A SUPPLEMENT TO "ON AN EIGENVALUE AND EIGENFUNCTION PROBLEM OF THE EQUATION $\Delta u+\lambda u=0$"

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In a previous paper [1], we have given a theorem on an eigenvalue and eigenfunction problem of the equation $\Delta u+\lambda u=0$ with fixed boundary condition for a certain plane domain, concerning the first eigenvalue and the corresponding first eigenfunction.

In the present paper, we shall first establish a more general theorem on the k-th eigenvalue and the corresponding k-th eigenfunction of the same problem for a plane domain. In the second place, we shall remark that our 2-dimensional theorem can be transferred into 3-dimensional one. Finally two illustrative examples in the 3-dimensional case will be given.

I. Theorem. *Let D be a bounded connected plane domain*[1] *and C its boundary. Let $\{D_n\}$ be a sequence of domains exhausting D:*

$$D_1\subset D_2\subset\cdots\subset D_n\subset\cdots,$$

i. e. a sequence such that $\lim_{n\to\infty}D_n=D$, where the boundary C_n of the domain D_n consists of a finite number of smooth curves. Let further $\lambda_{k,n}$ and $u_{k,n}$ be the k-th eigenvalue and k-th eigenfunction, respectively, of the problem

$$\begin{cases} \Delta u+\lambda u=0 & in \quad D_n, \\ \quad\quad u=0 & on \quad C_n \end{cases}$$

and $u_{k,n}$ be normalized by $\iint_{D_n} u_{k,n}^2 d\sigma=1$.

Then $\lim_{n\to\infty}\lambda_{k,n}=\lambda_k$ exists independently of the choice of an exhausting sequence and, for any infinite subsequence $\{u_{k,n'}\}$ of the corresponding sequence $\{u_{k,n}\}$ of the eigenfunctions, there exists a uniformly convergent subsequence $\{u_{k,n''}\}$. Putting $\lim_{n''\to\infty}u_{k,n''}=u_k$, $\iint_D u_k^2 d\sigma=1$ holds. Moreover the limit function u_k and limit value λ_k satisfy the equation $\Delta u_k+\lambda_k u_k=0$ in D together with the condition $u_k=0$ on the boundary of D except at most[2]

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1) In the previous paper the boundary of D consists of a smooth closed curve C and a closed set lying entirely in the interior of the domain surrounded by C, while in the present paper D is supposed to be a bounded connected plane domain.

2) If $k=1$ the exceptional points are identical with those of the ordinary Green's function for the same domain.

for the exceptional points of ordinary Green's function for the same domain, which is of capacity zero. Furthermore if we have the relation[3]

$$\lim_{n\to\infty}\lambda_{k-1,n}<\lim_{n\to\infty}\lambda_{k,n}<\lim_{n\to\infty}\lambda_{k+1,n},$$

then u_k is determined independently of the choice of exhausting sequence. Otherwise, that is, if we have

$$\lim_{n\to\infty}\lambda_{k-\nu-1,n}<\lim_{n\to\infty}\lambda_{k-\nu,n}=\lim_{n\to\infty}\lambda_{k-\nu+1,n}=\cdots=\lim_{n\to\infty}\lambda_{k-\nu+m-1,n}<\lim_{n\to\infty}\lambda_{k-\nu+m,n},$$

then u_k depends on the choice of an infinite subsequence of $\{u_{k,n}\}$. But, among possible limit functions u_k there exist only m linearly independent u_k.

Proof. Our proposition can be established with almost the same reasoning as in our previous paper except for a few steps there. To avoid the tediousness, we will not repeat the same reasonings as those in the previous paper. In §4 and §8 of the previous paper, there are some places to be modified, while in the other paragraphs, besides some trivial modifications, we need only to replace the first eigenvalues and the first eigenfunctions by the k-th eigenvalues and k-th eigenfunctions.

Before making modifications for §4 and §8, we notice that, from the reasoning of §2 in the previous paper, the uniform boundedness of the sequence $\{u_{k,n}\}$ in A can be shown, where A denotes a subdomain of D with a positive distance ε from the boundary of D.

