ON CONFORMAL MAPPING OF A HYPERELLIPTIC RIEMANN SURFACE ONTO ITSELF

BY RYŌHEI TSUJI

1. Let W be a bordered Riemann surface, i.e. a compact subregion of a Riemann surface, the relative boundary of which consists of a finite number of closed analytic curves, and $\mathfrak{G}$ be the group of all conformal mappings of W onto itself. For given integers g $(\geqq 0)$ and k $(\geqq 1)$ we put

$$N(g, k) = \max(\text{ord.}\,\mathfrak{G}).$$

where ord. $\mathfrak{G}$ means the order of $\mathfrak{G}$ and the maximum is taken with respect to all W having the genus g and k boundary components.

Next, on a closed Riemann surface W of genus g, we take k points $p_1, \cdots, p_k$ and consider the group $\mathfrak{G}'$ of all conformal mappings of the region $W - \{p_1, \cdots, p_k\}$ onto itself. For given integers g and k, we put

$$N'(g, k) = \max(\text{ord.}\,\mathfrak{G}'),$$

the maximum being taken with respect to all W of genus g and all sets of k points $p_1, \cdots, p_k \in W$.

The values of $N(0, k)$ and $N'(1, k)$ have been completely determined by Heins [1] and Oikawa [4], respectively. An estimation of $N(g, k)$ with $2g + k - 1 \geqq 2$, $g \geqq 0$ and $k \geqq 1$ has been given by Oikawa [4] as a supplement of Hurwitz's [3]. Oikawa [5] has further proved that

$$N(g, k) = N'(g, k)$$

for any $g \geqq 0$ and $k \geqq 0$ with $2g + k - 1 \geqq 2$.

In the present paper, we introduce the quantities of similar nature

$$N_h(g, k) \quad \text{and} \quad N_h'(g, k)$$

where the suffix h means that Riemann surfaces are restricted to hyperelliptic ones.

By carefully tracing the Oikawa's proof of $N(g, k) = N'(g, k)$, we can conclude $N_h(g, k) = N_h'(g, k)$. Therefore, it is sufficient to investigate $N_h'(g, k)$. For later discussion, we use the symbol $N_h(g, k)$ instead of $N_h'(g, k)$. It is to be noted that a surface with $g = 2$ is always hyperelliptic and hence

$$N_h(2, k) = N(2, k).$$

In the following lines, we shall give a general estimation for $N_h(g, k)$ and then show that its exact value can be obtained in case of $g = 2$.

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2. Let W be a hyperelliptic Riemann surface of genus g ($\geqq 2$) which is defined by the algebraic equation

$$(1) \qquad y^2 = \prod_{\nu=1}^{2g+2} (x - d_\nu), \qquad d_\nu \neq d_\mu \ (\nu \neq \mu).$$

The normalized differentials of the first kind are

$$du_\nu = \frac{x^{\nu-1}}{y} dx \qquad (\nu = 1, \cdots, g).$$

Let $\mathfrak{G}$ be a group of all transformations of W onto itself.

LEMMA 1. *A transformation which appears by a projection of $\mathfrak{T} \in \mathfrak{G}$ to x-plane is an elliptic linear transformation unless it is the identity map.*

Proof. Let u_ν be the value of the integral of du_ν at point $p = (x, y) \in W$, and u_ν' the value of u_ν at the point $\mathfrak{T}(p)$. Evidently du_ν' can be expressed as a linear combination of du_μ ($\mu = 1, \cdots, g$):

$$du_\nu' = \sum a_{\nu\mu} du_\mu \qquad (\nu = 1, \cdots, g).$$

We consider functions on W defined by

$$\frac{du_\nu}{du_1} = \frac{x^{\nu-1}/y\,dx}{1/y\,dx} = x^{\nu-1} \ (\nu = 2, \cdots, g), \qquad \frac{du_2'}{du_1'} = x'$$

where x' is the value of x at the point $\mathfrak{T}(p)$:

$$x' = \frac{du_2'}{du_1'} = \frac{a_{21}du_1 + a_{22}du_2 + \cdots + a_{2g}du_g}{a_{11}du_1 + a_{12}du_2 + \cdots + a_{2g}du_g} = \frac{a_{21} + a_{22}x + \cdots + a_{2g}x^{g-1}}{a_{11} + a_{12}x + \cdots + a_{1g}x^{g-1}}.$$

