ON A SEQUENCE OF FOURIER COEFFICIENTS

BY HIROSHI HIROKAWA

1. Introduction. Let $f(t)$ be an integrable function with period 2π and let

$$f(t) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) \equiv \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n(t). \tag{1.1}$$

Then the conjugate series of (1.1) is given by

$$\sum_{n=1}^{\infty}(b_n \cos nt - a_n \sin nt) \equiv \sum_{n=1}^{\infty} B_n(t). \tag{1.2}$$

Throughout this paper we use the following notations:

$$\varphi(t) = \varphi_x(t) = f(x+t) + f(x-t) - 2s,$$

where s is an assigned finite number;

$$\psi(t) = \psi_x(t) = f(x+t) - f(x-t) - l,$$

where l is an assigned finite number;

$$\theta(t) = \theta_x(t) = f(x+t) - f(x+t);$$

$$\Phi_\alpha(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}\varphi(u)\,du \qquad (\alpha > 0);$$

$\Psi_\alpha(t)$, $\Theta_\alpha(t)$ have similar meanings. We always suppose $\Delta \geqq 1$ and $0 < \xi < \pi$, and write $\alpha(n, k) = \bar{o}(1)$, for any function α of n and k, when and only when

$$\lim_{k\to\infty} \limsup_{n\to\infty} \alpha(n, k) = 0.$$

Recently, B. Singh [4] proved the following

THEOREM A. *If $\Psi_1(t) = o(t)$ as $t \to 0$, and*

$$\int_y^{\xi} \frac{|\psi(t+y) - \psi(t)|}{t}\,dt = o(1) \qquad as \quad y \to 0,$$

then

$$\frac{1}{n}\sum_{\nu=1}^{n} \nu B_\nu(x) \to \frac{l}{\pi} \qquad as \quad n \to \infty,$$

that is, the sequence $\{nB_n(x)\}$ is evaluable $(C, 1)$ to the value l/π.

The conditions of Theorem A are of Lebesgue type for the convergence

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of the conjugate series (1. 2). Therefore, we shall consider, under the conditions of Gergen type [1], the summability $(C, 1)$ of the sequence $\{nB_n(x)\}$. Under the conditions of this type, G. Sunouchi [5] and S. Izumi [2] proved the following theorems.

THEOREM B. *Let* $\gamma \geqq \beta > 0$ *and* $\Delta = \gamma/\beta$. *If*

$$\Phi_\beta(t) = o(t^\gamma) \qquad \text{as} \quad t \to 0, \tag{1.3}$$

and

$$\lim_{k\to\infty} \limsup_{y\to 0} \int_{(ky)^{1/\Delta}}^{\xi} \frac{|\varphi(t+y)-\varphi(t)|}{t} dt = 0, \tag{1.4}$$

then the Fourier series (1. 1) *converges to the value s at* $t = x$.

THEOREM C. *Let* $\Delta \geqq 1$. *If*

$$\int_0^t |\varphi(u)|\, du = o(t/\log t^{-1}) \qquad \text{as} \quad t \to 0,$$

and (1. 4) *holds, then the Fourier series* (1. 1) *converges to the value s at* $t = x$.

Concerning Theorem A, we shall prove the following theorems.

THEOREM 1. *Let* $\gamma \geqq \beta > 0$ *and* $\Delta = \gamma/\beta$. *If*

$$\Psi_\beta(t) = o(t^\gamma) \qquad \text{as} \quad t \to 0, \tag{1.5}$$

and

$$\lim_{k\to\infty} \limsup_{y\to 0} \int_{(ky)^{1/\Delta}}^{\xi} \frac{|\psi(t+y)-\psi(t)|}{t} dt = 0, \tag{1.6}$$

then the sequence $\{nB_n(x)\}$ *is evaluable* $(C, 1)$ *to the value* l/π.

