta{}^\kappa = (B_b{}^\kappa, C_i{}^\kappa), \qquad A^\alpha{}_\lambda = (B^a{}_\lambda, C^h{}_\lambda)$$

($\alpha, \beta, \gamma, \cdots = 1, 2, \cdots, n$) and write (1.1) and (1.2) in the following form:

(1.4) $$A_\beta{}^\kappa A^\alpha{}_\kappa = \delta_\beta^\alpha, \qquad A_\alpha{}^\kappa A^\alpha{}_\lambda = A_\lambda^\kappa.$$

We call the set of $A_\beta{}^\kappa = (B_b{}^\kappa, C_i{}^\kappa)$ the non-holonomic frame.

If we put

(1.5) $$B_a{}^\kappa B^a{}_\lambda = B_\lambda^\kappa, \qquad C_i{}^\kappa C^i{}_\lambda = C_\lambda^\kappa,$$

then we have from (1.1) and (1.2)

(1.6) $$B_\lambda^\kappa B_\mu^\lambda = B_\mu^\kappa, \quad B_\lambda^\kappa C_\mu^\lambda = 0, \quad C_\lambda^\kappa B_\mu^\lambda = 0, \quad C_\lambda^\kappa C_\mu^\lambda = C_\mu^\kappa$$

and

(1.7) $$A_\lambda^\kappa = B_\lambda^\kappa + C_\lambda^\kappa.$$

It will be easily verified that the tensors $B_\lambda^\kappa$ and $C_\lambda^\kappa$ do not depend on the choice of $B_b{}^\kappa$ in $B$ and $C_i{}^\kappa$ in $C$. These are projection tensors on $B$ and $C$ respectively, that is, an arbitrary vector $v^\kappa$ in the tangent space at $\xi \in M$ is decomposed into

$$v^\kappa = B_\lambda^\kappa v^\lambda + C_\lambda^\kappa v^\lambda,$$

$B_\lambda^\kappa v^\lambda$ and $C_\lambda^\kappa v^\lambda$ being respectively in $B$ and $C$. $A$ vector in $B$ is characterized by

$$v^\kappa = B_\lambda^\kappa v^\lambda \qquad \text{or} \qquad C_\lambda^\kappa v^\lambda = 0$$

and a vector in $C$ by

$$v^\kappa = C_\lambda^\kappa v^\lambda \qquad \text{or} \qquad B_\lambda^\kappa v^\lambda = 0.$$

We now define the tensor $F_\lambda{}^\kappa$ by

(1.8) $$F_\lambda{}^\kappa = B_\lambda^\kappa - C_\lambda^\kappa,$$

then (1.7) and (1.8) give

(1.9) $$B_\lambda^\kappa = \frac{1}{2}(A_\lambda^\kappa + F_\lambda{}^\kappa), \qquad C_\lambda^\kappa = \frac{1}{2}(A_\lambda^\kappa - F_\lambda{}^\kappa).$$

Taking account of (1.6) we can easily see that

(1.10) $$F_\lambda{}^\rho F_\rho{}^\kappa = A_\lambda^\kappa.$$

We notice here that if the distributions $B$ and $C$ are given globally then the tensor $F_\lambda{}^\kappa$ of rank $n$ will be defined also globally.

Conversely, if, in the manifold $M$, a tensor $F_\lambda{}^\kappa$ satisfying equation (1.10) is given globally, then we define $B_\lambda^\kappa$ and $C_\lambda^\kappa$ by (1.9) and we can easily see that these $B_\lambda^\kappa$ and $C_\lambda^\kappa$ satisfy (1.6) and (1.7). Thus $B_\lambda^\kappa$ and $C_\lambda^\kappa$ define two complementary distributions $B$ and $C$ globally.

Let $a$ be an eigenvalue of the matrix $F_\lambda{}^\kappa$ and $v^\kappa$ the corresponding eigenvector, then we have

(1.11) $$F_\mu{}^\lambda v^\mu = a v^\lambda,$$

from which, contracting with $F_\lambda{}^\kappa$

$$F_\lambda{}^\kappa F_\mu{}^\lambda v^\mu = a F_\lambda{}^\kappa v^\lambda$$

or

$$v^\kappa = a^2 v^\kappa,$$

because of (1.10) and (1.11). Thus we have $a^2 = 1$, which shows that the eigenvalues of the matrix $F_\lambda{}^\kappa$ are $+1$ or $-1$. For an eigenvector $v^\kappa$ corresponding to the eigenvalue $+1$, we have

$$F_\lambda{}^\kappa v^\lambda = v^\kappa \qquad \text{or} \qquad C_\lambda^\kappa v^\lambda = \frac{1}{2}(A_\lambda^\kappa - F_\lambda{}^\kappa) v^\lambda = 0,$$

which shows that $v^\kappa$ is in $B$. An eigenvector corresponding to the eigenvalue $-1$ is in $C$.

Thus if $F_\lambda{}^\kappa$ has eigenvalue $+1$ of multiplicity $p$ and eigenvalue $-1$ of multiplicity $q$, then the dimension of $B$ is $p$ and that of $C$ is $q$.

An $n$-dimensional manifold $M$ in which a tensor field $F_\lambda{}^\kappa$ ($\neq A_\lambda^\kappa$) satisfying (1.10) is given is called an *almost product space*.

## §2. Integrability conditions.

We put

(2.1) $$\Omega_{\gamma\beta}{}^\alpha = \frac{1}{2}(\partial_\gamma A_\beta{}^\kappa - \partial_\beta A_\gamma{}^\kappa) A^\alpha{}_\kappa$$

and call $\Omega_{\gamma\beta}{}^\alpha$ the non-holonomic object, where $\partial_\gamma$ denotes the non-holonomic or Pfaffian derivative with respect to $A_\gamma{}^\kappa$, that is,

(2.2) $$\partial_\gamma = A_\gamma{}^\kappa \partial_\kappa = A_\gamma{}^\kappa \frac{\partial}{\partial \xi^\kappa},$$

$\xi^\kappa$ being the local coordinates. Thus

(2.3) $$\partial_b = B_b{}^\kappa \partial_\kappa \qquad \text{and} \qquad \partial_i = C_i{}^\kappa \partial_\kappa.$$

The equation (2.1) can also be written in the form

(2.4) $$\Omega_{\gamma\beta}{}^{\alpha} = -\frac{1}{2} A_{\gamma}{}^{\mu} A_{\beta}{}^{\lambda} (\partial_{\mu} A^{\alpha}{}_{\lambda} - \partial_{\lambda} A^{\alpha}{}_{\mu}),$$

which shows that $\Omega_{\gamma\beta}{}^{\alpha}$ are scalars under a transformation of the local coordinates. But if we effect the transformation of the non-holonomic frame:

(2.5) $$A_{\beta'}{}^{\kappa} = A_{\beta'}^{\beta} A_{\beta}{}^{\kappa} \qquad (|A_{\beta'}^{\beta}| \neq 0),$$

then the non-holonomic object $\Omega_{\gamma\beta}{}^{\alpha}$ undergoes the transformation

(2.6) $$\Omega_{\gamma'\beta'}{}^{\alpha'} = A_{\gamma'}^{\gamma} A_{\beta'}^{\beta} A_{\alpha}^{\alpha'} \Omega_{\gamma\beta}{}^{\alpha} + \frac{1}{2} (\partial_{\gamma'} A_{\beta'}^{\alpha} - \partial_{\beta'} A_{\gamma'}^{\alpha}) A_{\alpha}^{\alpha'},$$

where $A_{\alpha}^{\alpha'}$ is the inverse matrix of $A_{\alpha'}^{\alpha}$. Equation (2.6) shows that the non-holonomic object $\Omega_{\gamma\beta}{}^{\alpha}$ is not a tensor under the transformation of non-holonomic frame.

Since we are considering a non-holonomic frame whose first $p$ vectors are in the distribution $B$ and whose second $q$ vectors are in the distribution $C$, the transformation (2.5) of the non-holonomic frame should split into

(2.7) $$B_{b'}{}^{\kappa} = A_{b'}^{b} B_{b}{}^{\kappa}, \qquad C_{i'}{}^{\kappa} = A_{i'}^{i} C_{i}{}^{\kappa},$$

which means

(2.8) $$A_{\alpha'}^{\alpha} = \begin{pmatrix} A_{b'}^{b} & 0 \\ 0 & A_{i'}^{i} \end{pmatrix}.$$

Thus equation (2.6) gives

(2.9) $$\Omega_{c'b'}{}^{h'} = A_{c'}^{c} A_{b'}^{b} A_{h}^{h'} \Omega_{cb}{}^{h}, \qquad \Omega_{j'i'}{}^{a'} = A_{j'}^{j} A_{i'}^{i} A_{a}^{a'} \Omega_{ji}{}^{a},$$

which show that $\Omega_{cb}{}^{h}$ and $\Omega_{ji}{}^{a}$ are tensors under a transformation of the non-holonomic frame.

We shall now consider the integrability condition of the distribution $B$.

An arbitrary contravariant vector $d\xi^{\kappa}$ in the tangent space at $\xi \in M$ can bewritten in the form

(2.10) $$d\xi^{\kappa} = B_{a}{}^{\kappa} (d\xi)^{a} + C_{h}{}^{\kappa} (d\xi)^{h}$$

because of (1.7), where

(2.11) $$(d\xi)^{a} = B^{a}{}_{\lambda} d\xi^{\lambda}, \qquad (d\xi)^{h} = C^{h}{}_{\lambda} d\xi^{\lambda}.$$

Thus the distribution $B$ is defined by

(2.12) $$(d\xi)^{h} = C^{h}{}_{\lambda} d\xi^{\lambda} = 0.$$

The condition for the distribution $B$ to be completely integrable is then that

$$(\partial_{\mu} C^{h}{}_{\lambda} - \partial_{\lambda} C^{h}{}_{\mu}) d\xi^{\mu} \wedge d\xi^{\lambda} = 0$$

be satisfied by any $d\xi^{\kappa}$ satisfying (2.12), that is,

(2.13) $$\Omega_{cb}{}^{h} = -\frac{1}{2} B_{c}{}^{\mu} B_{b}{}^{\lambda} (\partial_{\mu} C^{h}{}_{\lambda} - \partial_{\lambda} C^{h}{}_{\mu}) = 0.$$

This is the condition for the distribution $B$ to be completely integrable.

