A NOTE ON THE CROSS-NORM OF THE DIRECT PRODUCT OF OPERATOR ALGEBRA

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In [2], T. Turumaru introduced the idea of the direct product of operator algebras in the following sense. Let $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ be C^*-algebras and $\boldsymbol{M}_1 \odot \boldsymbol{M}_2$ their algebraic direct product. Then we define a cross-norm α on $\boldsymbol{M}_1 \odot \boldsymbol{M}_2$ as follows:

$$\alpha\Big(\sum_{i=1}^{n} x_i \otimes y_i\Big) = \sup \left\{ \frac{<\Big(\sum_{j=1}^{m} a_j^* \otimes b_j^*\Big)\Big(\sum_{i=1}^{n} x_i^* \otimes y_i^*\Big)\Big(\sum_{i=1}^{n} x_i \otimes y_i\Big)\Big(\sum_{j=1}^{m} a_j \otimes b_j\Big),\ \varphi \otimes \psi >}{<\Big(\sum_{j=1}^{m} a_j^* \otimes b_j^*\Big)\Big(\sum_{j=1}^{m} a_j \otimes b_j\Big),\ \varphi \otimes \psi >} \right\}^{1/2}$$

for any element $\sum_{i=1}^{n} x_i \otimes y_i \in \boldsymbol{M}_1 \odot \boldsymbol{M}_2$, where $\sum_{j=1}^{m} a_j \otimes b_j$ runs over the all non-zero elments of $\boldsymbol{M}_1 \odot \boldsymbol{M}_2$ and φ, ψ the all non-zero positive linear functionals on $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ respectively. $\boldsymbol{M}_1 \odot \boldsymbol{M}_2$ becomes an imcomplete C^*-algebra under this cross-norm α. We call the completion of $\boldsymbol{M}_1 \odot \boldsymbol{M}_2$ the C^*-direct product of $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ and denote by $\boldsymbol{M}_1 \widehat{\otimes}_\alpha \boldsymbol{M}_2$.

The purpose of this note is to study the relation between the α-norm defined above and λ-norm or γ-norm in the sense of R. Schatten [1].

Let $\boldsymbol{M}$ be a C^*-algebra with a unit and $\mathfrak{S}$ the state space of $\boldsymbol{M}$, then $\mathfrak{S}$ is the $\sigma(\boldsymbol{M}^*, \boldsymbol{M})$-convex closure of the space of all pure states on $\boldsymbol{M}$. So we call the $\sigma(\boldsymbol{M}^*, \boldsymbol{M})$-closure of the space of all pure states on $\boldsymbol{M}$ the pure state space of $\boldsymbol{M}$.

Proposition 1. *Let $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ be C^*-algebras with units, $\boldsymbol{M}$ their C^*-direct product and Ω_1, Ω_2, and Ω the pure state spaces of $\boldsymbol{M}_1$, $\boldsymbol{M}_2$ and $\boldsymbol{M}$ respectively. Then we have*

$$\Omega = \Omega_1 \times \Omega_2$$

if and only if either $\boldsymbol{M}_1$ or $\boldsymbol{M}_2$ is commutative, where $\Omega_1 \times \Omega_2$ means the cartesian product of Ω_1 and Ω_2.

Proof. Sufficiency. If $\boldsymbol{M}_1$ is commutative, then $\boldsymbol{M}_1 \otimes 1$ is contained in the center $\boldsymbol{Z}$ of $\boldsymbol{M}$. Let τ be a pure state on $\boldsymbol{M}$, then we have $<ax, \tau> = <a, \tau><x, \tau>$ for all $a \in \boldsymbol{Z}$ and $x \in \boldsymbol{M}$. If we define a linear functional σ on $\boldsymbol{M}_1$ such as

Received September 18, 1958.