THEOREM 2. *Let* $\Delta \geqq 1$. *If*

$$\int_0^t |\psi(u)|\, du = o(t/\log t^{-1}) \qquad \text{as} \quad t \to 0, \tag{1.7}$$

and (1. 6) *holds, then the sequence* $\{nB_n(x)\}$ *is evaluable* $(C, 1)$ *to the value* l/π.

Following the method of R. Mohanty and M. Nanda [3], we get the following convergence criteria for the conjugate series (1. 2) as corollaries of Theorems 1 and 2.

THEOREM 3. *Let* $\gamma \geqq \beta > 0$ *and* $\Delta = \gamma/\beta$. *If*

$$\Theta_\beta(t) = o(t^\gamma) \qquad \text{as} \quad t \to 0,$$

and

$$\lim_{k\to\infty} \limsup_{y\to 0} \int_{(ky)^{1/\Delta}}^{\xi} \frac{|\theta(t+y)-\theta(t)|}{t} dt = 0, \tag{1.8}$$

then the conjugate series (1.2), *at* $t=x$, *converges to the value*

$$\frac{1}{2\pi}\int_{\to 0}^{\pi} \theta(t)\cot\frac{t}{2}\,dt \tag{1.9}$$

provided that the integral exists as a Cauchy integral at the origin.

THEOREM 4. *Let* $\Delta \geqq 1$. If

$$\int_0^t |\theta(u)|\,du = o(t/\log t^{-1}) \qquad \text{as} \quad t\to 0,$$

and (1.8) *holds, then the conjugate series* (1.2), *at* $t=x$, *converges to the value* (1.9) *provided that the integral exists as a Cauchy integral at the origin.*

2. Preliminary Lemmas.

LEMMA 1. *Under the conditions of Theorem* 1, *we have, for integer* ν, $1\leqq\nu\leqq[\beta]+1$,

$$\Psi_\nu(t)=o(t^{1+(\nu-1)\Delta}). \tag{2.1}$$

This is due to S. Izumi [2; Lemma 1].

LEMMA 2. *Let* $0<\tau\leqq 1$ *and let* $0\leqq u<v<\infty$. *Then we have*

$$\int_u^v (t-u)^{\tau-1}e^{int}dt=o(n^{-\tau}). \tag{2.2}$$

This is due to G. Sunouchi [6; Lemma 1].

LEMMA 3. *Let* $\sigma(n,t)=\dfrac{\sin nt}{nt^2}-\dfrac{\cos nt}{t}$. *Then we have*

$$\frac{d^m}{dt^m}\sigma(n,t)=O\Big(\sum_{j=-1}^{m} n^j/t^{m-j+1}\Big) \qquad (m=0,1,2,\cdots), \tag{2.3}$$

and, for $0<\tau\leqq 1$, $0<u<v<\infty$,

$$\int_u^v (t-u)^{\tau-1}\frac{d^m}{dt^m}\sigma(n,t)\,dt=O\Big(\sum_{j=-1}^{m} n^{j-\tau}u^{j-m-1}\Big) \quad (m=0,1,2,\cdots). \tag{2.4}$$

Proof. By Leibniz formula, we have

$$\frac{d^m}{dt^m}\sigma(n,t)=\frac{1}{n}\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}(m-j+1)!\frac{n^j}{t^{m-j+2}}\sin(nt+j\pi/2)$$
$$-\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}(m-j)!\frac{n^j}{t^{m-j+1}}\cos(nt+j\pi/2).$$

