ON A COEFFICIENT PROBLEM FOR ANALYTIC FUNCTIONS TYPICALLY-REAL IN AN ANNULUS

BY HAN NISHIMIYA

Considering a class of analytic functions regular and univalent in an annulus whose Laurent coefficients are all real, Nehari and Schwarz [3] have given an estimation of Laurent coefficients for any function of the class. As they have also remarked there, the result remains valid for a slightly wider class, i. e. the class consisting of functions regular and typically-real in an annuls. Komatu [2] has then ameliorated the bound of this estimation by modifying Szász' method previously used for a similar problem on Taylor series. The result obtained is precise for all coefficients, as shown by extremal function.

In the present paper, we first introduce an integral representation valid for any function of the class under consideration.

Though it is a simple consequence of a well-known Villat's formula, we state its proof fully for the sake of completeness. By making use of this representation, we give an alternative proof for the estimation of the coefficients due to Komatu. Finally, it is shown that our present method enables us to determine all possible functions which are extremal for our coefficient problem.

THEOREM 1. *Let*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \tag{1}$$

be a single-valued analytic function which is regular and typically-real in an annulus $q < |z| < 1$ *and satisfies* $\Im f(z) \cdot \Im z > 0$ *for* $\Im z \neq 0$. *Then there holds an integral representation*

$$\begin{aligned} f(z) = &\int_0^\pi \frac{\zeta(i \lg z + \varphi) - \zeta(i \lg z - \varphi)}{2 \sin \varphi} d\rho(\varphi) \\ &- \int_0^\pi \frac{\zeta_3(i \lg z - \varphi) - \zeta_3(i \lg z - \varphi)}{2 \sin \varphi} d\tau(\varphi) + c \end{aligned} \tag{2}$$

where $\rho(\varphi)$ *and* $\tau(\varphi)$ *are real-valued functions satisfying the conditions*

$$\begin{aligned} d\rho(\varphi) &\geqq 0, \qquad \int_0^\pi d\rho(\varphi) = a_1 - a_{-1}; \\ d\tau(\varphi) &\geqq 0, \qquad \int_0^\pi d\tau(\varphi) = a_1 q - a_{-1} q^{-1} \end{aligned} \tag{3}$$

Received March 28, 1957.

and c is a real constant. The zeta-functions are those from Weierstrassian theory of elliptic functions depending on the primitive periods $2\omega_1 = 2\pi$ and $2\omega_3 = -2i \lg q$.

Proof.[1] Applying the Villat's formula to the function $-if(z)$ which is regular and single-valued in $t \leqq |z| \leqq r$ with $q < t < r < 1$, we get

$$f(z) = \frac{1}{\pi}\int_0^\pi \Im f(re^{i\varphi})\Big(\zeta\Big(i \lg \frac{z}{r} + \varphi; \frac{t}{r}\Big) - \zeta\Big(i \lg \frac{z}{r} - \varphi; \frac{t}{r}\Big)\Big) d\varphi$$

$$- \frac{1}{\pi}\int_0^\pi \Im f(te^{i\varphi})\Big(\zeta_3\Big(i \lg \frac{z}{r} + \varphi; \frac{t}{r}\Big) - \zeta_3\Big(i \lg \frac{z}{r} - \varphi; \frac{t}{r}\Big)\Big) d\varphi + c$$

where c is a real constant. Here the parameter t/r associated to zeta-functions means that they depend on the quasi-periods 2π and $-2i\lg(t/r)$. We put, for $q < r < 1$ and $0 \leqq \varphi \leqq \pi$,

$$\rho(r, \varphi) = \frac{2}{\pi}\int_0^\varphi \sin\theta\, \Im f(re^{i\theta})\, d\theta.$$

Then the assumption $\Im f(z) \cdot \Im z \geqq 0$ implies

$$d\rho(r, \varphi) \geqq 0 \qquad (0 \leqq \varphi \leqq \pi)$$

and it is readily shown that there holds, for any r with $q < r < 1$,

$$\rho(r, \pi) - \rho(r, 0) = \rho(r, \pi) = a_1 r - a_{-1} r^{-1} \quad (< a_1 - a_{-1}).$$

Namely $\rho(r, \varphi)$ is, for any fixed r, an increasing function with respect to φ. It is uniformly bounded in $q < r < 1$ and its total variation with respect to φ is also uniformly bounded. Accordingly, based on a theorem due to Helly, we can choose an increasing sequence $\{r_k\}$ with $r_k \to 1$ and a decreasing sequence $\{t_k\}$ with $t_k \to q$ such that there exist limit functions

$$\rho(\varphi) = \lim_{k\to\infty} \rho(r_k, \varphi) \qquad \text{and} \qquad \tau(\varphi) = \lim_{k\to\infty} \rho(t_k, \varphi)$$

for all values of φ with $0 \leqq \varphi \leqq \pi$. Evidently, $\rho(\varphi)$ and $\tau(\varphi)$ satisfy the condition (3). Now, on account of a Lebesgue's theorem, since the derivative of $\zeta(u + \varphi;\, q_k)$ with respect to φ is continuous in φ as well as in q_k, we obtain

