ON GENERALIZATION OF FROSTMAN'S LEMMA AND ITS APPLICATIONS

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1. Introduction.

The generalized Riemann-Liouvill's integral has been given by M. Riesz. By the use of a lemma which is a consequence of Riemann-Liouville's integral, Frostman [1] has proved his fundamental theorem on the energy integral. In 1938, M. Riesz [4] has given the relations between the Riemann-Liouville's integrals and potentials.

In the present paper, we shall give a generalization of Frostman's lemma in a higher dimensional space and some analogous lemmas in two dimensional space. Also we shall give some examples which show us how to apply them. The author is much indebted to Professor Y. Komatu who gives him many useful advices.

2. On generalized Riemann-Liouville's integrals.

Let $r_{PQ} = r_{QP}$ be the distance between P and Q in the m-dimensional euclidean space Ω_m $(m \geqq 1)$, and α, β be positive numbers. Then we define the integral by M. Riesz:

$$\text{(A)} \qquad I^{\alpha} f(\mathrm{P}) = \frac{1}{C_m(\alpha)} \int_{\Omega_m} f(\mathrm{Q}) r_{\mathrm{PQ}}^{\alpha-m} d\mathrm{Q},$$

where

$$\text{(B)} \qquad C_m(\alpha) = \pi^{\frac{m}{2}} \frac{2^{\alpha} \Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{m-\alpha}{2}\right)}$$

and $d\mathrm{Q}$ denotes the volume element. Here $f(\mathrm{Q})$ is continuous and satisfies the condition that the above integral (A) should converge absolutely.

For example, in order that (A) converges near the point $\mathrm{P} = \mathrm{Q}$, it is necessary that $\alpha > 0$ while the convergence near the point at infinity depends on the behavior of $f(\mathrm{Q})$. If $f(\mathrm{Q})$ is a continuous function which behaves like e^{-cr} at infinity, $I^{\alpha} f(\mathrm{Q})$ exists when $\alpha > 0$ and it represents a continuous function of α. If, however, $f(\mathrm{Q})$ is a continous function which behaves like $1/r^{\kappa}$ $(\kappa > 0)$ at infinity, $I^{\alpha} f(\mathrm{Q})$ exists when $0 < \alpha < \kappa$ and it represents a continuous function of α in the interval $0 < \alpha < \kappa$.

Concerning Riesz's operator I^{α}, the fundamental results are mentioned as follows:

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THEOREM. *If f be a continuous function such that $I^{\alpha}f$ exists absolutely in the interval $0<\alpha<\kappa$, then*

(C) $I^{\alpha}\{I^{\beta}f(\mathrm{P})\}=I^{\alpha+\beta}f(\mathrm{P})$ *for* $\alpha>0$, $\beta>0$, *and* $\alpha+\beta<m$,

(D) $\Delta I^{\alpha+2}f(\mathrm{P})=-I^{\alpha}f(\mathrm{P})$,

(E) $\lim\limits_{\alpha\to 0} I^{\alpha}f(\mathrm{P})=I^{0}f(\mathrm{P})=f(\mathrm{P})$,

where Δ indicates the Laplace's operator.

3. Generalization of Frostman's lemma.

LEMMA I (Frostman [1]). *Let $0<k<3$, $0<l<3$ and $k+l>3$, then*

$$\int_{\Omega_3}\frac{1}{r^k_{\mathrm{PM}}}\cdot\frac{1}{r^l_{\mathrm{MQ}}}d\mathrm{M}=H_3(k,l)\frac{1}{r^{k+l-3}_{\mathrm{PQ}}},\tag{I}$$

where

$$H_3(k,l)=\pi^{\frac{3}{2}}\frac{\Gamma\left(\frac{3-k}{2}\right)\Gamma\left(\frac{3-l}{2}\right)\Gamma\left(\frac{k+l-3}{2}\right)}{\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{l}{2}\right)\Gamma\left(\frac{6-k-l}{2}\right)}.\tag{I$'$}$$

Here, we shall give a generalization of this lemma, which is the main result of this paper.

LEMMA II. *If k and l are positive numbers and satisfy the conditions $m>k$, $m>l$ and $m<k+l<2m$, then*

$$\int_{\Omega_m}\frac{1}{r^k_{\mathrm{PM}}}\,\frac{1}{r^l_{\mathrm{MQ}}}d\mathrm{M}=H_m(k,l)\frac{1}{r^{k+l-m}_{\mathrm{PQ}}},\tag{II}$$

where

$$H_m(k,l)=\pi^{\frac{m}{2}}\frac{\Gamma\left(\frac{m-k}{2}\right)\Gamma\left(\frac{m-l}{2}\right)\Gamma\left(\frac{k+l-m}{2}\right)}{\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{l}{2}\right)\Gamma\left(\frac{2m-k-l}{2}\right)}.\tag{II$'$}$$