The same condition may be found also in the following way. An arbitrary contravariant vector $d\xi^\kappa$ in the tangent space at $\xi \in M$ can be written in the form:

$$d\xi^\kappa = B_\lambda^\kappa d\xi^\lambda + C_\lambda^\kappa d\xi^\lambda \tag{2.14}$$

because of (1.7). Thus the distribution $B$ is defined by

$$C_\lambda^\kappa d\xi^\lambda = 0. \tag{2.15}$$

The condition for $B$ to be completely integrable is then that

$$(\partial_\mu C_\lambda^\kappa - \partial_\lambda C_\mu^\kappa) d\xi^\mu \wedge d\xi^\lambda = 0$$

be satisfied by any $d\xi^\kappa$ satisfying (2.15), that is, by any vector satisfying $B_\lambda^\kappa d\xi^\lambda = d\xi^\kappa$. Thus we have

$$-\frac{1}{2} B_\mu^\tau B_\lambda^\sigma (\partial_\tau C_\sigma^\kappa - \partial_\sigma C_\tau^\kappa) = 0 \tag{2.16}$$

as the condition for $B$ to be completely integrable.

By a straightforward calculation, we can show that

$$B^c{}_\mu B^b{}_\lambda C_h{}^\kappa \Omega_{cb}{}^h = -\frac{1}{2} B_\mu^\tau B_\lambda^\sigma (\partial_\tau C_\sigma^\kappa - \partial_\sigma C_\tau^\kappa), \tag{2.17}$$

and the equivalenee of (2.13) and (2.16) is evident.

Now substituting (1.9) in the left hand member of equation (2.16), we find

$$-\frac{1}{2} B_\mu^\tau B_\lambda^\sigma (\partial_\tau C_\sigma^\kappa - \partial_\sigma C_\tau^\kappa) = \frac{1}{16}(N_{\mu\lambda}{}^\kappa - N_{\mu\lambda}{}^\rho F_\rho{}^\kappa), \tag{2.18}$$

where $N_{\mu\lambda}{}^\kappa$ is the so-called Nijenhuis tensor [1] formed with $F_\lambda{}^\kappa$:

$$N_{\mu\lambda}{}^\kappa = F_\mu{}^\rho(\partial_\rho F_\lambda{}^\kappa - \partial_\lambda F_\rho{}^\kappa) - F_\lambda{}^\rho(\partial_\rho F_\mu{}^\kappa - \partial_\mu F_\rho{}^\kappa). \tag{2.19}$$

The equation (2.18) shows that the conditions for $B$ to be completely integrable is expressed also in the form:

$$N_{\mu\lambda}{}^\kappa - N_{\mu\lambda}{}^\rho F_\rho{}^\kappa = 0. \tag{2.20}$$

Similarly we can find that the condition for $C$ to be completely integrable is

$$\Omega_{ji}{}^a = 0 \tag{2.21}$$

or

$$-\frac{1}{2} C_\mu^\tau C_\lambda^\sigma (\partial_\tau B_\sigma^\kappa - \partial_\sigma B_\tau^\kappa) = 0 \tag{2.22}$$

or

$$N_{\mu\lambda}{}^\kappa + N_{\mu\lambda}{}^\rho F_\rho{}^\kappa = 0. \tag{2.23}$$

Gathering the above results we have

THEOREM 1. *In order that the distribution $B$ ($C$) be completely integrable, it is necessary and sufficient that*

$$\Omega_{cb}{}^h = 0 \qquad (\Omega_{ji}{}^a = 0)$$

*or equivalently*

$$-\frac{1}{2}B^\tau_\mu B^\sigma_\lambda(\partial_\tau C^\kappa_\sigma - \partial_\sigma C^\kappa_\tau) = 0 \qquad \left(-\frac{1}{2}C^\tau_\mu C^\sigma_\lambda(\partial_\tau B^\kappa_\sigma - \partial_\sigma B^\kappa_\tau) = 0\right)$$

*or equivalently*

$$N_{\mu\lambda}{}^\kappa - N_{\mu\lambda}{}^\rho F_\rho{}^\kappa = 0 \qquad (N_{\mu\lambda}{}^\kappa + N_{\mu\lambda}{}^\rho F_\rho{}^\kappa = 0).$$

*Consequently in order that both of the distributions B and C be completely integrable, it is necessary and sufficient that*

$$\Omega_{cb}{}^h = 0, \qquad \Omega_{ji}{}^a = 0$$

*or equivalently*

$$-\frac{1}{2}B^\tau_\mu B^\sigma_\lambda(\partial_\tau C^\kappa_\sigma - \partial_\sigma C^\kappa_\tau) = 0, \qquad -\frac{1}{2}C^\tau_\mu C^\sigma_\lambda(\partial_\tau B^\kappa_\sigma - \partial_\sigma B^\kappa_\tau) = 0$$

*or equivalently*

$$N_{\mu\lambda}{}^\kappa = 0.$$

## §3. Affine connexions and distributions.

Let $\Gamma^\kappa_{\mu\lambda}$ be components of an affine connexion in $M$ and the covariant differentiation of a contravariant vector $v^\kappa$ be denoted by

$$\nabla_\mu v^\kappa = \partial_\mu v^\kappa + \Gamma^\kappa_{\mu\lambda} v^\lambda. \tag{3.1}$$

If we put

$$\Gamma^\alpha_{\gamma\beta} = (\partial_\gamma A_\beta{}^\kappa + A_\gamma{}^\mu A_\beta{}^\lambda \Gamma^\kappa_{\mu\lambda}) A^\alpha{}_\kappa, \tag{3.2}$$

then the components $\nabla_\gamma v^\alpha$ of $\nabla_\mu v^\kappa$ with respect to the non-holonomic frame $(A_\alpha{}^\kappa)$ are given by

$$\nabla_\gamma v^\alpha = \partial_\gamma v^\alpha + \Gamma^\alpha_{\gamma\beta} v^\beta. \tag{3.3}$$

From (3.2) we have

$$\frac{1}{2}(\Gamma^\alpha_{\gamma\beta} - \Gamma^\alpha_{\beta\gamma}) = \Omega_{\gamma\beta}{}^\alpha + S_{\gamma\beta}{}^\alpha, \tag{3.4}$$

where

$$S_{\gamma\beta}{}^\alpha = A_\gamma{}^\mu A_\beta{}^\lambda A^\alpha{}_\kappa S_{\mu\lambda}{}^\kappa \tag{3.5}$$

and

$$S_{\mu\lambda}{}^\kappa = \frac{1}{2}(\Gamma^\kappa_{\mu\lambda} - \Gamma^\kappa_{\lambda\mu}) \tag{3.6}$$

is the torsion tensor for the affine connexion $\Gamma^\kappa_{\mu\lambda}$.

From (3.4) we can see that the $\Gamma^\alpha_{\gamma\beta}$ is not necessarily symmetric with respect to $\gamma$ and $\beta$ even if the affine connexion $\Gamma^\kappa_{\mu\lambda}$ is without torsion.

Now if we effect a transformation (2.5) of the non-holonomic frame, the components $\Gamma^\alpha_{\gamma\beta}$ of the affine connexion undergo the transformation

$$\Gamma^{\alpha'}_{\gamma'\beta'} = (\partial_{\gamma'} A^\alpha_{\beta'} + A^{\gamma}_{\gamma'}{}^{\beta}_{\beta'} \Gamma^\alpha_{\gamma\beta}) A^{\alpha'}_\alpha. \tag{3.7}$$

Since the matrix $(A^\alpha_{\alpha'})$ has the form (2.8), the equation (3.7) gives

$$
(3.8)\qquad \begin{cases} \Gamma^{h'}_{c'b'} = A^c_{c'} A^b_{b'} A^{h'}_h \Gamma^h_{cb}, & \Gamma^{h'}_{j'b'} = A^j_{j'} A^b_{b'} A^{h'}_h \Gamma^h_{jb}, \\ \Gamma^{a'}_{j'i'} = A^j_{j'} A^i_{i'} A^{a'}_a \Gamma^a_{ji}, & \Gamma^{a'}_{c'i'} = A^c_{c'} A^i_{i'} A^{a'}_a \Gamma^a_{ci}, \end{cases}
$$

which show that $\Gamma^h_{cb}$, $\Gamma^h_{jb}$, $\Gamma^a_{ji}$ and $\Gamma^a_{ci}$ are components of tensors with respect to the transformation of the non-holonomic frame $A_{\alpha'}{}^\kappa = A^\alpha_{\alpha'} A_\alpha{}^\kappa$ having a special form (2.8). For example, the vanishing of one of these tensors should have a geometrical meaning independent of the choice of the vectors $B_a{}^\kappa$ in $B$ and $C_i{}^\kappa$ in $C$.

Now equation (3.2) can be written also in the form:

$$
(3.9)\qquad \partial_\gamma A_\beta{}^\kappa + A_\gamma{}^\mu A_\beta{}^\lambda \Gamma^\kappa_{\mu\lambda} = \Gamma^\alpha_{\gamma\beta} A_\alpha{}^\kappa .
$$

If we put $\gamma = c$ and $\beta = b$ in (3.9), we get

$$
\partial_c B_b{}^\kappa + B_c{}^\mu B_b{}^\lambda \Gamma^\kappa_{\mu\lambda} = \Gamma^a_{cb} B_a{}^\kappa + \Gamma^h_{cb} C_h{}^\kappa
$$

or

$$
\partial_c B_b{}^\kappa + B_c{}^\mu B_b{}^\lambda \Gamma^\kappa_{\mu\lambda} - \Gamma^a_{cb} B_a{}^\kappa = \Gamma^h_{cb} C_h{}^\kappa .
$$

We denote by $\nabla_c B_b{}^\kappa$ the left hand member of the above equation:

$$
(3.10)\qquad \nabla_c B_b{}^\kappa = \partial_c B_b{}^\kappa + B_c{}^\mu B_b{}^\lambda \Gamma^\kappa_{\mu\lambda} - \Gamma^a_{cb} B_a{}^\kappa = \Gamma^h_{cb} C_h{}^\kappa .
$$

Here $\nabla_c B_b{}^\kappa$ is the so-called van der Waerden-Bortolotti derivative of $B_b{}^\kappa$ (Schouten [1]) along the distribution $B$ and consequently equation (3.10) reduces to that of Gauss when $B$ is integrable.

If we put $\gamma = c$ and $\beta = i$ in (3.9), we get

$$
(3.11)\qquad \nabla_c C_i{}^\kappa = \partial_c C_i{}^\kappa + B_c{}^\mu C_i{}^\lambda \Gamma^\kappa_{\mu\lambda} = \Gamma^a_{ci} B_a{}^\kappa + \Gamma^h_{ci} C_h{}^\kappa ,
$$

which reduces to the equation of Weingarten when $B$ is integrable.