Now we shall mention in details how to modify the §4. For a fixed N, let the smooth boundary curves of the domain D_N be

$$C_N=\{C_N^{(0)},\ C_N^{(1)},\ \cdots,\ C_N^{(s-1)}\},$$

where $C_N^{(0)}$ denotes the outer boundary of D_N and the others are the inner ones. If N is taken sufficiently large, then the area of the part of the domain D cut off by $C_N^{(0)}$ and lying outside of the domoin enclosed by $C_N^{(0)}$ becomes as small as we wish. The area of the part of D cut off by C_N^{i} $(i=1,\ \cdots,\ s-1)$ and lying inside of the domain enclosed by $C_N^{(i)}$ then becomes also small.

We decompose the domain D_N into $s+1$ subdomains not overlapping each other:

$$D_N=\widetilde{D}_N+D_N^{(0)}+D_N^{(1)}+\cdots+D_N^{(s-1)},$$

the domain $D_N^{(i)}$ being surrounded by $C_N^{(i)}$ and a smooth closed curve $B^{(i)}$ with positive distance from boundaries of $D_N^{(j)}(j\neq i)$ and $\widetilde{D}_N$ being taken to be contained completely in the interior of D_N and also of A.

Let the area of the part of the domain D cut off by the curve $B^{(0)}$ and lying outside of the domain enclosed by $B^{(0)}$ be smaller than the area of

3) Especially, this is always the case for $k=1$.

the circle with $\lambda_{k,1}$ as the first eigenvalue for the same boundary value problem. For $n>N$, denote by $D_n^{(0)}$ the part of the domain D_n cut off by $B^{(0)}$ and lying outside of the domain enclosed by $B^{(0)}$. The area of $D_n^{(0)}$ is of course smaller than the area of the circle above mentioned.

Next apply the same procedure to each of curves $B^{(i)}(i=1, \cdots, s-1)$, denoting by $D_n^{(i)}$ the part of the domain D_n cut off by $B^{(i)}$ lying inside of the domain enclosed $B^{(i)}$. The area of $D_n^{(i)}$ $(i=1, \cdots, s-1)$ is also smaller than the area of the above circle.

Then for such a domain $D_n^{(i)}$ the Green's function $\Gamma_n(p, q)$ of the equation $\Delta u+\lambda_{k,n}u=0$ is uniquely determined, because of that, for the first eigenvalue of the fixed boundary condition, the first eigenvalue of the domain for the same problem is not less than $\lambda_{k,1}$, as shown by the isoperimetric inequality, and therefore it is greater than $\lambda_{k,n}$ since $\lambda_{k,1}>\lambda_{k,n}$. Using the domain $D_n^{(i)}$, instead of $\widetilde{D}_n$ in §4 of the previous paper, we get

$$|u_{k,n}|<\text{const}\equiv c \quad \text{in} \quad D_n^{(i)} \qquad\qquad (i=0, 1, \cdots, s-1)$$

by the same reasoning as in the §4, where the constant c is independent of n. Combining this with the uniform boundedness of $\{u_{k,n}\}$ in A, its uniform boundedness in D can be shown exactly in the same way as in the previous paper.

The reasoning for the last paragraph §8 in the previous paper, which has established the unique determination of the limit function independent of the choice of exhausting sequence, is available with trivial modifications for our limit function u_k if

$$(1) \qquad \lim_{n\to\infty} \lambda_{k-1,n} < \lim_{n\to\infty} \lambda_{k,n} < \lim_{n\to\infty} \lambda_{k+1,n}.$$

Otherwise, namely, if

$$(2) \quad \lim_{n\to\infty} \lambda_{k-\nu-1,n} < \lim_{n\to\infty} \lambda_{k-\nu,n} = \lim_{n\to\infty} \lambda_{k-\nu+1,n} = \cdots = \lim_{n\to\infty} \lambda_{k-\nu+m-1,n} < \lim_{n\to\infty} \lambda_{k-\nu+m,n},$$

u_k depends on the choice of a subsequence of $\{u_{k,n''}\}$, but among the various limit functions obtained from our limit process, there exist only m linearly independent limit functions.