Thus we get

$$\frac{d^m}{dt^m}\sigma(n,t)=O\Big(\sum_{j=0}^{m} n^{j-1}/t^{m-j+2}\Big)+O\Big(\sum_{j=0}^{m} n^j/t^{m-j+1}\Big)=O\Big(\sum_{j=-1}^{m} n^j/t^{m-j+1}\Big),$$

which is (2.3). Now, using the second mean value theorem, we have

$$\int_u^v (t-u)^{\tau-1}\frac{e^{int}}{t^{m-j+1}}dt=\frac{1}{u^{m-j+1}}\int_u^{v'}(t-u)^{\tau-1}e^{int}dt \qquad (u<v'<v)$$
$$=O(1/n^\tau u^{m-j+1})$$

by (2. 2). Hence, from (2. 5),

$$\int_u^v (t-u)^{\tau-1}\frac{d^m}{dt^m}\sigma(n,t)\,dt=\frac{1}{n}\sum_{j=0}^m(-1)^{m-j}\binom{m}{j}(m-j+1)!\,n^j\int_u^v (t-u)^{\tau-1}\frac{\sin(nt+j\pi/2)}{t^{m-j+2}}dt$$
$$-\sum_{j=0}^m(-1)^{m-j}\binom{m}{j}(m-j)!\,n^j\int_u^v (t-u)^{\tau-1}\frac{\cos(nt+j\pi/2)}{t^{m-j+1}}dt$$
$$=O\Big(\sum_{j=0}^m n^{j-\tau-1}u^{j-m-2}\Big)+O\Big(\sum_{j=0}^m n^{j-\tau}u^{j-m-1}\Big)=O\Big(\sum_{j=-1}^m n^{j-\tau}u^{j-m-1}\Big)$$

which is the result required.

The following three Lemmas can be proved analogously to the proofs of Singh's lemmas [4; Lemmas 1, 2 and 3].

LEMMA 4. *If $\Psi_1(t)=o(t)$ as $t\to 0$, then, for every positive integer k,*

$$\int_{(k\pi/n)^{1/\Delta}}^{\xi}\Big(\frac{1}{t^2}-\frac{1}{t(t+\pi/n)}\Big)\psi(t)\sin nt\,dt=o(n^{1/\Delta}) \qquad as \quad n\to\infty.$$

LEMMA 5. *$\Psi_1(t)=o(t)$, then, for every positive integer k,*

$$\int_{(k\pi/n)^{1/\Delta}}^{\eta}\frac{\psi(t)}{t}e^{int}dt=o(1) \qquad as \quad n\to\infty,$$

where $(k\pi/n)^{1/\Delta}<\eta\leqq(k\pi/n)^{1/\Delta}+\pi/n$.

LEMMA 6. *If $\Psi_1(t)=o(t)$ and (1. 6) holds, then*

$$\int_{(k\pi/n)^{1/\Delta}}^{\xi}\frac{\psi(t)}{t(t+\pi/n)}e^{int}dt=\bar{o}(n^{1/\Delta}) \qquad as \quad n\to\infty.$$

LEMMA 7. *If $\sum u_n$ is Abel evaluable, then a necessary and sufficient condition that it should be convergent is that the sequence $\{nu_n\}$ is evaluable* (C, 1) *to the value zero.*

This Lemma is well-known as Tauber's second theorem.

3. Proof of Theorem 1. From the method of Mohanty and Nanda [3], we have

$$\frac{1}{n}\sum_{\nu=1}^n \nu B_\nu(x)-\frac{l}{\pi}=\frac{1}{\pi}\int_0^\pi\{f(x+t)-f(x-t)-l\}g(n,t)\,dt+o(1)$$
$$=\frac{1}{\pi}\int_0^\pi\psi(t)g(n,t)\,dt+o(1)=\frac{1}{\pi}P+o(1),$$

say, where

$$g(n,t)=-\frac{1}{n}\frac{d}{dt}\{\cos t+\cos 2t+\cdots+\cos nt\}$$
$$=\Big\{\frac{1}{4n}\frac{\sin nt}{\sin^2 t/2}-\frac{1}{2}\frac{\cos nt}{\tan t/2}\Big\}+\frac{1}{2}\sin nt.$$