In fact, if we take the function $f(\mathrm{Q})$ with the above mentioned properties, we can apply Riesz's formulas which imply

$$\begin{aligned}I^{\alpha}\{I^{\beta}f(\mathrm{P})\}&=\frac{1}{C_m(\alpha)C_m(\beta)}\int_{\Omega_m}\left\{\int_{\Omega_m}f(\mathrm{Q})r^{\beta-m}_{\mathrm{QM}}d\mathrm{Q}\right\}r^{\alpha-m}_{\mathrm{MP}}d\mathrm{M}\\&=\frac{1}{C_m(\alpha)C_m(\beta)}\int_{\Omega_m}\left\{\int_{\Omega_m}r^{\beta-m}_{\mathrm{QM}}r^{\alpha-m}_{\mathrm{MP}}d\mathrm{M}\right\}f(\mathrm{Q})d\mathrm{Q}.\end{aligned}\tag{1}$$

In order that the integrals here converge and the inversion of the order of integration is legitimate, it is sufficient to suppose that $f(\mathrm{Q})=O(1/r^{\kappa})$ as $r\to\infty$ and, $\kappa>\alpha+\beta$ and $m>\alpha+\beta$ ($\kappa=m$, for example). But on the other hand, there holds

(2) $$I^{\alpha+\beta}f(\mathrm{P}) = \frac{1}{C_m(\alpha+\beta)}\int_{\Omega_m} f(\mathrm{Q}) r_{\mathrm{PQ}}^{\alpha+\beta-m} d\mathrm{Q}.$$

As the continuous function $f(\mathrm{Q})$ is arbitrary and

$$I^{\alpha}\{I^{\beta}f(\mathrm{P})\} = I^{\alpha+\beta}f(\mathrm{P}),$$

so that we get

(3) $$\int_{\Omega_m} r_{\mathrm{QM}}^{\beta-m} r_{\mathrm{MP}}^{\alpha-m} d\mathrm{M} = \frac{C_m(\alpha)C_m(\beta)}{C_m(\alpha+\beta)} r_{\mathrm{PQ}}^{\alpha+\beta-m}.$$

If we determine two positive numbers k and l by

$$m-\alpha = k \quad \text{and} \quad m-\beta = l,$$

then they satisfy

$$0<k<m, \quad 0<l<m \quad \text{and} \quad m<k+l<2m,$$

and (3) become

$$\int_{\Omega_m} \frac{1}{r_{\mathrm{PM}}^k} \frac{1}{r_{\mathrm{MQ}}^l} d\mathrm{M} = H_m(k, l)\frac{1}{r_{\mathrm{PQ}}^{k+l-m}},$$

where

$$H_m(k, l) = \pi^{\frac{m}{2}} \frac{\Gamma\left(\frac{m-k}{2}\right)\Gamma\left(\frac{m-l}{2}\right)\Gamma\left(\frac{k+l-m}{2}\right)}{\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{l}{2}\right)\Gamma\left(\frac{2m-k-l}{2}\right)}.$$

4. Consequences of lemma II.

In (II) and (II′) of the Lemma II, if put $m=3$, then we obtain the Frostman's lemma I. Next, we shall show that it is possible to give a direct proof of the formula

(III) $$\int_{\Omega_3} \frac{1}{r_{\mathrm{PM}}^2} \frac{1}{r_{\mathrm{MQ}}^2} d\mathrm{M} = \frac{\pi^3}{r_{\mathrm{PQ}}},$$

which is a special case of (I).

In fact, if we put P(0, 0, 0), Q(0, 0, a), M(x, y, z) and $x = r\cos\varphi\sin\theta$, $y = r\sin\varphi\sin\theta$, $z = r\cos\theta$ then the left-hand member of (III) becomes

$$\int_{\Omega_3} \frac{1}{r_{\mathrm{PM}}^2} \frac{1}{r_{\mathrm{MQ}}^2} d\mathrm{M} = \int_0^\infty\int_0^\pi\int_0^{2\pi} \frac{r^2 \sin\theta\, d\varphi d\theta dr}{r^2(r^2+a^2-2ar\cos\theta)} = \frac{2\pi}{a}\int_0^\infty \frac{1}{r}\log\left|\frac{a+r}{a-r}\right| dr.$$

Putting $r = at$ in the last integral, we get

(1) $$\begin{aligned}\int_{\Omega_3} \frac{1}{r_{\mathrm{PM}}^2} \frac{1}{r_{\mathrm{MQ}}^2} d\mathrm{M} &= -\frac{\pi}{a}\left[\int_0^a \frac{1}{r}\log\frac{a+r}{a-r} dr + \int_a^\infty \frac{1}{r}\log\frac{r-a}{r+a} dr\right] \\ &= \frac{8\pi}{a}\int_0^1 \frac{1}{1+t^2}\log\frac{1}{t} dt.\end{aligned}$$

Putting $t = e^{-y}$ in (1), then

$$\begin{aligned}(2)\qquad \int_0^1 \frac{1}{1-t^2}\log\frac{1}{t}dt &= \int_0^\infty \frac{ye^{-y}}{1-e^{-2y}}dy = \sum_{n=0}^\infty \int_0^\infty ye^{-(2n-1)y}dy \\ &= \sum_{n=0}^\infty \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.\end{aligned}$$

From (1) and (2), we have

$$\int_{\Omega_3} \frac{1}{r^2_{\mathrm{PM}}}\,\frac{1}{r^2_{\mathrm{MQ}}}d\mathrm{M} = \frac{\pi^3}{a} = \frac{\pi^3}{r_{\mathrm{PQ}}}.$$