Similarly, from (3.9) we get

$$
(3.12)\qquad \nabla_j B_b{}^\kappa = \partial_j B_b{}^\kappa + C_j{}^\mu B_b{}^\lambda \Gamma^\kappa_{\mu\lambda} = \Gamma^a_{jb} B_a{}^\kappa + \Gamma^h_{jb} C_h{}^\kappa
$$

and

$$
(3.13)\qquad \nabla_j C_i{}^\kappa = \partial_j C_i{}^\kappa + C_j{}^\mu C_i{}^\lambda \Gamma^\kappa_{\mu\lambda} - \Gamma^h_{ji} C_h{}^\kappa = \Gamma^a_{ji} B_a{}^\kappa ,
$$

which reduce respectively to the equation of Weingarten and to that of Gauss when the distribution $C$ is integrable.

We shall now consider various conditions which we can put on the distributions $B$ and $C$.

( i ) *The condition for $B$ to be flat.*

Let us consider a vector field $v^\kappa$. If the vector is parallel when we displace in any direction contained in $B$, we say that the vector is *parallel along $B$*. We can use the same terminology also for the distribution, that is, if a distribution is parallel when we displace in any direction contained in $B$, we say that the distribution is parallel along $B$.

Now when we displace a vector contained in $B$ parallelly along $B$, if the displaced vector is always contained in the distribution $B$, we say that the distribution $B$ is a *flat* distribution. (See Hayden [1].) Walker [2] calls such a distribution a semi-parallel distribution.

The equation (3.10) shows that the condition for the distribution $B$ to be flat is

(3.14) $$\Gamma^h_{cb} = 0.$$

(ii) *The condition for B to be geodesic.*

Take a point $\xi^\kappa$ and a direction $v^\kappa$ at $\xi^\kappa$ which is contained in $B$. The auto-parallel curve or path with respect to the affine connexion under consideration is uniquely determined by the inital point $\xi^\kappa$ and the initial direction $v^\kappa$. If the tangent to the path thus determined is always contained in $B$ for any initial point and for any initial direction contained in $B$, we say that the distribution is *geodesic*. (See Hayden [1].) Walker [2] calls such a distribution a path-parallel distribution.

The condition for the distribution $B$ to be geodesic is then that, if the equation

$$C^i{}_\lambda \frac{d\xi^\lambda}{ds} = 0$$

is satisfied at the initial point $\xi^\kappa_0$ and for the initial direction $(d\xi^\kappa/ds)_0$ at $\xi^\kappa_0$, it should always be satisfied along the path:

$$\frac{d^2\xi^\kappa}{ds^2} + \Gamma^\kappa_{\mu\lambda}\frac{d\xi^\mu}{ds}\frac{d\xi^\lambda}{ds} = 0$$

having $\xi^\kappa_0$ as the initial point and $(d\xi^\kappa/ds)_0$ as the initial direction at $\xi^\kappa_0$.

Thus differentiating $C^i{}_\lambda d\xi^\lambda/ds = 0$ along the path, we have

$$(\partial_\mu C^i{}_\lambda - C^i{}_\kappa \Gamma^\kappa_{\mu\lambda})\frac{d\xi^\mu}{ds}\frac{d\xi^\lambda}{ds} = 0,$$

or, $d\xi^\lambda/ds$ being always contained in $B$,

$$B_{(c}{}^\mu B_{b)}{}^\lambda(\partial_\mu C^i{}_\lambda - C^i{}_\kappa \Gamma^\kappa_{\mu\lambda}) = 0$$

or

$$(\partial_{(c} B_{b)}{}^\kappa + B_{(c}{}^\mu B_{b)}{}^\lambda \Gamma^\kappa_{\mu\lambda})C^i{}_\kappa = 0,$$

which is equivalent to

(3.15) $$\Gamma^h_{(cb)} = 0.$$

This is the condition for $B$ to be geodesic.

Thus we see that a flat distribution is always geodesic, but a geodesic distribution is not necessarily flat. The distinction between flat distributions and geodesic distributions goes back to Hayden [1].

(iii) *The condition for B to be parallel along C.*

The equation (3.12) shows that the condistribution $B$ to be parallel along $C$ is

(3.16) $$\Gamma^h_{jb} = 0.$$

(iv) *The condition for C to be parallel along B.*

(3.17) $$\Gamma^a_{ci} = 0.$$

(v) *The condition for C to be flat.*

(3.18) $$\Gamma^a_{ji} = 0.$$

(vi) *The condition for $C$ to be geodesic.*

$$\Gamma^{a}_{(ji)} = 0. \tag{3.19}$$

(vii) *The condition for $B$ to be parallel.*

From (i) and (iii) we have

$$\Gamma^{h}_{cb} = 0 \quad \text{and} \quad \Gamma^{h}_{jb} = 0 \qquad \text{or} \qquad \Gamma^{h}_{\gamma b} = 0. \tag{3.20}$$

(viii) *The condition for $C$ to be parallel.*

From (iv) and (v) we have

$$\Gamma^{a}_{ci} = 0 \quad \text{and} \quad \Gamma^{a}_{ji} = 0 \qquad \text{or} \qquad \Gamma^{a}_{\gamma i} = 0. \tag{3.21}$$

Suppose that there is given a symmetric affine connexion with respect to which the distribution $B$ is flat, then we have

$$\Gamma^{h}_{cb} = 0.$$

On the other hand, from (3.4), we find

$$\frac{1}{2}(\Gamma^{h}_{cb} - \Gamma^{h}_{bc}) = \Omega_{cb}{}^{h},$$

the torsion tensor $S_{\gamma\beta}{}^{\alpha}$ being zero. This shows that when the distribution $B$ is flat with respect to a symmetric affine connexion, $B$ is integrable. The same is true of course for the distribution $C$.

## §4. The determination of affine connexions.

Let us consider an almost product space $M$ of class $C^{\infty}$ in which two complementary distributions $B$ and $C$ of class $C^{\infty}$ are given globally. Walker [1], [2] studied the existence of global affine connexions with respect to which the given distributions are flat, geodesic or parallel and which are without torsion whenever possible. We shall study the same problems with the use of the existing theory of non-holonomic subspaces and of the Nijenhuis tensor which is related closely to the integrability conditions of the distributions.

Following Walker, we first choose a symmetric affine connexion $\overset{\circ}{\Gamma}{}^{\kappa}_{\mu\lambda}$ defined globally in the almost product space $M$. Since the space is of class $C^{\infty}$, we can introduce a global Riemannian metric of class $C^{\infty}$ in $M$ and construct the Levi-Civita affine connexion which can be taken as our $\overset{\circ}{\Gamma}{}^{\kappa}_{\mu\lambda}$.

Then for any global affine connexion $\Gamma^{\kappa}_{\mu\lambda}$, if we put

$$\Gamma^{\kappa}_{\mu\lambda} = \overset{\circ}{\Gamma}{}^{\kappa}_{\mu\lambda} + T_{\mu\lambda}{}^{\kappa}, \tag{4.1}$$

$T_{\mu\lambda}{}^{\kappa}$ is a tensor field defined globally in $M$. The problem of determination of a global affine connexion $\Gamma^{\kappa}_{\mu\lambda}$ satisfying certain conditions is then reduced to that of the global tensor $T_{\mu\lambda}{}^{\kappa}$ satisfying certain conditions.

Substituting (4.1) into (3.2) we find

$$\Gamma^{\alpha}_{\gamma\beta} = \overset{\circ}{\Gamma}{}^{\alpha}_{\gamma\beta} + T_{\gamma\beta}{}^{\alpha}, \tag{4.2}$$

where

$$\overset{\circ}{\Gamma}{}^{\alpha}_{\gamma\beta} = (\partial_{\gamma} A_{\beta}{}^{\kappa} + A_{\gamma}{}^{\mu} A_{\beta}{}^{\lambda} \overset{\circ}{\Gamma}{}^{\kappa}_{\mu\lambda}) A^{\alpha}{}_{\kappa} \tag{4.3}$$

are components of the affine connexion $\overset{\circ}{\Gamma}{}^{\kappa}_{\mu\lambda}$ with respect to the non-holonomic frame $(A_\alpha{}^\kappa)$ and

$$T_{\gamma\beta}{}^{\alpha} = A_\gamma{}^\mu A_\beta{}^\lambda A^\alpha{}_\kappa T_{\mu\lambda}{}^\kappa \tag{4.4}$$

are those of the tensor $T_{\mu\lambda}{}^\kappa$ with respect to the same frame.

The affine connexion $\overset{\circ}{\Gamma}{}^{\kappa}_{\mu\lambda}$ being symmetric, we have from (4.3)

$$\frac{1}{2}(\overset{\circ}{\Gamma}{}^{\alpha}_{\gamma\beta} - \overset{\circ}{\Gamma}{}^{\alpha}_{\beta\gamma}) = \Omega_{\gamma\beta}{}^{\alpha}. \tag{4.5}$$

We shall study the existence of global affine connexions with respect to which the given distributions are flat, geodesic or parallel and which are symmetric whenever possible.

(i) *Affine connexions with respect to which the distribution B is flat.*

The condition for $B$ to be flat is

$$\Gamma^{h}_{cb} = 0.$$

Thus we have from (4.2)

$$0 = \overset{\circ}{\Gamma}{}^{h}_{cb} + T_{cb}{}^{h}.$$

Thus *the distribution B is flat with respect to the affine connexion* $\Gamma^{\kappa}_{\mu\lambda} = \overset{\circ}{\Gamma}{}^{\kappa}_{\mu\lambda} + T_{\mu\lambda}{}^\kappa$ *if and only if the tensor* $T_{\mu\lambda}{}^\kappa$ *satisfies*

$$T_{cb}{}^{h} = -\overset{\circ}{\Gamma}{}^{h}_{cb}. \tag{4.6}$$

To get the simplest $T_{\mu\lambda}{}^\kappa$ which satisfies this condition, we define $T_{\mu\lambda}{}^\kappa$ by requiring that all the components $T_{\gamma\beta}{}^\alpha$ of $T_{\mu\lambda}{}^\kappa$ with respect to the non-holonomic frame other than $T_{cb}{}^h$ given by (4.6) are zero. Such a $T_{\mu\lambda}{}^\kappa$ is given by the formula

$$T_{\mu\lambda}{}^{\kappa} = -B^c{}_\mu B^b{}_\lambda C_h{}^\kappa \overset{\circ}{\Gamma}{}^{h}_{cb}. \tag{4.7}$$

As we remarked at the beginning of Section 3, the $\overset{\circ}{\Gamma}{}^{h}_{cb}$ are components of a tensor with respect to the transformation of the non-holonomic frame, and the $T_{\mu\lambda}{}^\kappa$ defined here does not depend on the choice of the vectors $B_a{}^\kappa$ in $B$ and $C_i{}^\kappa$ in $C$. Thus we can see that the tensor $T_{\mu\lambda}{}^\kappa$ is determined uniquely by the distributions $B$ and $C$ and the affine connexion $\overset{\circ}{\Gamma}{}^{\kappa}_{\mu\lambda}$. But these are given globally and consequently the $T_{\mu\lambda}{}^\kappa$ is defined globally and we can conclude that the affine connexion

$$\Gamma^{\kappa}_{\mu\lambda} = \overset{\circ}{\Gamma}{}^{\kappa}_{\mu\lambda} + T_{\mu\lambda}{}^{\kappa}$$

is also defined globally. Thus we have

THEOREM 2. *For any distribution given globally, there exists a global affine connexion with respect to which the distribution is flat.*

We can further require that the affine connexion $\Gamma^{\kappa}_{\mu\lambda}$ is symmetric whenever the distribution $B$ is integrable.