First suppose that $\lambda_{h,n}$ $(h=k-\nu, \cdots, k-\nu+m-1)$ are all distinct for every D_n, i. e.

$$(3) \qquad \lambda_{k-\nu,n} < \lambda_{k-\nu+1,n} < \cdots < \lambda_{k-\nu+m-1,n},$$

then for every D_n, the corresponding eigenfunctions

$$u_{k-\nu,n}, \quad u_{k-\nu+1,n}, \quad \cdots, \quad u_{k-\nu+m-1,n}$$

are all orthogonal each other, i. e., for every $h\neq j$ $(h, j=k-\nu, \cdots, k-\nu+m-1)$,

$$(4) \qquad \iint_{D_n} u_{h,n}\cdot u_{j,n}\, d\sigma = 0.$$

By the uniform boundedness in D of functions $u_{k-\nu,n}, \cdots, u_{k-\nu+m-1,n}$ the above integral relations (4) also hold in the limiting case, i. e.

$$\iint_D u_h u_j \, d\sigma = 0$$

for

$$u_h = \lim_{n\to\infty} u_{h,n}, \quad u_j = \lim_{n\to\infty} u_{j,n}.$$

Thus m functions u_h $(h=k-\nu, \cdots, k-\nu+m-1)$ are all orthogonal each other, therefore they are linearly independent.

Now consider the general case. If in the sequence

$$(S_0) \qquad D_1, D_2, \cdots, D_n, \cdots; \quad \lim_{n\to\infty} D_n = D,$$

there are infinite number of D_n for which $\lambda_{k-\nu,n}$ is distinct from $\lambda_{k-\nu+1,n}$, take them as a subsequence (S_1). If there are only finite number of D_n for which $\lambda_{k-\nu,n} \neq \lambda_{k-\nu+1,n}$, then there are infinite number of D_n for which $\lambda_{k-\nu,n} = \lambda_{k-\nu+1,n}$. In this case, take them as a subsequence. In every case, we denote them by

$$(S_1) \qquad D_{1'}, D_{2'}, \cdots, D_{n'}, \cdots.$$

Next, from this sequence (S_1), choose the second subsequence (S_2) such that (i) if there are infinite number of $D_{n'}$ for which $\lambda_{k-\nu+1,n'} \neq \lambda_{k-\nu+2,n'}$, take them as a subsequence (S_2), and (ii) if there are only finite number of $D_{n'}$ for which $\lambda_{k-\nu+1,n'} \neq \lambda_{k-\nu+2,n'}$, then there are infinite number of $D_{n'}$ for which $\lambda_{k-\nu+1,n'} = \lambda_{k-\nu+2,n'}$ and take them as the subsequence (S_2). Continuing the same process, we finally obtain the subsequence

$$(S_{m-1}) \qquad D_{1^*}, D_{2^*}, \cdots, D_{n^*}, \cdots.$$

In this sequence (S_{m-1}), for every D_{n^*}, each consecutive eigenvalues are all distinct or all equal, i. e. for every n^*

$$\lambda_{h,n^*} \neq \lambda_{h+1,n^*} \quad \text{or} \quad \lambda_{h,n^*} = \lambda_{h+1,n^*} \qquad (h=k-\nu, \cdots, k-\nu+m-2).$$

Consider the case where

$$\lambda_{h-1,n^*} < \lambda_{h,n^*} = \lambda_{h+1,n^*} = \cdots = \lambda_{h+\mu-1,n^*} < \lambda_{h+\mu,n^*}.$$

Then, for each n^*, there correspond for this multiple eigenvalue μ independent eigenfunctions $u_{h,n^*}, \cdots, u_{h+\mu-1,n^*}$ which are orthogonal each other, i. e.

$$(5) \qquad \iint_{D_{n^*}} u_{i,n^*} \cdot u_{j,n^*} \, d\sigma = 0, \qquad (i \neq j).$$

We can choose from the domains D_{n^*}, a subsequence for which the sets of above functions $(u_{h,n^*}, \cdots, u_{h+\mu-1,n^*})$ for the multiple eigenvalues tend to a definite set of functions such that

$$\lim_{n^*\to\infty} u_{i,n^*} = u_i, \qquad (i=h, \cdots, h+\mu-1).$$

According to the uniform boundedness of these functions, we can conclude that the above relations (5) also hold in the limiting case, i. e.

$$(6) \qquad \iint_D u_i \cdot u_j \, d\sigma = 0 \qquad (i \neq j).$$

Thus the limiting functions $u_h, \cdots, u_{h+\mu-1}$ are orthogonal each other, hence they are also linearly independent.