5. In (II), we put $m=2$ and obtain the lemma:

LEMMA III. *If* $0<k<2$, $0<l<2$ *and* $2<k+l<4$, *then*

$$\text{(IV)}\qquad \int_{\Omega_2} \frac{1}{r^k_{\mathrm{PM}}}\,\frac{1}{r^l_{\mathrm{MQ}}}d\mathrm{M} = H_2(k,l)\frac{1}{r^{k+l-2}_{\mathrm{PQ}}},$$

where

$$H_2(k,l) = \pi\cdot\frac{\Gamma\left(\frac{2-k}{2}\right)\Gamma\left(\frac{2-l}{2}\right)\Gamma\left(\frac{k+l-2}{2}\right)}{\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{l}{2}\right)\Gamma\left(\frac{4-l-k}{2}\right)}.$$

Now, in two dimensional case, we shall give lemmas supplementary to (IV). We describe about P a circle Σ with a sufficiently large radius R.

LEMMA IV. *Let* k *and* l *be positive numbers such that* $k+l=2$, *then*

$$\text{(V)}\qquad \int_{\Sigma} \frac{1}{r^k_{\mathrm{PM}}}\,\frac{1}{r^l_{\mathrm{MQ}}}d\mathrm{M} = A(k,l)\log\frac{1}{r_{\mathrm{PQ}}} + B(k,l) + O(\log R),$$

where $A(k,l)$, $B(k,l)$ *are symmetric functions of* k *and* l.

To prove this lemma, it is sufficient to consider the case when r_{PQ} is less than $\delta/2$, δ being any fixed positive constant. We describe about P the concentric circles σ_1 and σ_2 with radii $2a$ and 2δ as such $r_{\mathrm{PQ}}=a<\delta/2$ respectively. We put

$$(1)\qquad I=\int_{\Sigma} \frac{1}{r^k_{\mathrm{PM}}}\,\frac{1}{r^l_{\mathrm{MQ}}}d\mathrm{M} = \left(\int_{\Sigma-\sigma_2} + \int_{\sigma_2-\sigma_1} + \int_{\sigma_1}\right)\frac{1}{r^k_{\mathrm{PM}}}\,\frac{1}{r^l_{\mathrm{MQ}}}d\mathrm{M} = I_1+I_2+I_3.$$

When M belongs to $\Sigma-\sigma_2$, then $r_{\mathrm{PM}}\leqq 2\delta$, $r_{\mathrm{MQ}}\geqq\delta$. Hence I_1 is of the form:

$$(2)\qquad I_1 = O(\log R).$$

When M belongs to $\sigma_2-\sigma_1$, we have

$$r^2_{\mathrm{MQ}} = r^2_{\mathrm{PM}} + a^2 - 2ar_{\mathrm{PM}}\cos\theta \quad\text{and}\quad r_{\mathrm{PM}} > 2a,$$

where θ is the angle between the radius-vectors r_{PQ} and r_{PM}. In this case, we have easily

$$(3) \qquad \frac{1}{2} < \frac{r_{\mathrm{MQ}}}{r_{\mathrm{PM}}} < \frac{3}{2}.$$

Hence, putting $r_{\mathrm{PM}} = r$ and as we can suppose δ sufficiently small,

$$(4) \qquad \begin{aligned} I_2 &< \int_0^{2\pi}\int_{2a}^{2\delta} \frac{1}{r^{2-k}} \frac{1}{(r/2)^k} r dr d\theta = 2^{k+1}\pi\left(\log \frac{1}{2a} + \log 2\delta\right) \\ &< 2^{k+1}\pi \log \frac{1}{a} = 2^{k+1}\pi \log \frac{1}{r_{\mathrm{PQ}}}. \end{aligned}$$

Now about I_3, we divide the circle σ_1 in two parts by the bisector of the segment PQ. We denote by σ_1' the part of σ_1 which contains P and by σ_1'' the other part. Then we put

$$(5) \qquad I_3 = \left(\int_{\sigma_1'} + \int_{\sigma_1''}\right) \frac{1}{r_{\mathrm{PM}}^{2-k}} \frac{1}{r_{\mathrm{MQ}}^{k}} d\mathrm{M}.$$

If M belongs to σ_1', then $r_{\mathrm{MQ}} \geqq a/2$ so that

$$(6) \qquad \int_{\sigma_1'} \frac{1}{r_{\mathrm{PM}}^{2-k}} \frac{1}{r_{\mathrm{MQ}}^{k}} d\mathrm{M} < \frac{1}{(a/2)^k} \int_0^{2\pi}\int_0^{2a} \frac{r dr d\theta}{r^{2-k}} = \frac{2^{2k+1}\pi}{k}.$$

When M belongs to σ_1'', then $r_{\mathrm{PM}} \geqq a/2$. We describe about Q the circle τ with radius $3a$, which is contained in σ_2. Introducing the polar coordinates with pole at Q we get

$$(7) \qquad \begin{aligned} \int_{\sigma_1''} \frac{1}{r_{\mathrm{PM}}^{2-k}} \frac{1}{r_{\mathrm{MQ}}^{k}} d\mathrm{M} &< \frac{1}{(a/2)^{2-k}} \int_{\tau} \frac{1}{r_{\mathrm{MQ}}^{k}} d\mathrm{M} \\ &< \frac{2^{2-k}}{a^{2-k}} 2\pi \int_0^{3a} \frac{r dr}{r^k} = \frac{6^{2-k}\pi}{2-k}. \end{aligned}$$