In order that this is the case, we should have

$$S_{\gamma\beta}{}^{\alpha} = \frac{1}{2}(T_{\gamma\beta}{}^{\alpha} - T_{\beta\gamma}{}^{\alpha}) = 0,$$

whenever

$$\Omega_{cb}{}^{h} = 0.$$

On the other hand, we have, from (4.6),

$$S_{cb}{}^{h} = -\Omega_{cb}{}^{h}.$$

Thus, *the distribution B is flat with respect to the affine connexion* $\Gamma_{\mu\lambda}^{\kappa} = \overset{\circ}{\Gamma}{}_{\mu\lambda}^{\kappa} + T_{\mu\lambda}{}^{\kappa}$, *and the affine connexion* $\Gamma_{\mu\lambda}^{\kappa}$ *is symmetric whenever the distribution B is integrable, if and only if the tensor* $T_{\mu\lambda}{}^{\kappa}$ *satisfies*

(4.8) $$T_{cb}{}^{h} = -\overset{\circ}{\Gamma}{}_{cb}^{h}, \quad \text{the other } T\text{'s satisfying } T_{\gamma\beta}{}^{\alpha} - T_{\beta\gamma}{}^{\alpha} = 0.$$

The simplest $T_{\mu\lambda}{}^{\kappa}$ which satisfies these conditions is given by (4.7). Thus we have

THEOREM 3. *For any distribution given globally, there exists a global affine connexion with respect to which the given distribution is flat and which is symmetric whenever the distribution is integrable.*

(ii) *Affine connexions with respect to which the distribution B is geodesic.*

The condition for $B$ to be geodesic is

$$\Gamma_{(cb)}^{h} = 0.$$

Thus we have from (4.2)

$$0 = \overset{\circ}{\Gamma}{}_{(cb)}^{h} + T_{(cb)}{}^{h}.$$

Thus *the distribution B is geodesic with respect to the affine connexion* $\Gamma_{\mu\lambda}^{\kappa} = \overset{\circ}{\Gamma}{}_{\mu\lambda}^{\kappa} + T_{\mu\lambda}{}^{\kappa}$ *if and only if the tensor* $T_{\mu\lambda}{}^{\kappa}$ *satisfies*

$$T_{(cb)}{}^{h} = -\overset{\circ}{\Gamma}{}_{(cb)}^{h}.$$

The simplest symmetric $T_{\mu\lambda}{}^{\kappa}$ satisfying this condition is given globally by

(4.9) $$T_{\mu\lambda}{}^{\kappa} = -B^{c}{}_{\mu}B^{b}{}_{\lambda}C_{h}{}^{\kappa}\overset{\circ}{\Gamma}{}_{(cb)}^{h}.$$

Thus we have

THEOREM 4. *For any distribution given globally, there exists a global symmetric affine connexion with respect to which the distribqtion is geodesic.*

(iii) *Affine connexions with respect to which distribution B is parallel along C.*

The condition for $B$ to be parallel along $C$ is

(4.10) $$\Gamma_{jb}^{h} = 0.$$

Thus we have from (4.2)

$$0 = \overset{\circ}{\Gamma}{}_{jb}^{h} + T_{jb}{}^{h}.$$

Thus *the distribution B is parallel along C with respect to the affine connexion* $\Gamma^{\kappa}_{\mu\lambda} = \overset{\circ}{\Gamma}{}^{\kappa}_{\mu\lambda} + T_{\mu\lambda}{}^{\kappa}$ *if and only if the tensor* $T_{\mu\lambda}{}^{\kappa}$ *satisfies*

(4.11) $$T_{jb}{}^{h} = -\overset{\circ}{\Gamma}{}^{h}_{jb}.$$

The simplest $T_{\mu\lambda}{}^{\kappa}$ satisfying this condition is given globally by

(4.12) $$T_{\mu\lambda}{}^{\kappa} = -C^{j}{}_{\mu}B^{b}{}_{\lambda}C_{h}{}^{\kappa}\overset{\circ}{\Gamma}{}^{h}_{jb}.$$

Thus we have

THEOREM 5. *For any complementary distributions given globally, there exists a global affine connexion with respect to which one distribution is parallel along the other.*

(iv) *Affine connexions with respect to which the distribution C is parallel along B.*

Interchanging $B$ and $C$ in (iii), we see that *the distribution C is parallel along B with respect to the affine connexion* $\Gamma^{\kappa}_{\mu\lambda} = \overset{\circ}{\Gamma}{}^{\kappa}_{\mu\lambda} + T_{\mu\lambda}{}^{\kappa}$ *if and only if the tensor* $T_{\mu\lambda}{}^{\kappa}$ *satisfies*

(4.13) $$T_{ci}{}^{a} = -\overset{\circ}{\Gamma}{}^{a}_{ci}.$$

The simplest $T_{\mu\lambda}{}^{\kappa}$ satisfying this connition is given globally by

(4.14) $$T_{\mu\lambda}{}^{\kappa} = -B^{c}{}_{\mu}C^{i}{}_{\lambda}B_{a}{}^{\kappa}\overset{\circ}{\Gamma}{}^{a}_{ci}.$$

(v) *Affine connexions with respect to which the distribution C is flat.*

Interchanging $B$ and $C$ in (i), we see that *the distribution C is flat with respect to the affine connexion* $\Gamma^{\kappa}_{\mu\lambda} = \overset{\circ}{\Gamma}{}^{\kappa}_{\mu\lambda} + T_{\mu\lambda}{}^{\kappa}$ if and only if the tensor $T_{\mu\lambda}{}^{\kappa}$ satisfies

(4.15) $$T_{ji}{}^{a} = -\overset{\circ}{\Gamma}{}^{a}_{ji}.$$

The simplest $T_{\mu\lambda}{}^{\kappa}$ satisfying this condition is given globally by

(4.16) $$T_{\mu\lambda}{}^{\kappa} = -C^{j}{}_{\mu}C^{i}{}_{\lambda}B_{a}{}^{\kappa}\overset{\circ}{\Gamma}{}^{a}_{ji}.$$

The affine connexion defined here is symmetric whenever the distribution $C$ is integrable.

(vi) *Affine connexions with respect to which the distribution C is geodesic.*

Interchanging $B$ and $C$ in (ii) we see that *the distribution C is geodesic with respect to the affine connexion* $\Gamma^{\kappa}_{\mu\lambda} = \overset{\circ}{\Gamma}{}^{\kappa}_{\mu\lambda} + T_{\mu\lambda}{}^{\kappa}$ *if and only if the tensor* $T_{\mu\lambda}{}^{\kappa}$ *sayisfies*

(4.17) $$T_{(ji)}{}^{a} = -\overset{\circ}{\Gamma}{}^{a}_{(ji)}.$$

The simplest symmetric $T_{\mu\lambda}{}^{\kappa}$ satisfying this condition is given globally by

(4.18) $$T_{\mu\lambda}{}^{\kappa} = -C^{j}{}_{\mu}C^{i}{}_{\lambda}B_{a}{}^{\kappa}\overset{\circ}{\Gamma}{}^{a}_{(ji)}.$$

Now combining the results in (i) and (v) we have

THEOREM 6. *For any complementary distributions B and C given globally, there exists a global affine connexion with respect to which both of the distributions are flat and which is symmetric whenever both of the distributions are integrable.*

Denoting by $\Gamma^{\kappa}_{\mu\lambda} = \overset{\circ}{\Gamma}{}^{\kappa}_{\mu\lambda} + T_{\mu\lambda}{}^{\kappa}$ the components of such an affine connexion, the tensor $T_{\mu\lambda}{}^{\kappa}$ must satisfy

$$T_{cb}{}^{h} = -\overset{\circ}{\Gamma}{}^{h}_{cb}, \qquad T_{ji}{}^{a} = -\overset{\circ}{\Gamma}{}^{a}_{ji}, \tag{4.19}$$

all the other $T$'s satisfying $T_{\gamma\beta}{}^{\alpha} - T_{\beta\gamma}{}^{\alpha} = 0$.

The simplest $T_{\mu\lambda}{}^{\kappa}$ satisfying these conditions is given by

$$T_{\mu\lambda}{}^{\kappa} = -B^{c}{}_{\mu}B^{b}{}_{\lambda}C_{h}{}^{\kappa}\overset{\circ}{\Gamma}{}^{h}_{cb} - C^{j}{}_{\mu}C^{i}{}_{\lambda}B_{a}{}^{\kappa}\overset{\circ}{\Gamma}{}^{a}_{ji}. \tag{4.20}$$

Next combining (ii) and (vi) we have

THEOREM 7. *For any complementary distributions B and C given globally, there exists a global symmetric affine connexion with respect to which both of the distributions are geodesic.* (Walker [2], Theorem 1.)

Denoting by $\Gamma^{\kappa}_{\mu\lambda} = \overset{\circ}{\Gamma}{}^{\kappa}_{\mu\lambda} + T_{\mu\lambda}{}^{\kappa}$ the components of such an affine connexion, the tensor $T_{\mu\lambda}{}^{\kappa}$ should be symmetric and satisfy

$$T_{cb}{}^{h} = -\overset{\circ}{\Gamma}{}^{h}_{(cb)}, \qquad T_{ji}{}^{a} = -\overset{\circ}{\Gamma}{}^{a}_{(ji)}. \tag{4.21}$$

The simplest $T_{\mu\lambda}{}^{\kappa}$ satisfying these conditions is given by

$$T_{\mu\lambda}{}^{\kappa} = -B^{c}{}_{\mu}B^{b}{}_{\lambda}C_{h}{}^{\kappa}\overset{\circ}{\Gamma}{}^{h}_{(cb)} - C^{j}{}_{\mu}C^{i}{}_{\lambda}B_{a}{}^{\kappa}\overset{\circ}{\Gamma}{}^{a}_{(ji)}. \tag{4.22}$$

Also combining (ii), (iii), (iv) and (vi), we have

THEOREM 8. *For any complementary distributions B and C given globally, there exists a global symmetric affine connexion with respect to which both of the distributions are geodesic and one of the distributions is parallel along the other.* (Walker [2], Theorem 2.)