$$<x, \sigma> = <x \otimes 1, \tau> \qquad \text{for} \quad x \in \boldsymbol{M}_1,$$

then σ is a pure state on $\boldsymbol{M}_1$ by the multiplicativity of τ on $\boldsymbol{Z}$. Next we define a linear functional ρ on $\boldsymbol{M}_2$ such as

$$<y, \rho> = <1 \otimes y, \rho> \qquad \text{for} \quad y \in \boldsymbol{M}_2,$$

then ρ is a state on $\boldsymbol{M}_2$ and $\tau = \sigma \otimes \rho$. Suppose there exist two states ρ', ρ'' on $\boldsymbol{M}_2$ such as $\rho = (1/2)(\rho' + \rho'')$. Put $\tau' = \sigma \otimes \rho'$ and $\tau = \sigma \otimes \rho''$, then $\tau = (1/2)(\tau' + \tau'')$. Since τ is a pure state on $\boldsymbol{M}$, we get $\tau = \tau' = \tau''$. We have

$$<y, \rho> = <1 \otimes y, \tau> = <1 \otimes y, \tau'> = <y, \rho'> = <y, \rho''>,$$

i.e. ρ is a pure state on $\boldsymbol{M}_2$.

On the other hand, for any pure states σ on $\boldsymbol{M}_1$ and ρ on $\boldsymbol{M}_2$, $\tau = \sigma \otimes \rho$ is clearly a pure state on $\boldsymbol{M}$.

Combinning these arguments, the dense part of $\Omega_1 \times \Omega_2$ coincides with the dense part of Ω. Moreover, the mapping $(f, g) \to f \otimes g$ from $\boldsymbol{M}_1^* \times \boldsymbol{M}_2^*$ into $\boldsymbol{M}^*$ is weakly continuous, where the space $\boldsymbol{M}_1^* \times \boldsymbol{M}_2^*$ has the cartesian product topology of $\sigma(\boldsymbol{M}_1^*, \boldsymbol{M}_1)$ and $\sigma(\boldsymbol{M}_2^*, \boldsymbol{M}_2)$-topologies. Hence we get $\Omega = \Omega_1 \times \Omega_2$.

Necessity. If $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ are both non-commutative, then there exist pure states σ and ρ on $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ respectively such that the Hilbert spaces H_σ and H_ρ canonically constructed by σ and ρ respectively are at least two-dimensional. Put $\tau = \sigma \otimes \rho$, then we have $H_\tau = H_\sigma \otimes H_\rho$, where H_τ is the Hilbert space canonically constructed by τ. For canonical representation π_σ, π_ρ and π_τ by σ, ρ and τ, put $\pi_\sigma(\boldsymbol{M}_1) = \boldsymbol{M}_{1\sigma}$, $\pi_\rho(\boldsymbol{M}_2) = \boldsymbol{M}_{2\rho}$ and $\pi_\tau(\boldsymbol{M}) = \boldsymbol{M}_\tau$ then we get $\boldsymbol{M}_\tau = \boldsymbol{M}_{1\sigma} \widehat{\otimes}_\alpha \boldsymbol{M}_{2\rho}$. Now for an arbitrary vector $\zeta \in H_\tau$ with $\|\zeta\| = 1$, we define a state ω_ζ on $\boldsymbol{M}_\tau$ such as $<z, \omega_\zeta> = (z\zeta, \zeta)$ for $z \in \boldsymbol{M}_\tau$. Then ${}^t\pi_\tau(\omega_\zeta)$ is a pure state on $\boldsymbol{M}$. If ζ is not representable by the form $\xi \otimes \eta$ for any $\xi \in H_\sigma$ and $\eta \in H_\rho$, then ω_ζ is not multiplicative with respect to $\boldsymbol{M}_{1\sigma}$ and $\boldsymbol{M}_{2\rho}$. Hence we get ${}^t\pi_2(\omega_\zeta) \notin \Omega_1 \times \Omega_2$. And surely there exists such vector ζ of H_τ by the hypothesis on H_σ and H_ρ. We get $\Omega \not\subset \Omega_1 \times \Omega_2$. This concludes the proof.