Since for distinct eigenvalues their eigenfunctions are orthogonal, therefore the whole m limiting functions $u_{k-\nu}, u_{k-\nu+1}, \cdots, u_{k-\nu+m-1}$ constructed above are orthogonal each other, and hence linearly independent. Moreover they satisfy evidently the equation

$$\Delta u + \Lambda u = 0$$

where we put

$$\Lambda \equiv \lim_{n\to\infty} \lambda_{k-\nu,n} = \cdots = \lim_{n\to\infty} \lambda_{k-\nu+m-1,n}.$$

Let $\bar{u}$ be a limiting function chosen from another sequence $\{\bar{D}_{n'}\}$, which corresponds to the same eigenvalue Λ. Now let us show that this function $\bar{u}$ can be represented as a linear combination of the m functions $u_{k-\nu}, \cdots, u_{k-\nu+m-1}$, which we have got before.

Obviously

$$(7) \qquad \Delta\bar{u} + \Lambda\bar{u} = 0.$$

Now we will develop the function $\bar{u}$ in the series of the eigenfunctions of the domain D_{n^*}, which we have considered above. For the simplicity, we replace this n^* merely by n in the sequel. Let us introduce an auxiliary harmonic function f_n defined such that

$$(8) \qquad \Delta f_n = 0 \quad \text{in } D_n,$$

$$(9) \qquad f_n = \bar{u} \quad \text{onthe boundary of } D_n.$$

Put $v = \bar{u} - f_n$. From (7), (8) we obtain by simple calculation

$$(10) \qquad \Delta v = -\Lambda(v + f_n).$$

From (9) we see that $v=0$ on the boundary of D.

By the usual method, the above equation (10) can be transformed into the integral form

$$(11) \qquad v(p) = \Lambda \iint_{D_n} G_n(p, q) \cdot \{v(q) + f_n(q)\} d\sigma_q,$$

where $G_n(p, q)$ is the ordinary Green's function of the domain D_n. Put

$$(12) \qquad \Lambda \iint_{D_n} G_n(p, q) f_n(q)\, d\sigma_q \equiv F_n(p).$$

Then (11) becomes

$$(13) \qquad v(p) - \Lambda \iint_{D_n} G_n(p, q)\, v(q)\, d\sigma_q = F_n(p).$$

This non-homogeneous integral equation can be solved in the form

(14) $$v(p)=F_n(p)+\sum_{h=1}^{\infty} c_h^{(n)} u_{h,n}(p),$$

where

$$c_h^{(n)}=\frac{\Lambda}{\lambda_{h,n}-\Lambda} F_{h,n}$$

and $F_{h,n}$ is the Fourier coefficient in the expansion of $F_n(p)$ by the eigenfunctions $u_{h,n}(p)$, i. e.

(15) $$F_{h,n}=\iint_{D_n} F_n(p)\, u_{h,n}(p)\, d\sigma_p.$$

From (12)

$$|F_n(p)|^2 \leqq \Lambda^2\left(\iint_{D_n} G_n^2(p,q)\, d\sigma_p\right)\left(\iint_{D_n} f_n^2(q)\, d\sigma_q\right).$$

On account of the reasoning of §8 in the previous paper, if q tends to the boundary of D, $\bar{u}(q)$ remains bounded and moreover $\bar{u}(q)$ tends to 0 except for a set of capacity zero. Because the harmonic function $f_n(q)$ has the common boundary values with $\bar{u}(q)$, $f_n(q)$ becomes identically 0 in an arbitrary closed set contained in D, as n tends to ∞. Therefore,

$$e_n \equiv \iint_{D_n} f_n^2(q)\, d\sigma_q$$

tends to 0 as $n \to \infty$.