From (6) and (7), there follows

$$(8) \qquad I_3 < \left(\frac{2^{2k+1}}{k} + \frac{6^{2-k}}{2-k}\right)\pi.$$

Choosing suitable constants $A(k, l)$ and $B(k, l)$, from (2), (4) and (8) we obtain

$$I = A(k, l) \log \frac{1}{r_{\mathrm{PQ}}} + B(k, l) + O(\log R).$$

LEMMA V. *Let $k > 0$, $l > 0$ and $k + l < 2$, then the integral*

$$\int_{\Sigma} \frac{1}{r_{\mathrm{PM}}^{k}} \frac{1}{r_{\mathrm{MQ}}^{l}} d\mathrm{M}$$

is a continuous function of r_{PQ}.

In fact, using the same notations as before, we put

$$(1) \qquad \int_{\Sigma} \frac{1}{r_{\mathrm{PM}}^{k}} \frac{1}{r_{\mathrm{MQ}}^{l}} d\mathrm{M} = \left(\int_{\Sigma-\sigma_2} + \int_{\sigma_2-\sigma_1} + \int_{\sigma_1}\right) \frac{1}{r_{\mathrm{PM}}^{k}} \frac{1}{r_{\mathrm{MQ}}^{l}} d\mathrm{M} = I_1 + I_2 + I_3.$$

Evidently, I_1 is continuous with respect to r_{PQ}. For I_2, since M belongs to $\sigma_2-\sigma_1$, and $r_{QM}\geqq r_{PM}/2$ holds, then introducing the polar coordinates with pole at P, we get

$$
(2)\qquad \begin{aligned} I_2 &< \int_{\sigma_2-\sigma_1}\frac{1}{r_{PM}^k}\frac{1}{(r_{PM}/2)^l}d\mathrm{M} < 2^{l+1}\pi\int_{2a}^{2\delta}\frac{rdr}{r^{k+l}} \\ &< \frac{2^{3-k}\pi}{2-(k+l)}\delta^{2-(k+l)}. \end{aligned}
$$

Hence I_2 vanishes with δ. Finally, we put

$$
I_3=\left(\int_{\sigma_1'}+\int_{\sigma_1''}\right)\frac{1}{r_{PM}^k}\frac{1}{r_{MQ}^l}d\mathrm{M}.
$$

By the same method as in the former lemma, we have

$$
\int_{\sigma_1'}\frac{1}{r_{PM}^k}\frac{1}{r_{MQ}^l}d\mathrm{M}<\frac{2^{2+l-k}\pi}{2-k}a^{2-(k+l)}
$$

and

$$
\int_{\sigma_1''}\frac{1}{r_{PM}^k}\frac{1}{r_{MQ}^l}d\mathrm{M}<\frac{2^{k+1}\cdot 3^{2-l}\pi}{2-l}a^{2-(k+l)}.
$$

Therefore it becomes

$$
(3)\qquad I_3<\left(\frac{2^{2+l-k}}{2-k}+\frac{2^{k+1}\cdot 3^{2-l}}{2-l}\right)\pi\cdot r_{PQ}^{2-(k+l)},\qquad a=r_{PQ}.
$$

Thus the integral I_3 vanishes also with δ. From (1), (2) and (3) the integral is a continuous function of r_{PQ}.

6. Applications.

Next we shall illustrate some examples of the above mentioned results in this and following sections.

Let V be a closed region bounded by regular surfaces S and let V_i and V_e be the regions consisting of inner and outer points of V, respectively. A function φ which is defined and continuous in V is called regular when it satisfies the conditions:

1) it has the continuous first partial derivatives in V, and
2) it has the continuous second partial derivatives in V_i.

If φ, ψ are regular in V we have the Green's formula:

$$
(1)\qquad \int_V\varphi\Delta\psi dV+\int_V(\nabla\varphi,\nabla\psi)dV=\int_S\varphi\frac{d\psi}{dn}dS,
$$

where n indicates the outer normal to S and dV, dS denote the volume and surface elements, respectively. For a regular surfaces of m-dimensional space Ω_m, we consider the potentials

$$(2)\qquad U(\mathrm{P})=\int_S \mu(\mathrm{Q})\frac{d\mathrm{Q}}{r_{\mathrm{PQ}}},\qquad W(\mathrm{P})=\int_S \nu(\mathrm{Q})\frac{d\mathrm{Q}}{r_{\mathrm{PQ}}}$$

where μ and ν are the continuous functions on S.

REMARK. In Ω_2 we replace U and W by the logarithmic potentials. For the proof of the problem in Ω_2, the following proof remains valid by slight modifications.