Applying this proposition we can prove the following proposition about the cross norm of C^*-direct product of C^*-algebras.

PROPOSITION 2. *Let $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ be two C^*-algebras with units respectively. Then the C^*-norm α of their C^*-direct product $\boldsymbol{M} = \boldsymbol{M}_1 \widehat{\otimes}_\alpha \boldsymbol{M}_2$ coincides with λ-norm if and only if either $\boldsymbol{M}_1$ or $\boldsymbol{M}_2$ is commutative.*

Proof. Sufficiency. We assume the commutativity of $\boldsymbol{M}_1$. Let Ω_1 be the pure state space of $\boldsymbol{M}_1$, then we have $\boldsymbol{M}_1 = C(\Omega_1)$ by the well-known representation theorem of commutative C^*-algebra. For an arbitrary element $\sum_{i=1}^n x_i \otimes y_i \in \boldsymbol{M}_1 \odot \boldsymbol{M}_2$ the $\boldsymbol{M}_2$-valued function $\sum_{i=1}^n x_i(t) y_i$ is continuous on Ω_1, i.e. $\sum_{i=1}^n x_1(\cdot) y_i \in C_{\boldsymbol{M}_2}(\Omega_1)$, where $C_{\boldsymbol{M}_2}(\Omega_1)$ means the space of all $\boldsymbol{M}_2$-valued continuous functions with the uniform norm on Ω_1. And we get

$$\begin{aligned}\left\|\sum_{i=1}^{n} x_i(\cdot) y_i\right\| &= \sup\left\{\left\|\sum_{i=1}^{n} x_i(t) y_i\right\|;\ t\in\Omega_1\right\}\\ &= \sup\left\{\left|<\sum_{i=1}^{n} x_i(t) y_i, \psi>\right|;\ t\in\Omega_1,\ \psi\in M_2^*,\ \|\psi\|\leqq 1\right\}\\ &= \sup\left\{\left|\sum_{i=1}^{n} x_i(t) <y_i, \psi>\right|;\ t\in\Omega_1,\ \psi\in M_2^*,\ \|\psi\|\leqq 1\right\}\\ &= \sup\left\{\left\|\sum_{i=1}^{n} <y_i, \psi> x_i(\cdot)\right\|;\ \psi\in M_2^*,\ \|\psi\|\leqq 1\right\}\\ &= \sup\left\{\left|\sum_{i=1}^{n} <x_i, \varphi><y_i, \psi>\right|;\ \varphi\in M_1^*, \|\varphi\|\leqq 1, \psi\in M_2^*, \|\psi\|\leqq 1\right\}\\ &= \lambda\left(\sum_{i=1}^{n} x_i\otimes y_i\right).\end{aligned}$$

Moreover, if we define a multiplication and a $*$-operation of $C_{M_2}(\Omega_1)$ as follows:

$$(a\cdot b)(t)=a(t)\,b(t),\qquad (a)^*(t)=a(t)^*\qquad \text{for all}\quad a,\ b\in C_{M_2}(\Omega_1)$$

and $t\in\Omega_1$, then $C_{M_2}(\Omega_1)$ becomes a C^*-algebra since we have

$$\begin{aligned}\|a^*a\| &= \sup\{\|(a^*a)(t)\|;\ t\in\Omega_1\}\\ &= \sup\{\|a^*(t)\,a(t)\|;\ t\in\Omega_1\}\\ &= \sup\{\|a(t)\|^2;\ t\in\Omega_1\}\\ &= \sup\{\|a(t)\|;\ t\in\Omega_1\}^2\\ &= \|a\|^2\end{aligned}$$

for all $a\in C_{M_2}(\Omega_1)$.