The function x is the projection of p to x-plane. Therefore, from this equation we see that the $\mathfrak{T}$-images of two points with the same x-value have also the same x-value. The projection of $\mathfrak{T}(p)$ to x-plane is thus a univalent function $T(x)$ on x-plane. Hence, $(2g+2)$ branch-points are merely interchanged among them by $\mathfrak{T}$.

Since, by Hurwitz [2], $\mathfrak{T}(p)$ is a periodic mapping, the projection $T(x)$ is also periodic unless it is the identity map. Hence, $T(x)$ is a linear transformation. Otherwise, a univalent function $T(x)$ cannot map x-plane one to one. In fact, there were $x_1 \neq x_2$ such that $T(x_1) = T(x_2)$. From the periodicity of $T(x)$, there were a number n such that T^n is the identity E. Hence we would have $T^n(x_1) = T^n(x_2) = x_1$, which contradicts $T^n = E$. A linear transformation having a period is an elliptic linear transformation. Thus, Lemma 1 has been proved.

Our problem thus reduces to the investigation of a map $\mathfrak{T}(p)$ having an x-projection of an elliptic transformation or the identity map. If $T(x)$ is the identity map, $\mathfrak{T}(p)$ is evidently the involutory map which interchanges two sheets unless it is the identity map. Since an elliptic linear transformation $T(x)$ has two fixed points, $\mathfrak{T}(p)$ has at most four fixed points unless $T(x)$ is the identity map. This was proved by Hurwitz [3] in another way.

LEMMA 2. *A finite group G of linear transformations of x-plane which has an invariant point-set $(d_1, \cdots, d_{2g+2})$ can be extended to a finite transformation group $\mathfrak{G}$ of W defined by* (1) *in such a way that the x-projection of $\mathfrak{G}$ is G. By a suitable extension, the order of $\mathfrak{G}$ can be made twice that of G.*

Proof. Let T belong to G, and the order of T be n. Without loss of generality, we can assume that the fixed points of T lie at $x=0$ and $x=\infty$. Then T is of the form

$$T(x) = e^{2\pi i/n}x. \tag{2}$$

We divide two cases: (i) one of d_ν is zero, (ii) all d_ν are not zero.

(i) In this case, (1) has the form:

$$y^2 = \prod_{\nu=1}^{s}(x-d_\nu) \tag{3}$$

where $s=2g+2$ if $x=\infty$ does not coincides with any d, and $s=2g+1$ if $x=\infty$ coincides with a d_ν. The invariance of the set $(d_1, \cdots, d_{2g+2})$ under T implies that s is a multiple of n. Substituting (2) into (3), we get two possibilities, i.e.

$$\mathfrak{T}_1\begin{pmatrix} x \to e^{2\pi i/n}x \\ y \to y \end{pmatrix} \quad \text{and} \quad \mathfrak{T}_2\begin{pmatrix} x \to e^{2\pi i/n}x \\ x \to -y \end{pmatrix}. \tag{4}$$

Of course, these are admissible birational transformations. So, we have three possibilities of extension. The first consists of $\mathfrak{T}_1$ alone, the second of $\mathfrak{T}_2$ alone, and the third of $\mathfrak{T}_1$ and $\mathfrak{T}_2$. If $\mathfrak{T}_1$ and $\mathfrak{T}_2$ occur simultaneously, the extension $\mathfrak{T}$ of T admits the interchange of two sheets.