Denoting by $\Gamma^{\kappa}_{\mu\lambda} = \overset{\circ}{\Gamma}{}^{\kappa}_{\mu\lambda} + T_{\mu\lambda}{}^{\kappa}$ the components of such an affine connexion, the tensor $T_{\mu\lambda}{}^{\kappa}$ should be symmetric and satisfy

$$T_{cb}{}^{h} = -\overset{\circ}{\Gamma}{}^{h}_{(cb)}, \quad T_{jb}{}^{h} = T_{bj}{}^{h} = -\overset{\circ}{\Gamma}{}^{h}_{jb}, \quad T_{ci}{}^{a} = T_{ic}{}^{a} = -\overset{\circ}{\Gamma}{}^{a}_{ci}, \quad T_{ji}{}^{a} = -\overset{\circ}{\Gamma}{}^{a}_{(ji)}. \tag{4.23}$$

The simplest $T_{\mu\lambda}{}^{\kappa}$ satisfying these conditions is given by

$$\begin{aligned} T_{\mu\lambda}{}^{\kappa} = & -B^{c}{}_{\mu}B^{b}{}_{\lambda}C_{h}{}^{\kappa}\overset{\circ}{\Gamma}{}^{h}_{(cb)} - C^{j}{}_{\mu}B^{b}{}_{\lambda}C_{h}{}^{\kappa}\overset{\circ}{\Gamma}{}^{h}_{jb} - B^{b}{}_{\mu}C^{j}{}_{\lambda}C_{h}{}^{\kappa}\overset{\circ}{\Gamma}{}^{h}_{jb} \\ & - B^{c}{}_{\mu}C^{i}{}_{\lambda}B_{a}{}^{\kappa}\overset{\circ}{\Gamma}{}^{a}_{ci} - C^{i}{}_{\mu}B^{c}{}_{\lambda}B_{a}{}^{\kappa}\overset{\circ}{\Gamma}{}^{a}_{ci} - C^{j}{}_{\mu}C^{i}{}_{\lambda}B_{a}{}^{\kappa}\overset{\circ}{\Gamma}{}^{a}_{(ji)}. \end{aligned} \tag{4.24}$$

(vii) *Affine connexions with respect to which the distribution B is parallel.*

The condition for $B$ to be parallel is

$$\Gamma^{h}_{cb} = 0 \qquad \text{and} \qquad \Gamma^{h}_{jb} = 0.$$

From (4.2) we have

$$0 = \overset{\circ}{\Gamma}{}^h_{cb} + T_{cb}{}^h, \qquad 0 = \overset{\circ}{\Gamma}{}^h_{jb} + T_{jb}{}^h.$$

Thus *the distribution $B$ is parallel with respect to the affine connexion $\Gamma^\kappa_{\mu\lambda} = \overset{\circ}{\Gamma}{}^\kappa_{\mu\lambda} + T_{\mu\lambda}{}^\kappa$ if and only if the tensor $T_{\mu\lambda}{}^\kappa$ satisfies*

$$T_{cb}{}^h = -\overset{\circ}{\Gamma}{}^h_{cb} \quad \text{and} \quad T_{jb}{}^h = -\overset{\circ}{\Gamma}{}^h_{jb}. \tag{4.25}$$

The simplest $T_{\mu\lambda}{}^\kappa$ satisfying these conditions is given by

$$T_{\mu\lambda}{}^\kappa = -B^c{}_\mu B^b{}_\lambda C_h{}^\kappa \overset{\circ}{\Gamma}{}^h_{cb} - C^j{}_\mu B^b{}_\lambda C_h{}^\kappa \overset{\circ}{\Gamma}{}^h_{jb}. \tag{4.26}$$

Thus we have

THEOREM 9. *For any distribution given globally, there exists a global affine connexion with respect to which the given distribution is parallel.*

We can further require that the affine connexion $\Gamma^\kappa_{\mu\lambda}$ is symmetric whenever the given distribution is integrable.

*The distribution $B$ is parallel with respct to the affine connexion $\Gamma^\kappa_{\mu\lambda} = \overset{\circ}{\Gamma}{}^\kappa_{\mu\lambda} + T_{\mu\lambda}{}^\kappa$ and the affine connexion $\Gamma^\kappa_{\mu\lambda}$ is symmetric whenever the distribution $B$ is integrable if and only if the tensor $T_{\mu\lambda}{}^\kappa$ satisfies*

$$T_{cb}{}^h = -\overset{\circ}{\Gamma}{}^h_{cb}, \qquad T_{jb}{}^h = T_{bj}{}^h = -\overset{\circ}{\Gamma}{}^h_{jb}, \tag{4.27}$$

*all the other $T_{\gamma\beta}{}^\alpha$ being symmetric in $\gamma$ and $\beta$.*

The simplest $T_{\mu\lambda}{}^\kappa$ satisfying these conditions is given by

$$T_{\mu\lambda}{}^\kappa = -B^c{}_\mu B^b{}_\lambda C_h{}^\kappa \overset{\circ}{\Gamma}{}^h_{cb} - C^j{}_\mu B^b{}_\lambda C_h{}^\kappa \overset{\circ}{\Gamma}{}^h_{jb} - B^b{}_\mu C^j{}_\lambda C_h{}^\kappa \overset{\circ}{\Gamma}{}^h_{jb}. \tag{4.28}$$

Thus we have

THEOREM 10. *For any distribution given globally, there exists a global affine connexion with respect to which the given distribution is parallel and which is symmetric whenever the distribution is integrable.*

(viii) *Affine connexions with respect to which the distribution $C$ is parallel.*

Interchanging $B$ and $C$ in (vii), we see that *the distribution $C$ is parallel with respect to the affine connexion $\Gamma^\kappa_{\mu\lambda} = \overset{\circ}{\Gamma}{}^\kappa_{\mu\lambda} + T_{\mu\lambda}{}^\kappa$ if and only if the tensor $T_{\mu\lambda}{}^\kappa$ satisfies*

$$T_{ci}{}^a = -\overset{\circ}{\Gamma}{}^a_{ci} \quad \text{and} \quad T_{ji}{}^a = -\overset{\circ}{\Gamma}{}^a_{ji}. \tag{4.29}$$

The simplest $T_{\mu\lambda}{}^\kappa$ satisfying these conditions is given by

$$T_{\mu\lambda}{}^\kappa = -B^c{}_\mu C^i{}_\lambda B_a{}^\kappa \overset{\circ}{\Gamma}{}^a_{ci} - C^j{}_\mu C^i{}_\lambda B_a{}^\kappa \overset{\circ}{\Gamma}{}^a_{ji}. \tag{4.30}$$

*The distribution $C$ is parallel with respect to the affine connexion $\Gamma^\kappa_{\mu\lambda} = \overset{\circ}{\Gamma}{}^\kappa_{\mu\lambda} + T_{\mu\lambda}{}^\kappa$ and the affine connexion $\Gamma^\kappa_{\mu\lambda}$ is symmetric whenever the* *distribution C is integrable if and only if* $T_{\mu\lambda}{}^{\kappa}$ *satisfies*

$$(4.31)\qquad T_{c\imath}{}^{a}=T_{\imath c}{}^{a}=-\overset{\circ}{\Gamma}{}^{a}_{c\imath}\quad \textit{and}\quad T_{ji}{}^{a}=-\overset{\circ}{\Gamma}{}^{a}_{ji}$$

*all the other* $T_{\gamma\beta}{}^{\alpha}$ *being symmetric in* $\gamma$ *and* $\beta$.

The simplest $T_{\mu\lambda}{}^{\kappa}$ satifying these conditions is given by

$$(4.32)\qquad T_{\mu\lambda}{}^{\kappa}=-B^{c}{}_{\mu}C^{i}{}_{\lambda}B_{a}{}^{\kappa}\overset{\circ}{\Gamma}{}^{a}_{c\imath}-C^{i}{}_{\mu}B^{c}{}_{\lambda}B_{a}{}^{\kappa}\overset{\circ}{\Gamma}{}^{a}_{ci}-C^{j}{}_{\mu}C^{i}{}_{\lambda}B_{a}{}^{\kappa}\overset{\circ}{\Gamma}{}^{a}_{ji}.$$

Now combining the results in (i), (iii), (iv), (vi) and (vii), we see that *the distribution C is geodesic and C is parallel along B with respect to an affine connexion* $\Gamma^{\kappa}_{\mu\lambda}=\overset{\circ}{\Gamma}{}^{\kappa}_{\mu\lambda}+T_{\mu\lambda}{}^{\kappa}$ *if and only if the tensor* $T_{\mu\lambda}{}^{\kappa}$ *satsifies*

$$(4.33)\qquad T_{cb}{}^{h}=-\overset{\circ}{\Gamma}{}^{h}_{cb},\quad T_{jb}{}^{h}=-\overset{\circ}{\Gamma}{}^{h}_{jb},\quad T_{(ji)}{}^{a}=-\overset{\circ}{\Gamma}{}^{a}_{(ji)},\quad T_{ci}{}^{a}=-\overset{\circ}{\Gamma}{}^{a}_{ci}.$$

The simplest $T_{\mu\lambda}{}^{\kappa}$ satisfying these conditions is given by

$$(3.34)\qquad T_{\mu\lambda}{}^{\kappa}=-B^{c}{}_{\mu}B^{b}{}_{\lambda}C_{h}{}^{\kappa}\overset{\circ}{\Gamma}{}^{h}_{cb}-C^{j}{}_{\mu}B^{b}{}_{\lambda}C_{h}{}^{\kappa}\overset{\circ}{\Gamma}{}^{h}_{jb}-C^{j}{}_{\mu}C^{i}{}_{\lambda}B_{a}{}^{\kappa}\overset{\circ}{\Gamma}{}^{a}_{(ji)}-B^{c}{}_{\mu}C^{i}{}_{\lambda}B_{a}{}^{\kappa}\overset{\circ}{\Gamma}{}^{a}_{ci}.$$

Thus we have

THEOREM 11. *For any complementary distributions B and C given globally, there exists a global affine connexion* $\Gamma^{\kappa}_{\mu\lambda}=\overset{\circ}{\Gamma}{}^{\kappa}_{\mu\lambda}+T_{\mu\lambda}{}^{\kappa}$ *with respect to which the distribution B is parallel, the distribution C is geodesic and is parallel along B.*

We can require further that the affine connexion $\Gamma^{\kappa}_{\mu\lambda}$ is symmetric whenever the given distribution $B$ is integrable.