Now we can prove that *the potentials* $U(\mathrm{P})$ *and* $W(\mathrm{P})$ *also satisfy* (1):

$$(3)\qquad \int_V U\Delta W dV+\int_V (\nabla U,\ \nabla W)dV=\int_S U\frac{dW}{dn}dS.$$

Since $U(\mathrm{P})$ *and* $W(\mathrm{P})$ *are harmonic in* V, (3) *becomes*

$$(3')\qquad \int_V (\nabla U,\ \nabla W)dV=\int_S U\frac{dW}{dn}dS.$$

If the density function is merely continuous on S, the first derivatives of the potential of a single layer is not necessarily bounded on S. Therefore, we must prove the validity of (3). In this proof we shall use the abbreviation:

$$\mathrm{M}_i(x_i,\ y_i,\ z_i)=\mathrm{M}_i,\qquad \mathrm{M}(x,\ y,\ z)=\mathrm{M}_0=\mathrm{M},$$

and

$$r_{0i}^2=(x-x_i)^2+(y-y_i)^2+(z-z_i)^2,\qquad r_{ik}^2=(x_i-x_k)^2+(y_i-y_k)^2+(z_i-z_k)^2,$$
$$\sigma(0)=\sigma(x,\ y,\ z),\qquad \sigma(i)=\sigma(x_i,\ y_i,\ z_i),\qquad dS(i)=dS(x_i,\ y_i,\ z_i),$$

etc.

Proof. i) If M_1 and M_2 are inner points of V_e, then $1/r_{10}$ and $1/r_{20}$ have continuous partial derivative of any order. Hence U and W are regular in V and the identity (3′) holds.

ii) Let M_3 be a fixed inner point of V_e and M_1 be an arbitrary point on S. Now consider

$$(4)\qquad \begin{aligned}&\int_V\left(\frac{\partial\frac{1}{r_{10}}}{\partial x}\frac{\partial\frac{1}{r_{30}}}{\partial x}+\frac{\partial\frac{1}{r_{10}}}{\partial y}\frac{\partial\frac{1}{r_{30}}}{\partial y}+\frac{\partial\frac{1}{r_{10}}}{\partial z}\frac{\partial\frac{1}{r_{30}}}{\partial z}\right)dV(0)\\ &\qquad=\int_S\frac{1}{r_{10}}\frac{d\frac{1}{r_{30}}}{dn}dS(0)=-\int_S\frac{1}{r_{10}}\frac{\cos(r_{03},\ N_0)}{r_{03}^2}dS(0).\end{aligned}$$

Here, since

$$\int_V\frac{\partial\frac{1}{r_{10}}}{\partial x}\frac{\partial\frac{1}{r_{30}}}{\partial x}dV(0)\quad \text{etc.}$$

are the first partial derivatives of potentials with continuous densities of a single layer, and the last integral represents a potential of a double layer with a continuous density, they are both continuous, and the equality (3′) holds.

iii) In case where the point M_3 coincides with M_2 of S which is different from M_1, we investigate the relation (3′). When $M_3 \to M_2$ from V_e, (4) becomes

$$\int_V \left(\frac{\partial \frac{1}{r_{10}}}{\partial x} \frac{\partial \frac{1}{r_{20}}}{\partial x} + \frac{\partial \frac{1}{r_{10}}}{\partial y} \frac{\partial \frac{1}{r_{20}}}{\partial y} + \frac{\partial \frac{1}{r_{10}}}{\partial z} \frac{\partial \frac{1}{r_{20}}}{\partial z} \right) dV(0) = \frac{2\pi}{r_{12}} - \int_S \frac{1}{r_{10}} \frac{\cos(r_{02}, N_0)}{r_{02}^2} dS(0). \tag{5}$$

In fact, the right-hand member of (4) becomes, when $M_3 \to M_2$ from V_e,

$$\begin{aligned} \lim_{M_3 \to M_2} \left(- \int_S \frac{1}{r_{10}} \frac{\cos(r_{03}, N_0)}{r_{03}} dS(0) \right) &= - \int_S \frac{1}{r_{10}} \frac{\cos(r_{02}, N_0)}{r_{02}^2} dS(0) - \left(-2\pi \frac{1}{r_{12}} \right) \\ &= \frac{2\pi}{r_{12}} - \int_S \frac{1}{r_{10}} \frac{\cos(r_{02}, N_0)}{r_{02}^2} dS(0). \end{aligned}$$

Multiplying μ and ν and integrating, we get

$$\begin{aligned} &\int_S \mu(2) \left\{ \int_S \nu(1) \left[\int_V \left(\frac{\partial \frac{1}{r_{10}}}{\partial x} \frac{\partial \frac{1}{r_{20}}}{\partial x} + \frac{\partial \frac{1}{r_{10}}}{\partial y} \frac{\partial \frac{1}{r_{20}}}{\partial y} \right. \right. \right. \\ &\qquad\qquad \left. \left. \left. + \frac{\partial \frac{1}{r_{10}}}{\partial z} \frac{\partial \frac{1}{r_{20}}}{\partial z} \right) dV(0) \right] dS(1) \right\} dS(2) \\ &= \int_S \mu(2) \left\{ \int_S \nu(1) \left[\frac{2\pi}{r_{12}} - \int_S \frac{1}{r_{10}} \frac{\cos(r_{02}, N_0)}{r_{02}^2} dS(0) \right] dS(1) \right\} dS(2). \end{aligned} \tag{6}$$