(ii) In this case, (1) has the form:

$$y^2 = x\prod_{\nu=1}^{s}(x-d_\nu) \tag{5}$$

where $s=2g+1$ if $x=\infty$ does not coincide with any d_ν and $s=2g$ if $x=\infty$ coincides with a d_ν. As before, we can conclude that s is a multiple of n. Substituting (2) into (5), we get two possibilities, i.e.

$$\mathfrak{T}_3\begin{pmatrix} x \to e^{2\pi i/n}x \\ y \to e^{\pi i/n}y \end{pmatrix} \quad \text{and} \quad \mathfrak{T}_4\begin{pmatrix} x \to e^{2\pi i/n}x \\ y \to -e^{\pi i/n}y \end{pmatrix}. \tag{6}$$

If n is an even number, a cyclic group generated by $\mathfrak{T}_3$ coincides with that of $\mathfrak{T}_4$. But this is not true if n is an odd number. $(\mathfrak{T}_3)^n$ is then an involutory mapping which interchanges two sheets. And the group generated by $\mathfrak{T}_4$ does not contain it.

From these considerations, we see that T can be extended to $\mathfrak{T}$ in such a way that the order of $\mathfrak{T}$ is twice that of T, and that the extension which does not admit the interchange of sheets does not occur if $x=0$ is a branch-point and further n is an even number.

The extension of G to $\mathfrak{G}$ can be performed by extending all $T \in G$ in such a way that they may admit the interchange of sheets. The number of fundamental regions of W by $\mathfrak{G}$ is evidently twice that of x-plane by $\mathfrak{G}$. Thus Lemma 2 has been proved.

3. THEOREM 1. $N_h(g, k) \leqq 8(g+1)$ *for* $g \neq 2, 3, 5, 9$.

$$N_h(2, k),\ N_h(3, k) \leqq 48, \qquad N_h(5, k),\ N_h(9, k) \leqq 120.$$

The equality sign in these estimation is attained if and only if

$$\begin{array}{ll} k = 2(g+1)m + \{0 \text{ or } 4\} & \textit{for} \quad g \neq 2, 3, 5, 9, 14; \\ k = 24m + \{0, 6, 16 \text{ or } 22\} & \textit{for} \quad g = 2; \\ k = 12m + \{0 \text{ or } 8\} & \textit{for} \quad g = 3; \\ k = 60m + \{0, 12, 40 \text{ or } 52\} & \textit{for} \quad g = 5; \\ k = 60m + \{0, 20, 24 \text{ or } 44\} & \textit{for} \quad g = 9; \\ k = 120m + \{0, 4, 24, 30, 40, 54, 60, 64, 70 \text{ or } 94\} & \textit{for} \quad g = 14. \end{array}$$

Proof. Since a closed Riemann surface of genus $g \leqq 2$ admits only a finite number of conformal mappings onto itself, x-projection of $\mathfrak{G}$ is also a finite group. Let G be a finite group of x-plane and N be its order. Fixed points of $T \in G$ are divided to h congruent sets. Let n_i $(i = 1, \cdots, h)$ be the number of congruent points which belong to the i-th set. The order of T which have fixed points belong to the i-th set is N/n_i. In order that set of $(2g+2)$ branch-points are invariant nnder G the relation

$$2g + 2 = mN + \sum_{i=1}^{h} a_i n_i \tag{7}$$

must be satisfied, where m is the number of branch-points in a fundamental region of G and a_i the number of branch-points lying on a fixed point corresponding to the i-th set. a_i is eqaul to 0 or 1. By Lemma 2, if there are some numbers m, a_i satisfying (1), G can be extended to some $\mathfrak{G}$ of W with the order $2N$ or N.

We now consider the invariant point set $(p_1, \cdots, p_k) = \{p_i\}$. In order that G may be extend to some $\mathfrak{G}$, in other words, G may be x-projection of some $\mathfrak{G}$, k must satisfy certain conditions. We distinguish two cases:

(i) Let $\mathfrak{G}$ admit an involutory map which interchange two sheets. Then the p_i which do not lie on branch-points are pairly distributed on W, i.e. two points having the same x-projection are either belong to $\{p_i\}$ simultaneously or not. Then k must satisfy the relation

$$\begin{array}{ll} k = m'N + \sum n_i(2 - a_i)\delta_i & \text{for} \quad m > 0, \\ k = 2m'N + \sum n_i(2 - a_i)\delta_i & \text{for} \quad m = 0, \end{array} \tag{8}$$

where m' or $2m'$ is the number of points p_i in a two sheeted fundamental region of G, and $(2 - a_i)\delta_i$ the number of points p_i lying on one fixed point corresponding to the i-th set. δ_i is either 0 or 1. If there is a set of numbers g, k, a_i, m' and δ_i satisfying (7) and (8), G can be extended to $\mathfrak{G}$ with the order $2N$. We call this extension a double extension.