*The distribution B is parallel, the distribution C is geodesic and is parallel along B with respect to the affine connexion* $\Gamma^{\kappa}_{\mu\lambda}=\overset{\circ}{\Gamma}{}^{\kappa}_{\mu\lambda}+T_{\mu\lambda}{}^{\kappa}$ *and the affine connexion* $\Gamma^{\kappa}_{\mu\lambda}$ *is symmetric whenever the distribution B is integrable if and only if the tensor* $T_{\mu\lambda}{}^{\kappa}$ *satisfies*

$$(4.35)\qquad T_{cb}{}^{h}=-\overset{\circ}{\Gamma}{}^{h}_{cb},\quad T_{jb}{}^{h}=T_{bj}{}^{h}=-\overset{\circ}{\Gamma}{}^{h}_{jb},\quad T_{(ji)}{}^{a}=-\overset{\circ}{\Gamma}{}^{a}_{(ji)},\quad T_{c\imath}{}^{a}=T_{\imath c}{}^{a}=-\overset{\circ}{\Gamma}{}^{a}_{c\imath},$$

*all the other* $T_{\gamma\beta}{}^{\alpha}$ *being symmetric in* $\gamma$ *and* $\beta$.

The simplest $T_{\mu\lambda}{}^{\kappa}$ satisfying these conditions is given by

$$(4.36)\qquad \begin{aligned}T_{\mu\lambda}{}^{\kappa}=&-B^{c}{}_{\mu}B^{b}{}_{\lambda}C_{h}{}^{\kappa}\overset{\circ}{\Gamma}{}^{h}_{cb}-C^{j}{}_{\mu}B^{b}{}_{\lambda}C_{h}{}^{\kappa}\overset{\circ}{\Gamma}{}^{h}_{jb}-B^{b}{}_{\mu}C^{j}{}_{\lambda}C_{h}{}^{\kappa}\overset{\circ}{\Gamma}{}^{h}_{jb}\\&-C^{j}{}_{\mu}C^{i}{}_{\lambda}B_{a}{}^{\kappa}\overset{\circ}{\Gamma}{}^{a}_{(ji)}-B^{c}{}_{\mu}C^{i}{}_{\lambda}B_{a}{}^{\kappa}\overset{\circ}{\Gamma}{}^{a}_{c\imath}-C^{i}{}_{\mu}B^{c}{}_{\lambda}B_{a}{}^{\kappa}\overset{\circ}{\Gamma}{}^{a}_{c\imath}.\end{aligned}$$

Thus we have

THEOREM 12. *For any complementary distributions B and C given globally in M, there exists a global affine connexion with respect to which the distribution B is parallel, the distribution C is geodesic and is parallel along B and which is symmetric whenever the distribution B is integrable.* (Walker [2], Theorem 3.)

Combining (vii) and (iii), we can see that *the distributions B and C are both parallel with respect to the affine connexion* $\Gamma^{\kappa}_{\mu\lambda}=\overset{\circ}{\Gamma}{}^{\kappa}_{\mu\lambda}+T_{\mu\lambda}{}^{\kappa}$ *if and only if*

$$T_{cb}{}^{h}=-\overset{\circ}{\Gamma}{}^{h}_{cb},\quad T_{jb}{}^{h}=-\overset{\circ}{\Gamma}{}^{h}_{jb},\quad T_{ci}{}^{a}=-\overset{\circ}{\Gamma}{}^{a}_{ci},\quad T_{ji}{}^{a}=-\overset{\circ}{\Gamma}{}^{a}_{ji}. \tag{4.37}$$

The simplest $T_{\mu\lambda}{}^{\kappa}$ satisfying these conditions is given by

$$T_{\mu\lambda}{}^{\kappa}=-B^{c}{}_{\mu}B^{b}{}_{\lambda}C_{h}{}^{\kappa}\overset{\circ}{\Gamma}{}^{h}_{cb}-C^{j}{}_{\mu}B^{b}{}_{\lambda}C_{h}{}^{\kappa}\overset{\circ}{\Gamma}{}^{h}_{jb}-B^{c}{}_{\mu}C^{i}{}_{\lambda}B_{a}{}^{\kappa}\overset{\circ}{\Gamma}{}^{a}_{ci}-C^{j}{}_{\mu}C^{i}{}_{\lambda}B_{a}{}^{\kappa}\overset{\circ}{\Gamma}{}^{a}_{ji}. \tag{4.38}$$

We can further require that the affine connexion $\Gamma^{\kappa}_{\mu\lambda}$ is symmetric whenever the given distribution is integrable.

*The distributions B and C are both parallel with respect to the affine connexion* $\Gamma^{\kappa}_{\mu\lambda}=\overset{\circ}{\Gamma}{}^{\kappa}_{\mu\lambda}+T_{\mu\lambda}{}^{\kappa}$ *and the affine connexion is symmetric whenever the distributions B and C are both integrable if and only if the tensor* $T_{\mu\lambda}{}^{\kappa}$ *satisfies*

$$T_{cb}{}^{h}=-\overset{\circ}{\Gamma}{}^{h}_{cb},\ \ T_{jb}{}^{h}=T_{bj}{}^{h}=-\overset{\circ}{\Gamma}{}^{h}_{jb},\ \ T_{ci}{}^{a}=T_{ic}{}^{a}=-\overset{\circ}{\Gamma}{}^{a}_{ci},\ \ T_{ji}{}^{a}=-\overset{\circ}{\Gamma}{}^{a}_{ji}, \tag{4.39}$$

*all the other* $T_{\gamma\beta}{}^{\alpha}$ *being symmetric in* $\gamma$ *and* $\beta$.

The simplest $T_{\mu\lambda}{}^{\kappa}$ satisfying these conditions is given by

$$\begin{aligned}T_{\mu\lambda}{}^{\kappa}=&-B^{c}{}_{\mu}B^{b}{}_{\lambda}C_{h}{}^{\kappa}\overset{\circ}{\Gamma}{}^{h}_{cb}-C^{j}{}_{\mu}B^{b}{}_{\lambda}C_{h}{}^{\kappa}\overset{\circ}{\Gamma}{}^{h}_{jb}-B^{b}{}_{\mu}C^{j}{}_{\lambda}C_{h}{}^{\kappa}\overset{\circ}{\Gamma}{}^{h}_{jb}\\&-B^{c}{}_{\mu}C^{i}{}_{\lambda}B_{a}{}^{\kappa}\overset{\circ}{\Gamma}{}^{a}_{ci}-C^{i}{}_{\mu}B^{c}{}_{\lambda}B_{a}{}^{\kappa}\overset{\circ}{\Gamma}{}^{a}_{ci}-C^{j}{}_{\mu}C^{i}{}_{\lambda}B_{a}{}^{\kappa}\overset{\circ}{\Gamma}{}^{a}_{ji}.\end{aligned} \tag{4.40}$$

Thus we have

THEOREM 13. *For any complementary distributions given globally, there exists always a global affine connexion with respect to which both of the given distributions are parallel and which is symmetric whenever both of the distributions are integrable.*

## §5. Affine connexions in terms of the structure tensor.

In this Section we shall try to express various affine connexions obtained in Section 4 in terms of the structure tensor $F_{\lambda}{}^{\kappa}$ of the almost product space $M$.

( i ) *Affine connexions with respect to which the distribution B is flat.*

From (4.7) we have

$$\begin{aligned}T_{\mu\lambda}{}^{\kappa}&=-B^{c}{}_{\mu}B^{b}{}_{\lambda}C_{h}{}^{\kappa}\overset{\circ}{\Gamma}{}^{h}_{cb}\\&=-B^{c}{}_{\mu}B^{b}{}_{\lambda}C_{h}{}^{\kappa}(\partial_{c}B_{b}{}^{\rho}+B_{c}{}^{\tau}B_{b}{}^{\sigma}\overset{\circ}{\Gamma}{}^{\rho}_{\tau\sigma})C^{h}{}_{\rho},\end{aligned}$$

from which

$$T_{\mu\lambda}{}^{\kappa}=B^{\tau}_{\mu}B^{\sigma}_{\lambda}(\overset{\circ}{\nabla}_{\tau}C^{\kappa}_{\sigma}). \tag{5.1}$$

Substituting (1.9) into (5.1) we find

$$T_{\mu\lambda}{}^{\kappa} = -\frac{1}{8}[\overset{\circ}{\nabla}_{\mu}F_{\lambda}{}^{\kappa} + F_{\mu}{}^{\sigma}(\overset{\circ}{\nabla}_{\rho}F_{\lambda}{}^{\kappa}) + F_{\lambda}{}^{\rho}(\overset{\circ}{\nabla}_{\mu}F_{\rho}{}^{\kappa}) + F_{\mu}{}^{\tau}F_{\mu}{}^{\sigma}(\overset{\circ}{\nabla}_{\tau}F_{\sigma}{}^{\kappa})]. \tag{5.2}$$

If we put

$$M_{\mu\lambda}{}^{\kappa} = F_{\mu}{}^{\rho}(\overset{\circ}{\nabla}_{\rho}F_{\lambda}{}^{\kappa}) + F_{\lambda}{}^{\rho}(\overset{\circ}{\nabla}_{\mu}F_{\rho}{}^{\kappa}), \tag{5.3}$$

then equation (5.2) can be written as

$$T_{\mu\lambda}{}^{\kappa} = -\frac{1}{8}[M_{\mu\lambda}{}^{\kappa} - M_{\mu\lambda}{}^{\rho}F_{\rho}{}^{\kappa}]. \tag{5.4}$$

On the other hand it is easily verified that

$$M_{\mu\lambda}{}^{\kappa} - M_{\lambda\mu}{}^{\kappa} = N_{\mu\lambda}{}^{\kappa} \tag{5.5}$$

and consequently, from (5.4) we have

$$T_{\mu\lambda}{}^{\kappa} - T_{\lambda\mu}{}^{\kappa} = -\frac{1}{8}(N_{\mu\lambda}{}^{\kappa} - N_{\mu\lambda}{}^{\rho}F_{\rho}{}^{\kappa}). \tag{5.6}$$

Thus we have

THEOREM 14. *For any distribution B given globally in M, the affine connexion with respect to which the given distribution is flat and which is symmetric whenever the distribution is integrable is given by*

$$\Gamma^{\kappa}_{\mu\lambda} = \overset{\circ}{\Gamma}{}^{\kappa}_{\mu\lambda} - \frac{1}{8}[M_{\mu\lambda}{}^{\kappa} - M_{\mu\lambda}{}^{\rho}F_{\rho}{}^{\kappa}], \tag{5.7}$$

*$M_{\mu\lambda}{}^{\kappa}$ being given by* (5.3) *and $\overset{\circ}{\Gamma}{}^{\kappa}_{\mu\lambda}$ being an arbitrary symmetric affine connexion defined globally in M.*