Now, we have, when $n \to \infty$,

$$P=\int_0^\pi \psi(t)\left\{\frac{\sin nt}{4n\sin^2 t/2}-\frac{1}{2}\frac{\cos nt}{\tan t/2}\right\}dt+\frac{1}{2}\int_0^\pi \psi(t)\sin nt\,dt$$

$$(3.1)\qquad =\int_0^\pi \psi(t)\left\{\frac{\sin nt}{nt^2}-\frac{\cos nt}{t}\right\}dt+o(1)$$

$$=\left(\int_0^{k\pi/n}+\int_{k\pi/n}^{(k\pi/n)^{1/\Delta}}+\int_{(k\pi/n)^{1/\Delta}}^{\xi}\right)\psi(t)\sigma(n,t)\,dt+o(1)$$

$$=P_1+P_2+P_3+o(1),$$

say, where

$$\sigma(n,t)=\frac{\sin nt}{nt^2}-\frac{\cos nt}{t}.$$

Since, when $0<nt<C$, C being a positive constant,

$$\sigma(n,t)=O(n^2 t)$$

and

$$\frac{d}{dt}\sigma(n,t)=2\frac{\cos nt}{t^2}-2\frac{\sin nt}{nt^3}+\frac{n\sin nt}{t}=O(n^2),$$

we have, by (2.1),

$$P_1=[\Psi_1(t)\sigma(n,t)]_0^{k\pi/n}-\int_0^{k\pi/n}\Psi_1(t)\frac{d}{dt}\sigma(n,t)\,dt$$

$$=o\left(\frac{k\pi}{n}\cdot n^2\cdot\frac{k\pi}{n}\right)+o\left(\int_0^{k\pi/n} n^2 t\,dt\right)=o(1).$$

Applying Lemmas 4, 5 and 6, we have

$$P_3=\frac{1}{n}\int_{(k\pi/n)^{1/\Delta}}^{\xi}\frac{\psi(t)}{t^2}\sin nt\,dt-\int_{(k\pi/n)^{1/\Delta}}^{\xi}\frac{\psi(t)}{t}\cos nt\,dt$$

$$=\frac{1}{n}\int_{(k\pi/n)^{1/\Delta}}^{\xi}\frac{\psi(t)\sin nt}{t(t+\pi/n)}dt-\int_{(k\pi/n)^{1/\Delta}}^{\xi}\frac{\psi(t)}{t}\cos nt\,dt+o(1)$$

$$=-\int_{(k\pi/n)^{1/\Delta}}^{\xi}\frac{\psi(t)}{t}\cos nt\,dt+\bar{o}(1)$$

$$=-\left(\int_{(k\pi/n)^{1/\Delta}}^{(k\pi/n)^{1/\Delta}+\pi/n}+\int_{(k\pi/n)^{1/\Delta}+\pi/n}^{\xi+\pi/n}-\int_{\xi}^{\xi+\pi/n}\right)\frac{\psi(t)}{t}\cos nt\,dt+\bar{o}(1)$$

$$=-\int_{(k\pi/n)^{1/\Delta}+\pi/n}^{\xi+\pi/n}\frac{\psi(t)}{t}\cos nt\,dt+\bar{o}(1)$$

$$=\int_{(k\pi/n)^{1/\Delta}}^{\xi}\frac{\psi(t+\pi/n)}{t+\pi/n}\cos nt\,dt+\bar{o}(1)$$

$$=\frac{1}{2}\int_{(k\pi/n)^{1/\Delta}}^{\xi}\left\{\frac{\psi(t+\pi/n)}{t+\pi/n}-\frac{\psi(t)}{t}\right\}\cos nt\,dt+\bar{o}(1).$$