(ii) Let $\mathfrak{G}$ do not admit the involutory map which interchanges two sheets. By previous consideration, this case occurs only when there is no branch-point lying on a fixed point of an elliptic transformation of an even order. A fundamental region of $\mathfrak{G}$ is two-sheeted one of G. Except the fixed points, two distinct points having the same x-value are not congruent with respect to $\mathfrak{G}$. Hence the order of $\mathfrak{G}$ is N. Then k must satisfy relation

$$k = m'N + \sum n_i \alpha_i \tag{9}$$

where m' has the same meaning as in (1), and α_i is the number of p_i lying on a fixed point. α_i is equal to 0, 1 or 2. Though some numbers g, k, m, a, m' and α_i are suitably chosen within the conditions (7) and (9), we must not conclude that G can be extended to $\mathfrak{G}$. As shown by the following example, α_i is restricted in some cases. Let G be a cyclic group generated by $T(x) = e^{\pi i/3}x$. Then $n_1 = 1$ and $n_2 = 1$. Let the branch-points coincide with $p_i = e^{\nu\pi i/3}$ $(\nu = 0, 1, \cdots, 5)$ and $g = 2$. Then $a_1 = a_2 = 0$ and $m = 1$. If $\alpha_1 = 1$, only $\alpha_2 = 0$, 2 are admissible numbers. In general, two fixed points corresponding to a transformation cause a relation between two α_i.

But it is important that the ranges of α_i are certainly determined, whenever G is given and the positions of branch-points are fixed by (7). And for admissible numbers α_i, there exists one $\mathfrak{G}$ which has G as its x-projection. We call this extension a single extension.

It is well known that a finite group of linear transformations is one of the following five groups. We list the groups together with their numbers N and n.

		N	n_1	n_2	n_3
(E)	Elliptic cyclic group of order n:	n	1	1	
(D)	Dihedral group of order $2n$:	$2n$	n	n	2
(T)	Tetrahedral group:	12	6	4	4
(O)	Octahedral group:	24	12	8	6
(I)	Icosahedral group:	60	30	20	12

We now study the possibilities of values of the number g for each group, and if necessary we determine the values of the number k, using the fact that g, n_i, a_i, and m will have been already determined.

(E) Elliptic cyclic group of order n. (7) becomes

$$2g + 2 = nm + a_1 + a_2. \tag{10}$$

From $2g + 2 \geqq 6$, $m \geqq 1$. We get $n = (2g + 2 - a_1 - a_2)/m \leqq 2g + 2$. Equallity occurs only when $a_1 = a_2 = 0$ and $m = 1$. Then, ord. $\mathfrak{G} \leqq 2N = 4(g + 1)$.

(D) Dihedral group of order $2n$. (7) becomes

$$2g + 2 = 2nm + na_1 + na_2 + 2a_3. \tag{11}$$

From $2g + 2 \geqq 6$, $m + a_1 + a_2 \geqq 1$. Hence $n = (2g + 2 - 2a)/(2m + a_1 + a_2) \geqq 2g + 2$. Equallity occurs only when $a_1 + a_2 = 1$ and $a_3 = m = 0$. From $a_1 + a_2 = 0$, only double extensions occur. Then ord. $\mathfrak{G} = 2N = 4n = 8(g + 1)$. (8) becomes

$$k=4(2g+2)m'+(2g+2)\delta_1+2(2g+2)\delta_2+2\times 2\delta_3=2(g+1)m_1'+\{0 \text{ or } 4\},$$

where m_1' is an arbitrary non-negative integer.