(ii) *Affine connexions with respect to which the distribution B is geodesic.*

From (4.9) we have

$$T_{\mu\lambda}{}^{\kappa} = B^{\tau}_{(\mu}B^{\sigma}_{\lambda)}(\overset{\circ}{\nabla}_{\tau}C^{\kappa}_{\sigma}). \tag{5.8}$$

Substituting (1.9) into (5.8) we find

$$T_{\mu\lambda}{}^{\kappa} = -\frac{1}{8}[M_{(\mu\lambda)}{}^{\kappa} - M_{(\mu\lambda)}{}^{\rho}F_{\rho}{}^{\kappa}]. \tag{5.9}$$

Thus we have

THEOREM 15. *For any distribution B given globally in M, an affine connexion with respect to which the given distribution is geodesic is given by*

$$\Gamma^{\kappa}_{\mu\lambda} = \overset{\circ}{\Gamma}{}^{\kappa}_{\mu\lambda} - \frac{1}{8}[M_{(\mu\lambda)}{}^{\kappa} - M_{(\mu\lambda)}{}^{\rho}F_{\rho}{}^{\kappa}]. \tag{5.10}$$

(iii) *Affine connexions with respect to which the distribution B is parallel along C.*

From (4.12) we have

$$T_{\mu\lambda}{}^{\kappa} = C^{\tau}_{\mu}B^{\sigma}_{\lambda}(\overset{\circ}{\nabla}_{\tau}C^{\kappa}_{\sigma}). \tag{5.11}$$

Substituting (1.9) into (5.11) we find

$$T_{\mu\lambda}{}^{\kappa} = -\frac{1}{8}[\overset{\circ}{\nabla}_{\mu}F_{\lambda}{}^{\kappa} - F_{\mu}{}^{\rho}(\overset{\circ}{\nabla}_{\rho}F_{\lambda}{}^{\kappa}) + F_{\lambda}{}^{\rho}(\overset{\circ}{\nabla}_{\mu}F_{\rho}{}^{\kappa}) - F_{\mu}{}^{\tau}F_{\lambda}{}^{\sigma}(\overset{\circ}{\nabla}_{\tau}F_{\sigma}{}^{\kappa})]. \tag{5.12}$$

Thus we have

THEOREM 16. *For any complementary distributions B and C given globally in M, an affine connexion with respect to which B is parallel along C is given by*

$$\Gamma^{\kappa}_{\mu\lambda} = \overset{\circ}{\Gamma}{}^{\kappa}_{\mu\lambda} - \frac{1}{8}[\overset{\circ}{\nabla}_{\mu}F_{\lambda}{}^{\kappa} - F_{\mu}{}^{\rho}(\overset{\circ}{\nabla}_{\rho}F_{\lambda}{}^{\kappa}) + F_{\lambda}{}^{\rho}(\overset{\circ}{\nabla}_{\mu}F_{\rho}{}^{\kappa}) - F_{\mu}{}^{\tau}F_{\lambda}{}^{\sigma}(\overset{\circ}{\nabla}_{\tau}F_{\sigma}{}^{\kappa})]. \tag{5.13}$$

(iv) *Affine connexions with respect to which the distribution C is parallel along B.*

Interchanging $B$ and $C$ or changing the sign of $F_{\lambda}{}^{\kappa}$ in the above theorem, we can see that *for any complementary distributions B and C given globally in M, an affine connexion with respect to which C is parallel along B is given by*

$$\Gamma^{\kappa}_{\mu\lambda} = \overset{\circ}{\Gamma}{}^{\kappa}_{\mu\lambda} + \frac{1}{8}[\overset{\circ}{\nabla}_{\mu}F_{\lambda}{}^{\kappa} + F_{\mu}{}^{\rho}(\overset{\circ}{\nabla}_{\rho}F_{\lambda}{}^{\kappa}) - F_{\lambda}{}^{\rho}(\overset{\circ}{\nabla}_{\mu}F_{\rho}{}^{\kappa}) - F_{\mu}{}^{\tau}F_{\lambda}{}^{\sigma}(\overset{\circ}{\nabla}_{\tau}F_{\sigma}{}^{\kappa})]. \tag{5.14}$$

(v) *Affine connexions with respect to which the distribution C is flat.*

Interchanging $B$ and $C$ or changing the sign of $F_{\lambda}{}^{\kappa}$ in Theorem 14 we can see that *for any distribution C given globally in M, an affine connexion with respect to which the given distribution is flat and which is symmetric whenever the distribution C is integrabele is given by*

$$\Gamma^{\kappa}_{\mu\lambda} = \overset{\circ}{\Gamma}{}^{\kappa}_{\mu\lambda} - \frac{1}{8}(M_{\mu\lambda}{}^{\kappa} + M_{\mu\lambda}{}^{\rho}F_{\rho}{}^{\kappa}). \tag{5.15}$$

(vi) *Affine connexions with respect to which the distribution C is geodesic.*

Interchanging $B$ and $C$ or changing the sign of $F_{\lambda}{}^{\kappa}$ in Theorem 15 we can see the *for any distribution C given globally in M, an affine connexion with respect to which the given distribution is geodesic is given by*

$$\Gamma^{\kappa}_{\mu\lambda} = \overset{\circ}{\Gamma}{}^{\kappa}_{\mu\lambda} - \frac{1}{8}(M_{(\mu\lambda)}{}^{\kappa} + M_{(\mu\lambda)}{}^{\rho}F_{\rho}{}^{\kappa}). \tag{5.16}$$

Now from (4.20) we find

$$T_{\mu\lambda}{}^{\kappa} = -\frac{1}{4}M_{\mu\lambda}{}^{\kappa} = -\frac{1}{4}[F_{\mu}{}^{\rho}(\overset{\circ}{\nabla}_{\rho}F_{\lambda}{}^{\kappa}) + F_{\lambda}{}^{\rho}(\overset{\circ}{\nabla}_{\mu}F_{\rho}{}^{\kappa})], \tag{5.17}$$

form which

$$T_{\mu\lambda}{}^{\kappa} - T_{\lambda\mu}{}^{\kappa} = -\frac{1}{4}N_{\mu\lambda}{}^{\kappa}. \tag{5.18}$$

Thus we have

THEOREM 17. *For any complementary distributions B and C given globally, an affine connexion with respect to which both of the distributions are flat and which is symmetric whenever both of the distributions are integrable is given by*

$$\Gamma^{\kappa}_{\mu\lambda} = \overset{\circ}{\Gamma}{}^{\kappa}_{\mu\lambda} - \frac{1}{4}[F_{\mu}{}^{\rho}(\overset{\circ}{\nabla}_{\rho}F_{\lambda}{}^{\kappa}) + F_{\lambda}{}^{\rho}(\overset{\circ}{\nabla}_{\mu}F_{\rho}{}^{\kappa})]. \tag{5.19}$$

Also from (4.22) we find

$$T_{\mu\lambda}{}^{\kappa} = \frac{1}{4}M_{(\mu\lambda)}{}^{\kappa}. \tag{5.20}$$

Thus we have

THEOREM 18. *For any complementary distributions B and C given globally in M, a symmetric affine connexion with respect to which both of the distaibutions are geodesic is given by*

$$\Gamma^{\kappa}_{\mu\lambda} = \overset{\circ}{\Gamma}{}^{\kappa}_{\mu\lambda} + \frac{1}{4}M_{(\mu\lambda)}{}^{\kappa}. \tag{5.21}$$

Also from (4.24), we find

$$\begin{aligned} T_{\mu\lambda}{}^{\kappa} = &-\frac{1}{4}M_{(\mu\lambda)}{}^{\kappa} + \frac{1}{4}[F_{\mu}{}^{\rho}(\overset{\circ}{\nabla}_{\rho}F_{\lambda}{}^{\kappa}) - F_{\lambda}{}^{\rho}(\overset{\circ}{\nabla}_{\mu}F_{\rho}{}^{\kappa})] \\ &+ \frac{1}{4}[F_{\lambda}{}^{\rho}(\overset{\circ}{\nabla}_{\rho}F_{\mu}{}^{\kappa}) - F_{\mu}{}^{\rho}(\overset{\circ}{\nabla}_{\lambda}F_{\rho}{}^{\kappa})]. \end{aligned} \tag{5.22}$$

Thus we have

THEOREM 19. *For any complementary distributions B and C given globally a symmetric affine connexion with respect to which both of the distributions are geodesic and one of the distributions is parallel along the other is given by*

$$\begin{aligned} \Gamma^{\kappa}_{\mu\lambda} = \overset{\circ}{\Gamma}{}^{\kappa}_{\mu\lambda} &- \frac{1}{4}M_{(\mu\lambda)}{}^{\kappa} + \frac{1}{4}[F_{\mu}{}^{\rho}(\overset{\circ}{\nabla}_{\rho}F_{\lambda}{}^{\kappa}) - F_{\lambda}{}^{\rho}(\overset{\circ}{\nabla}_{\mu}F_{\rho}{}^{\kappa})] \\ &+ \frac{1}{4}[F_{\lambda}{}^{\rho}(\overset{\circ}{\nabla}_{\rho}F_{\mu}{}^{\kappa}) - F_{\mu}{}^{\rho}(\overset{\circ}{\nabla}_{\lambda}F_{\rho}{}^{\kappa})]. \end{aligned} \tag{5.23}$$

(vii) *Affine connexions with respect to which the distribution B is parallel.*

From (4.26) we have

$$T_{\mu\lambda}{}^{\kappa} = B^{\rho}_{\lambda}(\overset{\circ}{\nabla}_{\mu}C^{\kappa}_{\rho}). \tag{5.24}$$

Substituting (1.9) into (5.24) we find

$$T_{\mu\lambda}{}^{\kappa} = -\frac{1}{4}[(\overset{\circ}{\nabla}_{\mu}F_{\lambda}{}^{\kappa}) - (\overset{\circ}{\nabla}_{\mu}F_{\lambda}{}^{\rho})F_{\rho}{}^{\kappa}]. \tag{5.25}$$

Thus we have

THEOREM 20. *For any distribution B given globally in M, a global affine connexion with respect to which the distribution B is parallel is given by*

$$\Gamma^{\kappa}_{\mu\lambda}=\overset{\circ}{\Gamma}{}^{\kappa}_{\mu\lambda}-\frac{1}{4}[(\overset{\circ}{\nabla}_{\mu}F_{\lambda}{}^{\kappa})-(\overset{\circ}{\nabla}_{\mu}F_{\lambda}{}^{\rho})F_{\rho}{}^{\kappa}]. \tag{5.26}$$