Let $G(p,q)$ be the ordinary Green's function of the domain D, then

$$G(p,q) > G_n(p,q)$$

in D_n, because $D_n \subset D$. Therefore

$$\iint_{D_n} G_n^2(p,q)\, d\sigma_q < \iint_D G^2(p,q)\, d\sigma_q \equiv \alpha$$

and

(16) $$|F_n(p)|^2 < \Lambda^2 \alpha e_n.$$

From the orthonormal property of $u_{h,n}$, we obtain

(17) $$\iint_{D_n} [(v(p)-F_n(p))-\textstyle\sum' c_h^{(n)} u_{h,n}(p)]^2\, d\sigma_p=\textstyle\sum''(c_h^{(n)})^2$$

where the summation $\sum'$ runs over the index h for which $\lim_{n\to\infty}\lambda_{h,n}=\Lambda$, and $\sum''$ over h for which $\lambda_{h,n}$ does not tend to Λ. Because of the bounded convergence of functions $f_n(p)$, $F_n(p)$ and $u_{h,n}(p)$, the left hand member of (17) tends to

(18) $$\iint_D [\bar{u}(p)-\textstyle\sum' c_h u_h(p)]^2 d\sigma_p$$

where

$$c_h=\lim_{n\to\infty}\iint \bar{u}(p) u_{h,n}(p)\, d\sigma_p.$$

From (15) and (16)

$$(F_{h,n})^2 \leqq \iint_{D_n} |F_n(p)|^2 d\sigma_p \iint_{D_n} (u_{h,n})^2 d\sigma_p = \iint_{D_n} |F_n(p)|^2 d\sigma_p < \Lambda^2\alpha\,|D|\,e_n,$$

where $|D|$ is the area of the domain D. Hence, as for the right hand member of (11), we obtain

$$\sum{}''(c_h^{(n)})^2 = \sum{}'' \frac{\Lambda^2}{(\lambda_{h,n}-\Lambda)^2}(F_{h,n})^2 < e_n \Lambda^2 \alpha\,|D|\sum{}'' \frac{\Lambda^2}{(\lambda_{h,n}-\Lambda)^2}.$$

In the summation $\sum''$, $\lambda_{h,n}-\Lambda$ is greater than a certain positive number, therefore as n tends to ∞, from the above inequality we obtain

$$\lim_{n\to\infty}\sum{}''(c_h^{(n)})^2 = 0.$$

Thus we finally obtain

$$\iint_D [\bar{u}(p) - \sum{}' c_h u_h(p)]^2 d\sigma_p = 0.$$

Therefore $\bar{u}(p)$ can be represented as a linear combination of m functions $u_{k-\nu}, \cdots, u_{k-\nu+m-1}$.

The remaining facts given in the theorem about which we have not discussed in the proof can be readily established just in the same way as in our previous paper.

II. Our theorem could be transferred into the 3-dimensional case, using the mean value theorem for 3-dimensional case:

$$u(p)\frac{\sin kr}{r} = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} u \sin\theta\, d\theta\, d\varphi,$$

here the integration in the right hand member being taken over the sphere around p with radius r, instead of that for 2-dimensional one:

$$u(p)\cdot J(kr) = \frac{1}{2\pi}\int_0^{2\pi} u\, d\theta$$

in the previous paper, replacing $Y_0(\sqrt{\lambda_n}\, r_{pq})$ which appeared in the expression for the Green's function by $\cos(\sqrt{\lambda_n}\, r_{pq})/r_{pq}$, and remembering Newtonian capacity for this time. The circumstances will be illustrated by following examples.

EXAMPLE 1.

Let ξ, η, φ be the prolate spheroidal coordinates such that

(19)
$$\begin{cases} x = c\sqrt{(\xi^2-1)(1-\eta^2)}\cos\varphi, \\ y = c\sqrt{(\xi^2-1)(1-\eta^2)}\sin\varphi, \\ z = c\xi\eta \end{cases}$$

in which coordinate surfaces $\xi=$const. and $\eta=$const. represent confocal spheroids and confocal hyperboloids of revolution, respectively, $2c$ being the distance between common foci. ξ varies from 1 to ∞. $\xi=1$ represents the needle $x=y=0$, $-c\leqq z\leqq c$. η varies from -1 to $+1$. $\eta=-1$ represents the part of the z-axis where $z\leqq -c$ and $\eta=+1$ represents the part $z\geqq c$ of the same axis.

Let us consider a three dimensional domain which has a prolate spheroid $\xi=\xi_1$ as its outer boundary and a needle, $x=y=0$, $-c\leqq z\leqq c$ as its inner one. Next take a sequence of confocal prolate spheroids as its exhausting boundaries.

Let C_1 and C_2 be two prolate spheroids represented by $\xi=\xi_1$ and $\xi=\xi_2$, respectively, where ξ_1 and ξ_2 are two constants such that $1<\xi_2<\xi_1$.