(T) Tetrahedral group. (7) becomes

$$2g+2=12m+a_1+4a_1+4a_3=2m_1 \qquad (m_1\geqq 3).$$

ord. $\mathfrak{G}\leqq 12\times 2<8(g+1)$ for $g\geqq 3$.

(O) Octahedral group. (7) becomes

$$2g+2=24m+12a_1+8a_2+6a_3=6m_1+\{0 \text{ or } 2\}.$$

$$g=2,\ 3,\ 5,\ 6,\ 8,\ 9,\ 11,\ 12,\ \cdots.\ \text{ord.}\ \mathfrak{G}\leqq 2\times 24<8(g+1)$$

for $g\neq 2, 3, 5$.

When $g=2$, $m=a_1=a_2=0$, $a_3=1$.

$$k=2m'\times 24+12\times 2\delta_1+8\times 2\delta_2+6\delta_3=24m_1'+\{0,\ 6,\ 16 \text{ or } 22\}.$$

When $g=3$, $m=a_1=a_3=0$, $a_2=1$. For double extensions,

$$k=2m'\times 24+12\times 2\delta_1+8\times \delta_2+6\times 2\delta_3=12m_1'+\{0 \text{ or } 8\}.$$

(I) Icosahedral group. (7) becomes

$$2g+2=60m+30a_1+20a_2+12a_3=60m+\{0,\ 12,\ 20,\ 30,\ 32,\ 42,\ 50\ 62\}.$$

$$g=5,\ 9,\ 14,\ 15,\ 20,\ 24,\ 29,\ 30,\ \cdots.\ \text{ord.}\ \mathfrak{G}\leqq 2\times 60<8(g+1)$$

for $g\neq 5, 9, 14$.

When $g=5$, $m=a_1=a_2=0$, $a_3=1$. For double extensions,

$$k=2\times 60m'+30\times 2\delta_1+20\times 2\delta_2+12\delta_3=60m_1'+\{0,\ 12,\ 40 \text{ or } 52\}.$$

When $g=9$, $m=a_1=a_3=0$, $a_2=1$. For double extensions,

$$k=2\times 60m'+30\times 2\delta_1+20\delta_2+12\times 2\delta_3=60m_1'+\{0,\ 20,\ 24 \text{ or } 44\}.$$

When $g=14$, $m=a_2=a_3=0$, $a_1=1$.

$$\begin{aligned}k&=2\times 60m'+30\delta_1+2\times 20\delta_2+12\times 2\delta_3\\&=120m_1'+\{0,\ 12,\ 15,\ 20,\ 27,\ 32,\ 35 \text{ or } 47\}.\end{aligned}$$

All cases have been examined and the estimation $N_h(g, k)\leqq 8(g+1)$ has been established for $g\neq 2, 3, 5, 9, 14$. Equalities occur only when k is a number assigned in the theorem and the corresponding group is dihedral. For $g=2, 3$, the maximum of $N_h(g, k)$ is 48 and the group is octahedral. For $g=5, 9$, the maximum of $N_h(g, k)$ is 120 and the group is icosahedral. For $g=14$, the maximum $N_h(g, k)=120$ is attained both by a dihedral gronp and by the icosahedral one. Summing up, Theorem 1 has been proved.

4. THEOREM 2.
$$8\leqq N_h(g, 2k_1),$$
$$4\leqq N_h(g, 2k_1+1)\leqq 4(g+1).$$

Proof. The exact upper bound of $N_h(g, 2k)$ is given by the estimation in Theorem 1. By any choice of g and k, an elliptic cyclic group of order 2 which satisfies the branch condition $a_1=a_2=1$ can be extended to a $\mathfrak{G}$ of order 4. For the double extension, we get the equations

$$2g+2=2m+2 \quad \text{and} \quad k=2m'+\delta_1+\delta_2$$

which are always solved by g and k. And ord. $\mathfrak{G}=2N=4$.