From (4.28) we have

$$\begin{aligned} T_{\mu\lambda}{}^{\kappa}=&-\frac{1}{4}[(\overset{\circ}{\nabla}_{\mu}F_{\lambda}{}^{\kappa})+(\overset{\circ}{\nabla}_{\lambda}F_{\mu}{}^{\kappa})+F_{\lambda}{}^{\rho}(\overset{\circ}{\nabla}_{\mu}F_{\rho}{}^{\kappa})+F_{\mu}{}^{\rho}(\overset{\circ}{\nabla}_{\lambda}F_{\rho}{}^{\kappa})]\\ &+\frac{1}{8}[(\overset{\circ}{\nabla}_{\lambda}F_{\mu}{}^{\kappa})+F_{\mu}{}^{\rho}(\overset{\circ}{\nabla}_{\lambda}F_{\rho}{}^{\kappa})+F_{\lambda}{}^{\rho}(\overset{\circ}{\nabla}_{\rho}F_{\mu}{}^{\kappa})+F_{\mu}{}^{\tau}F_{\lambda}{}^{\sigma}(\overset{\circ}{\nabla}_{\sigma}F_{\tau}{}^{\kappa})], \end{aligned} \tag{5.27}$$

from which

$$T_{\mu\lambda}{}^{\kappa}-T_{\lambda\mu}{}^{\kappa}=\frac{1}{8}(N_{\mu\lambda}{}^{\kappa}-N_{\mu\lambda}{}^{\rho}F_{\rho}{}^{\kappa}). \tag{5.28}$$

Thus we have

THEOREM 21. *For any distribution B given globally in M, a global affine connexion with respect to which the given distribution is parallel and which is symmetric whenever the distribution is integrable is given by*

$$\begin{aligned} \Gamma^{\kappa}_{\mu\lambda}=\overset{\circ}{\Gamma}{}^{\kappa}_{\mu\lambda}&-\frac{1}{4}[(\overset{\circ}{\nabla}_{\mu}F_{\lambda}{}^{\kappa})+(\overset{\circ}{\nabla}_{\lambda}F_{\mu}{}^{\kappa})+F_{\lambda}{}^{\rho}(\overset{\circ}{\nabla}_{\mu}F_{\rho}{}^{\kappa})+F_{\mu}{}^{\rho}(\overset{\circ}{\nabla}_{\lambda}F_{\rho}{}^{\kappa})]\\ &+\frac{1}{8}[(\overset{\circ}{\nabla}_{\lambda}F_{\mu}{}^{\kappa})+F_{\mu}{}^{\rho}(\overset{\circ}{\nabla}_{\lambda}F_{\rho}{}^{\kappa})+F_{\lambda}{}^{\rho}(\overset{\circ}{\nabla}_{\rho}F_{\mu}{}^{\kappa})+F_{\mu}{}^{\tau}F_{\lambda}{}^{\sigma}(\overset{\circ}{\nabla}_{\sigma}F_{\tau}{}^{\kappa})]. \end{aligned} \tag{5.29}$$

(viii) *Affine connexions with respect to which the distribution C is parallel.*

Interchanging $B$ and $C$ or changing the sign of $F_{\lambda}{}^{\kappa}$ we can see from the above theorem that *for the distribution C given globally in M, a global affine connexion with respect to which C is integrable is given by*

$$\begin{aligned} \Gamma^{\kappa}_{\mu\lambda}=\overset{\circ}{\Gamma}{}^{\kappa}_{\mu\lambda}&+\frac{1}{4}[(\overset{\circ}{\nabla}_{\mu}F_{\lambda}{}^{\kappa})+(\overset{\circ}{\nabla}_{\lambda}F_{\mu}{}^{\kappa})-F_{\lambda}{}^{\rho}(\overset{\circ}{\nabla}_{\mu}F_{\rho}{}^{\kappa})-F_{\mu}{}^{\rho}(\nabla_{\lambda}F_{\rho}{}^{\kappa})]\\ &-\frac{1}{8}[(\overset{\circ}{\nabla}_{\lambda}F_{\mu}{}^{\kappa})-F_{\mu}{}^{\rho}(\overset{\circ}{\nabla}_{\lambda}F_{\rho}{}^{\kappa})-F_{\lambda}{}^{\rho}(\overset{\circ}{\nabla}_{\rho}F_{\mu}{}^{\kappa})+F_{\mu}{}^{\tau}F_{\lambda}{}^{\sigma}(\overset{\circ}{\nabla}_{\sigma}F_{\tau}{}^{\kappa})]. \end{aligned} \tag{5.30}$$

From (4.34) we find

$$\begin{aligned} T_{\mu\lambda}{}^{\kappa}=&-\frac{1}{4}[(\overset{\circ}{\nabla}_{\mu}F_{\lambda}{}^{\kappa})-(\overset{\circ}{\nabla}_{\mu}F_{\lambda}{}^{\rho})F_{\rho}{}^{\kappa}]\\ &-\frac{1}{8}[M_{(\mu\lambda)}{}^{\kappa}+M_{(\mu\lambda)}{}^{\rho}F_{\rho}{}^{\kappa}]\\ &+\frac{1}{8}[\overset{\circ}{\nabla}_{\mu}F_{\lambda}{}^{\kappa}+F_{\mu}{}^{\rho}(\overset{\circ}{\nabla}_{\rho}F_{\lambda}{}^{\kappa})-F_{\lambda}{}^{\rho}(\overset{\circ}{\nabla}_{\mu}F_{\rho}{}^{\kappa})-F_{\mu}{}^{\tau}F_{\lambda}{}^{\sigma}(\overset{\circ}{\nabla}_{\tau}F_{\sigma}{}^{\kappa})]. \end{aligned} \tag{5.31}$$

Thus we have

THEOREM 22. *For any complementary distributions B and C the affine connexion with respect to which B is parallel, C is geodesic and C is parallel along B is given by*

$$\begin{aligned}\Gamma^{\kappa}_{\mu\lambda} = \overset{\circ}{\Gamma}{}^{\kappa}_{\mu\lambda} &- \frac{1}{4}[(\overset{\circ}{\nabla}_\mu F_\lambda{}^\kappa) - (\overset{\circ}{\nabla}_\mu F_\lambda{}^\rho)F_\rho{}^\kappa] \\ &- \frac{1}{8}[M_{(\mu\lambda)}{}^\kappa + M_{(\mu\lambda)}{}^\rho F_\rho{}^\kappa] \\ &+ \frac{1}{8}[\overset{\circ}{\nabla}_\mu F_\lambda{}^\kappa + F_\mu{}^\rho(\overset{\circ}{\nabla}_\rho F_\lambda{}^\kappa) - F_\lambda{}^\rho(\overset{\circ}{\nabla}_\mu F_\rho{}^\kappa) - F_\mu{}^\tau F_\lambda{}^\rho(\overset{\circ}{\nabla}_\tau F_\sigma{}^\kappa)].\end{aligned} \tag{5.32}$$

From (4.36) we have

$$\begin{aligned}T_{\mu\lambda}{}^\kappa = -\frac{1}{8}[M_{(\mu\lambda)}{}^\kappa + M_{(\mu\lambda)}{}^\rho F_\rho{}^\kappa] &- \frac{1}{8}[M_{\mu\lambda}{}^\kappa - M_{\mu\lambda}{}^\sigma F_\sigma{}^\kappa] \\ &+ \frac{1}{2}[F_{(\mu}{}^\rho \nabla_{|\rho|} F_{\lambda)}{}^\kappa + \nabla_{(\mu} F_{\lambda)}{}^\rho F_\rho{}^\kappa],\end{aligned} \tag{5.33}$$

from which

$$T_{\mu\lambda}{}^\kappa - T_{\lambda\mu}{}^\kappa = -\frac{1}{8}(N_{\mu\lambda}{}^\kappa - N_{\mu\lambda}{}^\rho F_\rho{}^\kappa). \tag{5.34}$$

Thus we have

THEOREM 23. *For the complementary distributions B and C given globally in M, the affine connexion with respect to which B is parallel, C is geodesic and C ia parallel along B and which is smmetric whenever B is integrable is given by*

$$\begin{aligned}\Gamma^{\kappa}_{\mu\lambda} = \overset{\circ}{\Gamma}{}^{\kappa}_{\mu\lambda} - \frac{1}{8}[M_{(\mu\lambda)}{}^\kappa + M_{(\mu\lambda)}{}^\rho F_\rho{}^\kappa] &- \frac{1}{8}[M_{\mu\lambda}{}^\kappa - M_{\mu\lambda}{}^\rho F_\rho{}^\kappa] \\ &+ \frac{1}{2}[F_{(\mu}{}^\rho \nabla_{|\rho|} F_{\lambda)}{}^\kappa + \nabla_{(\mu} F_{\lambda)}{}^\rho F_\rho{}^\kappa].\end{aligned} \tag{5.35}$$

From (4.38) we have

$$T_{\mu\lambda}{}^\kappa = +\frac{1}{2}(\overset{\circ}{\nabla}_\mu F_\lambda{}^\rho)F_\rho{}^\kappa. \tag{5.36}$$

Thus we have

THEOREM 24. *For the distributions B and C given globally in M, an affine connexion with respect to which B and C are both parallel is given by*

$$\Gamma^{\kappa}_{\mu\lambda} = \overset{\circ}{\Gamma}{}^{\kappa}_{\mu\lambda} + \frac{1}{2}(\overset{\circ}{\nabla}_\mu F_\lambda{}^\rho)F_\rho{}^\kappa. \tag{5.37}$$

From (4.40) we have

$$T_{\mu\lambda}{}^\kappa = -\frac{1}{4}[F_\mu{}^\rho(\overset{\circ}{\nabla}_\lambda F_\rho{}^\kappa) - F_\lambda{}^\rho(\overset{\circ}{\nabla}_\rho F_\mu{}^\kappa)] + (\overset{\circ}{\nabla}_\mu F_\lambda{}^\rho)F_\rho{}^\kappa], \tag{5.38}$$

from which

$$T_{\mu\lambda}{}^{\kappa} - T_{\lambda\mu}{}^{\kappa} = -\frac{1}{4} N_{\mu\lambda}{}^{\kappa}. \tag{5.39}$$

Thus we have

THEOREM 25. *For the distributions B and C given globally in M, an affine connexion with respect to which B and C are both parallel and which is symmetric whenever both of B and C are integrable is given by*

$$\Gamma^{\kappa}_{\mu\lambda} = \mathring{\Gamma}^{\kappa}_{\mu\lambda} - \frac{1}{4}[F_{\mu}{}^{\rho}(\mathring{\nabla}_{\lambda}F_{\rho}{}^{\kappa}) - F_{\lambda}{}^{\rho}(\mathring{\nabla}_{\rho}F_{\mu}{}^{\kappa})] + \frac{1}{2}(\mathring{\nabla}_{\mu}F_{\lambda}{}^{\rho})F_{\rho}{}^{\kappa}. \tag{5.40}$$

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