If the inversion of order of integration on the left side is possible, we have

$$\begin{aligned} &\int_S \mu(2) \left\{ \int_S \nu(1) \left[\int_V \frac{\partial \frac{1}{r_{10}}}{\partial x} \frac{\partial \frac{1}{r_{20}}}{\partial x} dV(0) \right] dS(1) \right\} dS(2) \\ &= \int_V \left\{ \left[\int_S \nu(1) \frac{\partial \frac{1}{r_{10}}}{\partial x} dS(1) \right] \left[\int_S \mu(2) \frac{\partial \frac{1}{r_{20}}}{\partial x} dS(2) \right] \right\} dV(0) \\ &= \int_V \frac{\partial U}{\partial x} \frac{\partial W}{\partial x} dV(0), \end{aligned} \tag{7}$$

and there hold analoguous formulas with respect to y and z. If the inversion of the order of integral of the right-hand member of (6) is legitimate, then

$$(8)\quad \begin{aligned} &2\pi\int_S \mu(2)\left\{\int_S \nu(1)\frac{1}{r_{12}}dS(1)\right\}dS(2) \\ &\quad -\int_S\left[\int_S \nu(1)\frac{1}{r_{10}}dS(1)\right]\left[\int_S \mu(2)\frac{\cos(r_{02}, N_0)}{r_{02}^2}dS(2)\right]dS(0) \\ &= 2\pi\int_S \mu(0)U(0)dS(0) - \int_S U(0)\left[\int_S \mu(2)\frac{\cos(r_{02}, N_0)}{r_{02}^2}dS(2)\right]dS(0) \\ &= \int_S U(0)\left\{2\pi\mu(0) - \int_S \mu(2)\frac{\cos(r_{02}, N_0)}{r_{02}^2}dS(2)\right\}dS(0) \\ &= \int_S U(0)\frac{dW_i}{dn_0}dS(0), \end{aligned}$$

where dW_i/dn_0 denotes the limiting value of dW/dn at M_0 from V_i. From (7) and (8), there follows

$$(9)\quad \int_V \left(\frac{\partial U}{\partial x}\frac{\partial W}{\partial x} + \frac{\partial U}{\partial y}\frac{\partial W}{\partial y} + \frac{\partial U}{\partial z}\frac{\partial W}{\partial z}\right)dV(0) = \int_S U(0)\frac{dW_i}{dn_0}dS(0).$$

Now it is sufficient, in order to invert the order of the integration in the left-hand member of (6), to show the existence of the integral

$$(10)\quad \int_S |\mu(2)|\left\{\int_S |\nu(1)|\left[\int_V \left|\frac{\partial \frac{1}{r_{10}}}{\partial x}\right|\left|\frac{\partial \frac{1}{r_{20}}}{\partial x}\right| dV(0)\right]dS(1)\right\}dS(2).$$

We have evidently

$$(11)\quad \begin{aligned} \int_V \left|\frac{\partial \frac{1}{r_{10}}}{\partial x}\cdot\frac{\partial \frac{1}{r_{20}}}{\partial x}\right| dV(0) &\leqq \int_V \frac{1}{r_{10}^2}\frac{1}{r_{20}^2}dV(0) \\ &\leqq \int_{\Omega_3}\frac{1}{r_{01}^2}\frac{1}{r_{20}^2}dV(0). \end{aligned}$$

By the use of (III), i.e.

$$\int_{\Omega_3}\frac{1}{r_{10}^2}\frac{1}{r_{20}^2}dV(0) = \frac{\pi^3}{r_{12}},$$

if we take a positive constant A such that $A \geqq \operatorname{Max}|\nu(1)|$, (10) becomes

$$(12)\quad \begin{aligned} &\int_S |\nu(1)|\left[\int_V \left|\frac{\partial \frac{1}{r_{10}}}{\partial x}\cdot\frac{\partial \frac{1}{r_{20}}}{\partial x}\right| dV(0)\right]dS(1) \\ &\qquad \leqq A\int_S \frac{\pi^3}{r_{12}}dS(1) = A\pi^3\int_S \frac{1}{r_{12}}dS(1). \end{aligned}$$

The last term is bounded with respect to M_2 in Ω_3, and the same is true for similar terms. Therefore (10) converges uniformly.

Next, in order to invert the order of the integration in the right-hand member of (6), it is sufficient to show that the inequality

$$\int_S \frac{1}{r_{10}} \frac{|\cos(r_{20}, N_0)|}{r_{20}^2} dS(0) < \frac{C}{r_{12}^{1-\lambda}} \tag{13}$$

holds, where C is a positive constant. Since S is a regular surface, we can choose a constant a and λ such that

$$|\cos(r_{20}, N_0)| < a r_{20}^{\lambda} \qquad (0 < \lambda < 1)$$

hold. Now let σ be the part of S which lies in the sphere with center M_0 and radius δ. We put

$$\int_S \frac{1}{r_{10}} \frac{|\cos(r_{20}, N_0)|}{r_{20}^2} dS(0) = \left(\int_{S-\sigma} + \int_\sigma\right) \frac{1}{r_{10}} \frac{|\cos(r_{20}, N_0)|}{r_{20}^2} dS(0). \tag{14}$$