Considering the natural correspondence of $\sum_{i=1}^{n} x_i\otimes y_i\in M_1\widehat{\otimes}_\lambda M_2$ and $\sum_{i=1}^{n} x_i(\cdot)y_i\in C_{M_2}(\Omega_1)$, $M_1\widehat{\otimes}_\lambda M_2$ becomes a C^*-algebra and the totality of linear functionals of the form $\varphi\otimes\psi$, where φ and ψ are positive linear functionals on M_1 and M_2 respectively, is total on $M_1\widehat{\otimes}_\lambda M_2$. Hence we get

$$\begin{aligned}\lambda\left(\sum_{i=1}^{n} x_i\otimes y_i\right) &= \sup\left\{\frac{<\left(\sum_{j=1}^{m} a_j^*\otimes b_j^*\right)\left(\sum_{i=1}^{n} x_i^*\otimes y_i^*\right)\left(\sum_{i=1}^{n} x_i\otimes y_i\right)\left(\sum_{j=1}^{m} a_j\otimes b_j\right),\ \varphi\otimes\psi>}{<\left(\sum_{j=1}^{m} a_j^*\otimes b_j^*\right)\left(\sum_{j=1}^{m} a_j\otimes b_j\right),\ \varphi\otimes\psi>}\right\}^{1/2}\\ &= \alpha\left(\sum_{i=1}^{n} x_i\otimes y_i\right)\end{aligned}$$

for all $\sum_{i=1}^{n} x_i\otimes y_i\in M_1\odot M_2$, where $\sum_{j=1}^{m} a_j\otimes b_j$ runs over the all non-zero elements of $M_1\odot M_2$ and φ, ψ the all non-zero positive linear functionals on M_1 and M_2 respectively. Therefore we get $\alpha=\lambda$.

Indeed, we have $M_1\widehat{\otimes}_\lambda M_2=C_{M_2}(\Omega_1)$ by the decomposition of identity.

Necessity. We assume $\alpha=\lambda$. Then we have

$$\alpha(u)=\sup\ \{|<u, f>|;\ f\in\textstyle\sum_1\otimes\sum_2\}\qquad\text{for all}\quad u\in M,$$

where $\sum_1$ and $\sum_2$ are unit spheres of $\boldsymbol{M}_1^*$ and $\boldsymbol{M}_2^*$ respectively, and $\sum_1 \otimes \sum_2$ means the totality of $\varphi \otimes \psi$ $(\varphi \in \sum_1, \psi \in \sum_2)$. Hence if $\sum$ is the unit sphere of $\boldsymbol{M}^*$, we get

$$\sum_1 \otimes \sum_2)^{\circ\circ} = \sum,$$

where $(\sum_1 \otimes \sum_2)^{\circ\circ}$ means the bipolar of $\sum_1 \otimes \sum_2$. $\sum_1 \otimes \sum_2$ is $\sigma(\boldsymbol{M}^*, \boldsymbol{M})$-compact so that all extreme points of $\sum$ belong to $\sum_1 \otimes \sum_2$ by Krein-Milman's Theorem. Hence we have $\Omega \subset \sum_1 \otimes \sum_2$, where Ω is the pure state space on $\boldsymbol{M}$. So the pure state on $\boldsymbol{M}$ is multiplicative with respect to $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$. For a pure state τ on $\boldsymbol{M}$, put

$$<x, \sigma> = <x \otimes 1, \tau> \qquad \text{for} \quad x \in \boldsymbol{M}_1,$$
$$<y, \rho> = <1 \otimes y, \rho> \qquad \text{for} \quad y \in \boldsymbol{M}_2,$$

then σ and ρ are pure states on $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ respectively and $\tau = \sigma \otimes \rho$. That is, we get

$$\Omega = \Omega_1 \times \Omega_2,$$

where Ω_2 is the pure state space $\boldsymbol{M}_2$. Therefore either $\boldsymbol{M}_1$ or $\boldsymbol{M}_2$ is commutative by Proposition 1. This concludes the proof.

REFERENCES

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