For $k=2k_1$, a dihedral extension of order 4 satisfying the condition $a_1=a_2=1$ can be extended to a $\mathfrak{G}$ of order 8. For the double extension,

$$2g+2=4m+4+2a_3 \quad \text{and} \quad k=4m'+2\delta_1+2\delta_2+2(2-a_3)\delta_3$$

which are always solved by g and k. So ord. $\mathfrak{G}=2N=8$.

If $k=2k_1+1$ the relations (8) and (9) imply that tetra-, ocata- and icosahedral groups can never occur. In cases of elliptic cyclic and dihedral groups having single extension, we get $N_h(g, k) \leqq 4(g+1)$. Hence, only dihedral groups having double extension must be examined. From (11), $n(a_1+a_2)$ is an even number, and from (8), $k=2mn+n(2-a_1)\delta_1+n(2-a_2)\delta_2+2(2-a_3)\delta_3$. As k is an odd number, $n(a_1\delta_1+a_2\delta_2)$ and n are odd numbers so that we have $a_1=a_2=1$. Thus from (11), we get $n=(2g+2-2a_3)/(2m+a_1+a_2)=(g+1-a_3)/(m+1) \leqq g+1$. Consequently, ord. $\mathfrak{G}=2N=4n \leqq 4(g+1)$.

THEOREM 3. *For fixed g, $N_n(g, k)$ is a periodic function of k for $k \geqq 124$. And $P=120g(g+1)(2g+1)$ is surely a period.*

Proof. In the case of an elliptic group, the relation (10) implies $Nm=2g+2-a_1-a_2$, and either $2g+2$, $2g+1$ or $2g$ is a multiple of N. For a dihedral group, the relation (10) implies $(2m+a_1+a_2)N=2(2g+2-2a_3)$ and either $4g$ or $4(g+1)$ is a multiple of N. Hence, $P/2$ is a multiple of N in each case.

For given g and k we consider a distribution of branch-points such that the order of the extension $\mathfrak{G}$ is exactly $N_h(g, k)$. This distribution is not always unique. From (7), (8) and (9), there exist a_ι, δ_i, m, m', α_ι and n_ι such that the relations

$$2g+2=mN+\sum a_i n_\iota \quad \text{and} \quad k=m'N+\sum n_i\beta_i$$

are satisfied. Here any n_ι which is a multiple of N is gathered in the term of $m'N$. So, $\sum n_\iota\beta_i$ is a constant independent of N. m' does not always run over all non-positive integers, and the variation range depends on the position of branch-points. From the definition of $N_h(g, k)$ there is no such distribution which can be extended to a group $\mathfrak{G}'$ of order greater than $N_h(g, k)$. Let $P=2sN$. We have

$$k+P=m'N+\sum n_i\beta_i+2sN=(m'+2s)N+\sum n_\iota\beta_i.$$

Here, $m'+2s$ is an admissible number in any case, so that this distribution can be extended to $\mathfrak{G}$. Hence, $N_h(g, k) \leqq N_h(g, k+P)$.

In order to show that the equality in the last inequallity must hold, suppose the contrary. Then, there exists another distribution experessed by

$$2g+2=\bar{m}\bar{N}+\sum \bar{a}_i\bar{n}_\iota \quad \text{and} \quad k+P=\bar{m}'\bar{N}+\sum \bar{n}_\iota\beta_i.$$

And this distribution can be extended to a group $\bar{\mathfrak{G}}$ of order greater than

$N_h(g, k)$. Let $P = 2\bar{s}\bar{N}$. We have $k = \bar{N}(\bar{m}' - 2\bar{s}) + \sum \bar{n}_i \bar{\alpha}_i$. If $\bar{m}' - 2\bar{s} \geqq 0$, this number is admissible for this distribution. From the previous consideration we get $\max \sum \bar{n}_i \bar{\alpha}_i = 124$, and this is attained by the icosahedral group with $a_1 = a_2 = a_3 = 0$. Hence our hypothesis implies $\bar{m} - 2s \geqq 0$, and it is admissible. We then have $N_h(g, k) \geqq \text{ord.}\,\mathfrak{G} > N_h(g, k)$, which is absurd.