The first integral on the right-hand member is bounded at M_0. Let σ' be the projection of σ onto the tangential plane at M_0 and r'_{10} and r'_{20} denote the projection of r_{10} and r_{20}, respectively. Since S is regular, it is possible to take a positive constant δ such that between any vector $r \in \sigma$ and its projection $r' \in \sigma'$ we have $r < 2r'$ and in particular $dS(0) < 4dS'(0)$ where $dS'(0)$ denotes the projection of $dS(0)$. Then

$$\int_\sigma \frac{1}{r_{10}} \frac{|\cos(r_{20}, N_0)|}{r_{20}^2} dS(0) < \int_\sigma \frac{1}{r_{10}} \frac{1}{r_{20}^{2-\lambda}} dS(0) < \int_{\sigma'} \frac{4}{r'_{10} \cdot r'^{2-\lambda}_{20}} dS'(0)$$

and hence

$$\int_S \frac{1}{r_{10}} \frac{|\cos(r_{20}, N_0)|}{r_{20}^2} dS(0) < \frac{C'}{r'^{1-\lambda}_{12}}$$

where C' is a constant. Therefore the inequality (13) holds, and consequently

$$\int_S |\mu(2)| \left\{ \int_S |\nu(1)| \left[\int_S \frac{1}{r_{10}} \frac{|\cos(r_{20}, N_0)|}{r_{20}^2} dS(0) \right] dS(1) \right\} dS(2)$$

exists.

7. Iterated nuclei of integral equations.

We consider an integral equation in space Ω_m which is of the form

$$\mu(0) = \zeta \int_S K(1, 0)\mu(1) dS(1) + f(0), \tag{1}$$

where

$$K(1,0)=-\frac{1}{2\pi}\frac{\cos(r_{10},N_0)}{r_{01}^{m-1}}. \tag{2}$$

We investigate the properties of the iterated nuclei of the equation (1). We put

$$K_1(1,0)=K(1,0),\qquad K_n(1,0)=\int_S K_1(1,2)K_{n-1}(2,0)dS(2)\qquad (n\geqq 2). \tag{3}$$

Since S is regular, we have

$$|\cos(r_{10},N_0)|<ar_{10}^{\lambda},\qquad 0<\lambda<1, \tag{4}$$

a being a positive constant. Then we have

$$\frac{1}{2\pi}\frac{|\cos(r_{10},N_0)|}{r_{10}^{m-1}}r_{10}^{m-1-\lambda}<\frac{a}{2\pi}.$$

Therefore we can write

$$K(1,0)=\frac{C_1(1,0)}{r_{10}^{m-1-\lambda}}, \tag{5}$$

where $C_1(1,0)$ is a continuous function of M_0 and M_1.

In order to investigate $K_2(1,0)$, we describe a sphere about M_1 with a fixed radius δ, and denote by τ the part of S in the sphere. τ_1 indicates the part of S within the sphere about M_1 with radius $2r_{10}$ $(r_{10}<\delta/2)$. Besides we denote by τ', the projection of τ onto the tangential (hyper-)plane at M_1. We put

$$K_2(1,0)=\left(\int_{S-\tau}+\int_{\tau}\right)K_1(1,2)K_1(2,0)dS(2)=I_1+I_2. \tag{6}$$

Let M_1, $M_2\in\tau_1$ and $M_2\in S-\tau$, then I_1 is a continuous function of M_0 and M_1. By similar reasoning as in §6,

$$\begin{aligned}|I_2|&=\int_\tau\frac{1}{2\pi}\left|\frac{\cos(r_{12},N_2)}{r_{12}^{m-1}}\frac{1}{2\pi}\frac{\cos(r_{20},N_0)}{r_{20}}\right|dS(2)\\&<\frac{4}{4\pi^2}\int_\tau\frac{1}{r_{12}'^{m-1-\lambda}}\frac{1}{r_{20}'^{m-1-\lambda}}dS'(2).\end{aligned}$$

By the lemma II, we obtain

$$\begin{aligned}|I_2|&<\frac{1}{\pi^2}\int_{\tau'}\frac{1}{r_{12}'^{m-1-\lambda}}\frac{1}{r_{20}'^{m-1-\lambda}}dS(2)<\frac{1}{\pi^2}\int_{\Omega_{m-1}}\frac{1}{r_{12}'^{m-1-\lambda}}\frac{1}{r_{20}'^{m-1-\lambda}}dS'(2)\\&=\frac{1}{\pi^2}H_{m-1}(m-1-\lambda,m-1-\lambda)\frac{1}{r_{10}'^{m-1-2\lambda}}.\end{aligned} \tag{7}$$

Therefore, by suitably choosing $C_2(1, 0)$, we have

$$(8) \qquad K_2(1, 0) = C_2(1, 0) \frac{1}{r_{10}^{m-1-2\lambda}}.$$

The repetition of the above process implies

$$(9) \qquad K_n(1, 0) = C_n(1, 0) \frac{1}{r_{10}^{m-1-n\lambda}}.$$

Hence if we choose a positive integer n such that

$$m - 1 - n\lambda < 0 \qquad \text{i.e.} \qquad n > \frac{n-1}{\lambda},$$

then $K_n(1, 0)$ is a continuous function of r_{10}. Hence we can write the equation (1) in the form:

$$(10) \qquad \mu(0) = \zeta \int_S K_n(1, 0)\mu(1)\, dS(1) + \Sigma_n(0),$$

and

$$(11) \qquad \Sigma_n(0) = f(0) + \zeta \int_S K(1, 0)\mu(1) dS(1) + \cdots + \zeta^{n-1} \int_S K_{n-1}(1, 0)\mu(1) dS(1),$$

where $K_n(1, 0)$ is a continuous function of r_{10}.