Then we have, again using Lemma 6,

$$|P_3| \leqq \frac{1}{2}\left|\int_{(k\pi/n)^{1/\Delta}}^{\xi} \frac{\psi(t+\pi/n)-\psi(t)}{t+\pi/n}\cos nt\,dt\right| + \frac{\pi}{2n}\left|\int_{(k\pi/n)^{1/\Delta}}^{\xi} \frac{\psi(t)\cos nt}{t(t+\pi/n)}dt\right| + \bar{o}(1)$$

$$\leqq \frac{1}{2}\int_{(k\pi/n)^{1/\Delta}}^{\xi} \frac{|\psi(t+\pi/n)-\psi(t)|}{t}dt + \frac{\pi}{2n}\left|\int_{(k\pi/n)^{1/\Delta}}^{\xi} \frac{\psi(t)\cos nt}{t(t+\pi/n)}dt\right| + \bar{o}(1)$$

$$= \bar{o}(1) + \frac{1}{2n}\bar{o}(n) + \bar{o}(1) = \bar{o}(1).$$

Thus, when $\Delta=1$, we have $P=\bar{o}(1)$, since P_2 does not appear in P. When $\Delta>1$, we shall prove $P_2=\bar{o}(1)$. Let β be not an integer and let $[\beta]=\mu$. Then, by integration by parts, we have

$$P_2 = \int_{k\pi/n}^{(k\pi/n)^{1/\Delta}} \psi(t)\sigma(n,t)\,dt$$

$$= \left[\sum_{\nu=1}^{\mu+1}(-1)^{\nu+1}\Psi_\nu(t)\frac{d^{\nu-1}}{dt^{\nu-1}}\sigma(n,t)\right]_{k\pi/n}^{(k\pi/n)^{1/\Delta}} + (-1)^\mu\int_{k\pi/n}^{(k\pi/n)^{1/\Delta}} \Psi_{\mu+1}(t)\frac{d^{\mu+1}}{dt^{\mu+1}}\sigma(n,t)\,dt$$

$$= P_{21} + (-1)^\mu P_{22},$$

say, where, by (2. 1) and (2. 3),

$$P_{21} = o\left(\sum_{\nu=1}^{\mu+1} n^{-\{1+(\nu-1)\Delta\}/\Delta}\sum_{j=-1}^{\nu-1} n^{j+(\nu-j)/\Delta}\right) + o\left(\sum_{\nu=1}^{\mu+1} n^{-1-(\nu-1)\Delta+\nu}\right)$$

$$= o\left(\sum_{\nu=1}^{\mu+1}\sum_{j=-1}^{\nu-1} n^{(j+1-\nu)(1-1/\Delta)}\right) + o\left(\sum_{\nu=1}^{\mu+1} n^{(\nu-1)(1-\Delta)}\right) = o(1).$$

Now, omitting the constant factor, we have

$$P_{22} = \int_{k\pi/n}^{(k\pi/n)^{1/\Delta}} \frac{d^{\mu+1}}{dt^{\mu+1}}\sigma(n,t)\,dt\int_0^t \Psi_\beta(u)(t-u)^{\mu-\beta}\,du$$

$$= \int_0^{k\pi/n} \Psi_\beta(u)du\int_{k\pi/n}^{(k\pi/n)^{1/\Delta}} (t-u)^{\mu-\beta}\frac{d^{\mu+1}}{dt^{\mu+1}}\sigma(n,t)\,dt$$

$$+ \int_{k\pi/n}^{(k\pi/n)^{1/\Delta}} \Psi_\beta(u)\,du\int_u^{(k\pi/n)^{1/\Delta}} (t-u)^{\mu-\beta}\frac{d^{\mu+1}}{dt^{\mu+1}}\sigma(n,t)\,dt$$

$$= P_{221} + P_{222},$$

say. Then we have, by (1. 5) and (2. 4),

$$P_{221} = o\left(\int_0^{k\pi/n} n^{\beta+1}u^\gamma\,du\right) = o(n^{\beta+1-(\gamma+1)}) = o(1)$$

and

$$P_{222} = o\left(\sum_{j=-1}^{\mu+1} n^{j-\mu+\beta-1}\int_{k\pi/n}^{(k\pi/n)^{1/\Delta}} u^{\gamma+j-\mu-2}du\right)$$