As is seen from the above discussions, it is sufficient to take P equal to L.C.M. of $\{2(g+1), 2(2g+1), 8g, 120, 48\}$.

5. Let us examine the case $g = 2$. Since $g_h(2, k) = g(2, k)$ in this case, our interest is profound. Other cases $g \geqq 3$ can also be treated by a similar method so that we do not enter in detail.

THEOREM 4. *$N(2, k)$ has the following value:*

48 *for* $k = 24m + \{0, 6, 16$ *or* $22\}$,
24 *for* $k = 24m + \{4, 8, 10, 12, 14$ *or* $18\}$,
16 *for* $k = 24m + \{2$ *or* $20\}$,
12 *for* $k = 12m + \{3$ *or* $7\}$,
10 *for* $k = 60m + \{1, 5, 11, 13, 17, 21, 23, 25, 33, 35, 37, 41, 45, 47, 53$ *or* $57\}$,
8 *for* $k = 60m + \{9, 29$ *or* $49\}$,
6 *for* $k = 60m + 59$,

where m is an arbitrary non-positive integer.

Proof. As already seen, the octahedral group occurs only when $k = 24m + \{0, 6, 16 \text{ or } 22\}$, because single extension does not occur for $a_3 = 1$. For these numbers, we have $N(2, k) = 48$, which is equal to the maximal value of $N(2, k)$. The tetrahedral group occurs only when $a_1 = 1$, $m = a_2 = a_3 = 0$. As before, there occur only double extensions, and $\text{ord.}\,\mathfrak{G} = 12 \times 2$. By (8),

$$k = 12 \times 2m' + 6\delta_1 + 2 \times 4\delta_2 + 2 \times 4\delta_3 = 8m_1' + \{0 \text{ or } 6\}.$$

For dihedral groups, we have from (11) $n = (6 - 2a_3)/(2m + a_1 + a_2)$. We list all the cases of double extensions.

	a_1+a_2	a_3	m	n	N	ord. $\mathfrak{G}$	k
D_1	1	0	0	6	12	24	$24m' + 12\delta_1 + 6\delta_2 + 4\delta_3 = 6m_1' + \{0 \text{ or } 4\}$
D_2	1	1	0	4	8	16	$16m' + 8\delta_1 + 4\delta_2 + 2\delta_3 = 2m_1'$
D_3	2	0	0	3	6	12	$12m' + 3\delta_1 + 3\delta_2 + 4\delta_3 = 12m_1' + \{0, 3, 4, 6, 7 \text{ or } 10\}$
D_4	0	0	1	3	6	12	$6m' + 6\delta_1 + 6\delta_2 + 4\delta_3 = 6m_1' + \{0 \text{ or } 4\}$
D_5	1	0	1	2	4	8	$4m' + 4\delta_1 + 2\delta_2 + 4\delta_3 = 2m_1'$
D_6	0	1	1	2	4	8	$4m' + 4\delta_1 + 4\delta_2 + 2\delta_3 = 2m_1'$
D_7	2	1	0	2	4	8	$8m' + 2\delta_1 + 2\delta_2 + 2\delta_3 = 2m_1'$

The possibility of the single extension occurs only for D_4 and D_6. In the case of a single extension of D_4, each branch-point does not lie on any fixed point. Hence we take the extension of two generators corresponding to a_1 and a_3 in the type of $\mathfrak{T}_1$ of (4). From the position of the branch-points the extension of one remained generator must be of the type $\mathfrak{T}_2$ of (4). Then,

$$k = 6m' + 3\alpha_1 + 3\alpha_2 + 2\alpha_3$$

where $\alpha_1,\ \alpha_3 = 0$, 1 or 2 and $\alpha_2 = 0$ or 2. There may be other possibilities of k other than these. But we need not examine them. From this relation k may be an arbitrary integer for $k \geqq 2$. From this example only, we can immediately conclude $N(2, k) \geqq 6$ for $k \neq 1$. We need not examine the single extension of D_6, because ord. $\mathfrak{G}$ is only 4.