8. Integral equation of the Abel's type.

In space Ω_m, we consider the integral equation

$$(1) \qquad \int_{\Omega_m} \frac{1}{r_{PQ}^{\lambda}} dQ = \varphi(P)$$

where $\varphi(P)$ is a continuous given function.

To solve (1), we multiply (1) by $1/r_{PM}^{\mu}$ $(0 < \mu < m)$ and integrate, then

$$\int_{\Omega_m} \left\{ \int_{\Omega_m} \frac{f(Q)}{r_{PQ}^{\lambda}} dQ \right\} \frac{1}{r_{PM}^{\mu}} dP = \int_{\Omega_m} \varphi(P) \frac{1}{r_{PM}^{\mu}} dP.$$

By inverting the order of the integration, it becomes

$$(2) \qquad \int_{\Omega_m} \left\{ \int_{\Omega_m} \frac{1}{r_{PQ}^{\lambda}} \frac{1}{r_{PM}^{\mu}} dP \right\} f(Q) dQ = \int_{\Omega_m} \varphi(P) \frac{1}{r_{PM}^{\mu}} dP.$$

By the lemma II, we have

$$(3) \qquad H_m(\lambda, \mu) \int_{\Omega_m} f(Q) \frac{1}{r_{QM}^{\lambda+\mu-m}} dQ = \int_{\Omega_m} \varphi(P) \frac{1}{r_{PM}^{\mu}} dP.$$

Applying the Riesz's operator (A), there follows

$$C_m(2m - \lambda - \mu) H_m(\lambda, \mu) I^{2m-\lambda-\mu} f(M) = \int_{\Omega_m} \varphi(P) \frac{1}{r_{PM}^{\mu}} dP.$$

Then

$$
(4) \qquad \pi^m \cdot 2^{2m-\lambda-\mu} \frac{\Gamma\left(\frac{m-\lambda}{2}\right)\Gamma\left(\frac{m-\mu}{2}\right)}{\Gamma\left(\frac{\lambda}{2}\right)\Gamma\left(\frac{\mu}{2}\right)} \Delta I^{2(m-1)-\lambda-\mu+2} f(\mathrm{M}) = \int_{\Omega_m} \varphi(\mathrm{P}) \Delta\left(\frac{1}{r^{\mu}_{\mathrm{PM}}}\right) d\mathrm{P}.
$$

By the property (D) of Riesz's operatior, it becomes

$$
(5) \qquad -\pi^m \cdot 2^{2m-\lambda-\mu} \frac{\Gamma\left(\frac{m-\lambda}{2}\right)\Gamma\left(\frac{m-\mu}{2}\right)}{\Gamma\left(\frac{\lambda}{2}\right)\Gamma\left(\frac{\mu}{2}\right)} I^{2(m-1)-\lambda-\mu} f(\mathrm{M}) = \int_{\Omega_m} \varphi(\mathrm{P}) \Delta\left(\frac{1}{r^{\mu}_{\mathrm{PM}}}\right) d\mathrm{P}.
$$

Now since $I^0 f = f$, if we put $2m-2-\lambda-\mu \to 0$, i.e. $\mu \to 2m-2-\lambda$, we then obtain from (5)

$$
K(m, \lambda) f(\mathrm{M}) = \int_{\Omega_m} \varphi(\mathrm{P}) \Delta\left(\frac{1}{r^{2m-2-\lambda}_{\mathrm{PM}}}\right) d\mathrm{P}.
$$

where

$$
K(m, \lambda) = -\pi^m \cdot \frac{\Gamma\left(\frac{m-\lambda}{2}\right)\Gamma\left(\frac{2+\lambda-m}{2}\right)}{\Gamma\left(\frac{\lambda}{2}\right)\Gamma\left(\frac{2m-2-\lambda}{2}\right)}.
$$

Hence

$$
(6) \qquad f(\mathrm{M}) = \frac{1}{K(m, \lambda)} \int_{\Omega_m} \varphi(\mathrm{P}) \Delta\left(\frac{1}{r^{2m-2-\lambda}_{\mathrm{PM}}}\right) d\mathrm{P}.
$$

In order that the integral in (6) exists, the function φ must behaves conveniently at the origin and at infinity. But, for example, it is sufficient for this purpose to suppose that $\varphi = o(r^{\alpha})$, $\alpha > 0$ at the origin and $\psi = r^{-\kappa}$, $\kappa > m-\lambda$ at infinity. The expression (6) then gives the solution of the equation (1).

References

[1] Frostman, O., Potentiel d'équilibre et capacité de ensembles avec quelques applications à la théorie des fonctions. Med. Lund. Univ. Math. Sem. **3** (1935).

[2] Gunter, M., Die Potentialtheorie und ihre Anwendung auf Grundaufgaben der mathematischen Physik. Leibzig 1957, pp. 129–134.

[3] RIESZ, M., Comptes rendus du Congrès International des Mathématiciens (Oslo, 1936), Vol. II. pp. 44–45, 62–63.

[4] ———, Intégrales de Riemann-Liouville et potentiels. Acta Sci. Math. Szeged **9** (1938), 1–42.

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