Let us find non-trival solution u satisfying the conditions

$$u=0 \qquad \text{on } C_1 \text{ and } C_2, \tag{20}$$

$$\Delta u+\lambda u=0 \quad \text{in} \quad D \tag{21}$$

where D denotes the domain surrounded by C_1 and C_2. In terms of coordinates ξ, η, φ, the equation (21) becomes

$$\frac{1}{c^2(\xi^2-\eta^2)}\left\{\frac{\partial}{\partial\xi}\left((\xi^2-1)\frac{\partial u}{\partial\xi}\right)+\frac{\partial}{\partial\eta}\left((1-\eta^2)\frac{\partial u}{\partial\eta}\right)\right\}+\frac{1}{c^2}\frac{1}{(\xi^2-1)(1-\eta^2)}\frac{\partial^2 u}{\partial\varphi^2}+\lambda u=0. \tag{22}$$

This equation can be solved by the method of separation of variables in putting

$$u=u_1(\xi)\,u_2(\eta)e^{il\varphi}.$$

The functions $u_1(\xi)$ and $u_2(\eta)$ satisfy the respective differential equations

$$\frac{d}{d\xi}\left((1-\xi^2)\frac{du_1}{d\xi}\right)+\left[\mu+c^2\lambda(1-\xi^2)-\frac{l^2}{1-\xi^2}\right]u_1=0 \tag{23}$$

and

$$\frac{d}{d\eta}\left((1-\eta^2)\frac{du_2}{d\eta}\right)+\left[\mu+c^2\lambda(1-\eta^2)-\frac{l^2}{1-\eta^2}\right]u_2=0 \tag{24}$$

where l is a non-negative integer and μ is a certain constant which will be determined in following considerations.

The function u_2 should be finite for $\eta=\pm1$ (that is, on the z-axis). As $\eta=\pm1$ are the singularities of the equation (24), the above condition is satisfied only for certain particular values of μ which are the eigenvalues for this singular eigenvalue problem. For a given integer l, these eigenvalues are functions of λ.

On the other hand, from the given boundary condition, we must have

$$u_1(\xi_1)=u_1(\xi_2)=0. \tag{25}$$

This leads to the fixed boundary problem for the equation (23). To solve this problem, we will make some preliminary considerations on the equation (23). This equation has two fundamental solutions represented by

$$y_{\mathrm{I}}(\xi)=(\xi-1)^{l/2}P_1(\xi-1), \tag{26}$$

$$y_{\mathrm{II}}(\xi)=(\xi-1)^{-l/2}P_2(\xi-1)+\mathrm{const}\cdot y_{\mathrm{I}}(\xi)\log(\xi-1) \tag{27}$$

where $P_1(\xi-1)$ and $P_2(\xi-1)$ are Taylor series in $\xi-1$ converging in the circle $|\xi-1|<2$ with $P_1(0)=1$ and $P_2(0)\neq 0$ [2]. First we notice that

$$\lim_{\xi\to 1} y_{\mathrm{I}}(\xi)=\begin{cases} 0 & \text{for} \quad l\geqq 1, \\ 1 & \text{for} \quad l=0, \end{cases} \tag{28}$$

$$\lim_{\xi\to 1} y_{\mathrm{II}}(\xi)=\infty. \tag{29}$$

The solution which satisfies the condition (25) can be obtained by putting

$$u=c_1 y_{\mathrm{I}}(\xi)+c_2 y_{\mathrm{II}}(\xi)$$

where the constants c_1 and c_2 are determined by the equations

$$\begin{cases} c_1 y_{\mathrm{I}}(\xi_1)+c_2 y_{\mathrm{II}}(\xi_1)=0, \\ c_1 y_{\mathrm{I}}(\xi_2)+c_2 y_{\mathrm{II}}(\xi_2)=0. \end{cases} \tag{30}$$

In order to obtain non-trivial coefficients c_1, c_2, the relation

$$\begin{vmatrix} y_{\mathrm{I}}(\xi_1) & y_{\mathrm{II}}(\xi_1) \\ y_{\mathrm{I}}(\xi_2) & y_{\mathrm{II}}(\xi_2) \end{vmatrix}=0 \tag{31}$$

must hold. For a given integer l, the left hand member of the equation (31) is a function of λ and μ. Hence, the above relation together with the condition that the functions u_2 remains finite for $\eta=\pm 1$, gives us the eigenvalues of λ and at the same time the corresponding values of μ. We then get the corresponding c_1/c_2 from the equation (30) by inserting the values of λ and μ obtained above.