$$= o\left(\sum_{j=-1}^{\mu+1} n^{j-\mu+\beta-1-(\gamma+j-\mu-1)/\Delta} - \sum_{j=-1}^{\mu+1} n^{j-\mu+\beta-1-(\gamma+j-\mu-1)}\right)$$

$$= o\left(\sum_{j=-1}^{\mu+1} n^{(j-\mu-1)(1-1/\Delta)}\right) + o(1) = o(1).$$

Thus we get $P_2=o(1)$ and the proof of theorem is complete when β is not an integer. When β is an integer, the proof is similar to the above argument. For the proof, it is sufficient to prove that $P_2=o(1)$. By integration by parts, we have

$$P_2=\left[\sum_{\nu=1}^{\beta}(-1)^{\nu+1}\Psi_\nu(t)\frac{d^{\nu-1}}{dt^{\nu-1}}\sigma(n,t)\right]_{k\pi/n}^{(k\pi/n)^{1/\Delta}}+(-1)^{\beta+1}\int_{k\pi/n}^{(k\pi/n)^{1/\Delta}}\Psi_\beta(t)\frac{d^\beta}{dt^\beta}\sigma(n,t)\,dt$$

$$=o(1)+(-1)^{\beta+1}P_2',$$

where, by (1. 5) and (2. 3),

$$P_2'=o\left(\sum_{j=-1}^{\beta}n^j\int_{k\pi/n}^{(k\pi/n)^{1/\Delta}}t^{\gamma-\beta+j-1}dt\right)$$

$$=o\left(\sum_{j=-1}^{\beta}n^{j-(\gamma-\beta+j)/\Delta}\right)+o\left(\sum_{j=-1}^{\beta}n^{j-\gamma+\beta-j}\right)$$

$$=o\left(\sum_{j=-1}^{\beta}n^{j-\beta-(j-\beta)/\Delta}\right)+o(n^{\beta-\gamma})=o(1),$$

which is the result required and the proof of theorem is complete.

4. Proof of Theorem 2. The method of proof is similar to that of Theorem 1. Since (1. 7) implies that $\Psi_1(t)=o(t)$, for the proof, it is sufficient to prove that $P_2=o(1)$, where P_2 is found in (3. 1). Integration by parts gives

$$P_2=\int_{k\pi/n}^{(k\pi/n)^{1/\Delta}}\psi(t)\sigma(n,t)\,dt$$

$$=O\left(\int_{k\pi/n}^{(k\pi/n)^{1/\Delta}}|\psi(t)|\,t^{-1}dt\right)$$

$$=O\left(\left[t^{-1}\int_0^t|\psi(u)|\,du\right]_{k\pi/n}^{(k\pi/n)^{1/\Delta}}+\int_{k\pi/n}^{(k\pi/n)^{1/\Delta}}t^{-2}\left(\int_0^t|\psi(u)|\,du\right)dt\right)$$

$$=o(1)+o\left(\int_{k\pi/n}^{(k\pi/n)^{1/\Delta}}\frac{t}{\log(1/t)}\cdot\frac{1}{t^2}dt\right)$$

$$=o(1)+o\left(\log\frac{(1/\Delta)\log(k\pi/n)}{\log(k\pi/n)}\right)=o(1),$$

which is the result required and theorem is proved.

5. Proof of Theorems 3 and 4. The existence of the integral (1. 9) as a Cauchy integral at the origin implies the Abel summability of the conjugate series (1. 2) at $t=x$. (See [7; p. 55].) By Theorems 1 and 2, we find that the conditions of Theorems 3 and 4 imply the summability $(C,1)$ of the sequence $\{nB_n(x)\}$ to the value zero. Now, the convergence of the conjugate series (1. 2) at $t=x$ is a consequence of Lemma 7.

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CHIBA INSTITUTE OF TECHNOLOGY,
CHIBA, JAPAN.