We next treat the elliptic cyclic groups. From (10), we have $n = (6 - a_1 - a_2)/m$. Since a single extension produces $\mathfrak{G}$ of order $\leqq 6$, we need only double extensions of order 6, 5 or 4. Concerning these possibilities we list for the cases of elliptic cyclic groups.

	$a_1 + a_2$	m	$n = N$	ord. $\mathfrak{G}$	k
E_1	0	1	6	12	$6m' + 2\delta_1 + 2\delta_2 = 2m_1'$
E_2	1	1	5	10	$5m' + \delta_1 + 2\delta_2 = 5m_1' + (0, 1, 2 \text{ or } 3)$
E_3	2	1	4	8	$4m' + \delta_1 + \delta_2 = 4m_1' + (0, 1 \text{ or } 2)$

Each k is thus contained somewhere. And the groups which are extended to a $\mathfrak{G}$ of order > 6 are all examined. Hence, for fixed k, in order to determine the number $N(2, k)$, we should take all possibilities of k, and should take the maximal order of $\mathfrak{G}$. We further note that 120 is a period of $N(2, k)$. The following table give the number of $N(2, k)$ $(k = 0, 1, 2, \cdots, 119)$ together with the corresponding groups.

k	$N(2, k)$	The type of $\mathfrak{G}$	k	$N(2, k)$		k	$N(2, k)$		k	$N(2, k)$		k	$N(2, k)$	
0	48	O	14	24	T	28	24	D_1	42	24	D_1	56	24	T
1	10	E_2	15	12	D_3	29	8	E_3	43	12	D_3	57	10	E_2
2	16	D_2	16	48	O	30	48	O	44	16	D_2	58	24	D_1
3	12	D_3	17	10	E_2	31	12	D_3	45	10	E_2	59	6	D_4'
4	24	D_1	18	24	D_1	32	24	T	46	48	O	60	24	D_1
5	10	E_2	19	12	D_3	33	10	E_2	47	10	E_2	61	10	E_2
6	48	O	20	16	D_2	34	24	D_1	48	48	O	62	24	T
7	12	D_3	21	10	E_2	35	10	E_2	49	8	E_3	63	12	D_3
8	24	T	22	48	O	36	24	D_1	50	16	D_2	64	48	O
9	8	E_3	23	10	E_2	37	10	E_2	51	12	D_3	65	10	E_2
10	24	D_1	24	48	O	38	24	T	52	24	D_1	66	24	D_1
11	10	E_2	25	10	E_2	39	12	D_3	53	10	E_2	67	12	D_3
12	24	D_1	26	16	D_2	40	48	O	54	48	O	68	16	D_2
13	10	E_2	27	12	D_3	41	10	E_2	55	12	D_3	69	8	E_3

k	$N(2, k)$	The type of $\mathfrak{G}$	k	$N(2, k)$		k	$N(2, k)$		k	$N(2, k)$		k	$N(2, k)$	
70	48	O	80	24	T	90	24	D_1	100	24	D_1	110	24	T
71	10	E_2	81	10	E_2	91	12	D_3	101	10	E_2	111	12	D_3
72	48	O	82	24	D_1	92	16	D_2	102	48	O	112	48	O
73	10	E_2	83	10	E_2	93	10	E_2	103	12	D_3	113	10	E_2
74	16	D_2	84	24	D_1	94	48	O	104	24	T	114	24	D_1
75	12	D_3	85	10	E_2	95	10	E_2	105	10	E_2	115	12	D_3
76	24	D_1	86	24	T	96	48	O	106	24	D_1	116	16	D_2
77	10	E_2	87	12	D_3	97	10	E_2	107	10	E_2	117	10	E_2
78	48	O	88	48	O	98	16	D_2	108	24	D_1	118	48	O
79	12	D_3	89	8	E_3	99	12	D_3	109	8	E_3	119	6	D_4'

O ... Octahedral D ... Dihedral E ... Elliptic cyclic
T ... Tetrahedral D_4'... Single extension of D_4

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DEPARTMENT OF MATHEMATICS,
TOKYO INSTITUTE OF TECHNOLOGY.