Now let the inner ellipsoid $\xi=\xi_2$ tend to the needle $\xi=1$. Because $y_{\mathrm{I}}(\xi)\to\infty$ as $\xi_2\to 1$, c_2 has to tend to 0 as $\xi_2\to 1$. Therefore as $\xi_2\to 1$, the limit of eigenfunctions becomes $y_{\mathrm{I}}(\xi)$. For $l=0$, we see from (28) that the first eigenfunction tends to the function for which $u_2(1)\neq 0$. Thus our limit function of the first eigenfunction does not satisfy the zero boundary condition on the inner boundary which degenerates to a needle of capacity zero.

EXAMPLE 2.

For the example 2, we use the oblate spheroidal coordinates

$$\begin{cases} x=c\sqrt{(\xi^2+1)(1-\eta^2)}\cos\varphi, \\ y=c\sqrt{(\xi^2+1)(1-\eta^2)}\sin\varphi, \\ z=c\xi\eta \end{cases} \tag{32}$$

in which coordinate surfaces ξ=const. and η=const. represent confocal oblate spheroids and confocal hyperboloids of revolution, respectively. ξ varies from 0 to $+\infty$, $\xi=0$ corresponding to the circular disc $z=0$, $x^2+y^2\leqq c$. η varies from -1 to $+1$, $\eta=\mp 1$ corresponding to the z-axis, lower or upper, respectively.

Now let us consider the same problem concerning the domain which has an oblate spheroid $\xi=\xi_1$ as its outer boundary and a circular disc $z=0$, $x^2+y^2\leqq c$ as its inner boundary, where the inner boundary is of Newtonian capacity positive. Take a subsequence of confocal oblate spheroids as exhausting domains. We first solve the equation

$$\Delta u+\lambda u=0 \quad \text{in} \quad D \tag{33}$$

under the condition

$$u=0 \quad \text{on} \quad \xi=\xi_1, \quad \xi=\xi_2 \tag{34}$$

where $0<\xi_2<\xi_1$ and D is a domain surrounded by $\xi=\xi_1$ and $\xi=\xi_2$.

Using a method analogous to that employed in example 1, we put

$$u=u_1(\xi)\,u_2(\eta)\,e^{il\varphi}$$

where $u_1(\xi)$ and $u_2(\eta)$ satisfy the respective differential equations

$$-\frac{d}{d\xi}\left[(1+\xi^2)\frac{du_1}{d\xi}\right]+\left[\mu-c^2\lambda(1+\xi^2)-\frac{l^2}{1+\xi^2}\right]u_1=0 \tag{35}$$

and

$$\frac{d}{d\eta}\left[(1-\eta^2)\frac{du_2}{d\eta}\right]+\left[\mu-c^2\lambda(1-\eta^2)-\frac{l^2}{1-\eta^2}\right]u_2=0. \tag{36}$$

The reasoning in the example 1 is also valid except for the fact that the singularities of the equation (35) are all imaginary, that is $\xi=\pm i$. The last fact gives two fundamental solutions of (35), which are regular over our whole range $0\leqq\xi<\infty$. Let them be $y_{\mathrm{I}}(\xi)$ and $y_{\mathrm{II}}(\xi)$. The relation which determines the coefficients c_1, c_2 for the eigenfunction becomes

$$\begin{cases} c_1 y_{\mathrm{I}}(\xi_1)+c_2 y_{\mathrm{II}}(\xi_1)=0, \\ c_1 y_{\mathrm{I}}(\xi_2)+c_2 y_{\mathrm{II}}(\xi_2)=0. \end{cases} \tag{37}$$

In this case, since $y_{\mathrm{I}}(\xi)$ and $y_{\mathrm{II}}(\xi)$ remain finite for $\xi=0$, the above equation can be solved even when $\xi_2=0$. Thus, as $\xi_2\to 0$, the limiting function of the eigenfunction also satisfy the zero boundary condition on the inner boundary which degenerates to a circular disc of positive capacity. Indeed what we expected has been again checked explicitly.

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