$$\lim_{k\to\infty}\int_0^\pi \frac{1}{2\sin\varphi}\Big(\zeta\Big(i \lg \frac{z}{r_k} + \varphi; \frac{t_k}{r_k}\Big) - \zeta\Big(i \lg \frac{z}{r_k} - \varphi; \frac{t_k}{r_k}\Big)\Big) d\rho(r_k, \varphi)$$

$$= \lim_{k\to\infty}\Big\{\Big[\frac{1}{2\sin\varphi}\Big(\zeta\Big(i \lg \frac{z}{r_k} + \varphi; \frac{t_k}{r_k}\Big) - \zeta\Big(i \lg \frac{z}{r_k} - \varphi; \frac{t_k}{r_k}\Big)\Big)\rho(r_k, \varphi)\Big]_0^\pi\Big\}$$

$$- \int_0^\pi \rho(r_k, \varphi)\frac{\partial}{\partial\varphi}\frac{1}{2\sin\varphi}\Big(\zeta\Big(i \lg \frac{z}{r_k} + \varphi; \frac{t_k}{r_k}\Big) - \zeta\Big(i \lg \frac{z}{r_k} - \varphi; \frac{t_k}{r_k}\Big)\Big) \cdot d\varphi$$

$$= \Big[\frac{\zeta(i \lg z + \varphi) - \zeta(i \lg z - \varphi)}{2\sin\varphi}\rho(\varphi)\Big]_0^\pi$$

1) The proof given in the following lines is a modification of the proof which has been given by Komatu [1] for deriving a similar integral representation.

$$-\int_0^{\pi}\rho(\varphi)\frac{\partial}{\partial\varphi}\frac{\zeta(i\lg z+\varphi)-\zeta(i\lg z-\varphi)}{2\sin\varphi}\,d\varphi$$

$$=\int_0^{\pi}\frac{\zeta(i\lg z+\varphi)-\zeta(i\lg z-\varphi)}{2\sin\varphi}\,d\rho(\varphi).$$

Similarly we get

$$\lim_{k\to\infty}\int_0^{\pi}\frac{1}{2\sin\varphi}\left(\zeta_3\Big(i\lg\frac{z}{r_k}+\varphi;\frac{t_k}{r_k}\Big)-\zeta_3\Big(i\lg\frac{z}{r_k}-\varphi;\frac{t_k}{r_k}\Big)\right)d\rho(t_k,\varphi)$$

$$=\int_0^{\pi}\frac{\zeta_3(i\lg z+\varphi)-\zeta_3(i\lg z-\varphi)}{2\sin\varphi}\,d\tau(\varphi).$$

Thus the representation (2) has been proved.

From theorem 1 there follows readily an integral representation for the coefficients of any function of the class, as mentioned in the following theorem.

THEOREM 2. *Let $\mathfrak{F}$ be the class of functions*

$$f(z)=\sum_{n=-\infty}^{\infty}a_nz^n$$

which are regular and typically-real in an annulus $q<|z|<1$ and satisfy $\Im f(z)\cdot\Im z>0$ for $\Im z\neq 0$. Then, for any function $f(z)\in\mathfrak{F}$, the Laurent coefficients are represented by

$$a_n=\frac{1}{1-q^{2n}}\left\{\int_0^{\pi}\frac{\sin n\varphi}{\sin\varphi}\,d\rho(\varphi)-q^n\int_0^{\pi}\frac{\sin n\varphi}{\sin\varphi}\,d\tau(\varphi)\right\}\qquad(n\neq 0)\tag{4}$$

where $\rho(\varphi)$ and $\tau(\varphi)$ are functions satisfying the condition (3).

Proof. Two integrands involved in the representation (2) are developed in the Laurent series

$$\frac{\zeta(i\lg z+\varphi)-\zeta(i\lg z-\varphi)}{2\sin\varphi}=\frac{\eta_1\varphi}{\pi\sin\varphi}+\sum_{n=-\infty}^{\infty}{}'\frac{1}{1-q^{2n}}\frac{\sin n\varphi}{\sin\varphi}z^n\tag{5}$$

and

$$\frac{\zeta_3(i\lg z+\varphi)-\zeta_3(i\lg z-\varphi)}{2\sin\varphi}=\frac{\eta_1\varphi}{\pi\sin\varphi}+\sum_{n=-\infty}^{\infty}{}'\frac{q^n}{1-q^{2n}}\frac{\sin n\varphi}{\sin\varphi}z^n,\tag{6}$$

respectively, every summation extending over all integers n except $n=0$. By substituting (5) and (6) into (2) and then taking into account that the termwise integration is legitimate, we obtain

$$f(z)=\sum_{n=-\infty}^{\infty}{}'z^n\frac{1}{1-q^{2n}}\left\{\int_0^{\pi}\frac{\sin n\varphi}{\sin\varphi}\,d\rho(\varphi)-q^n\int_0^{\pi}\frac{\sin n\varphi}{\sin\varphi}\,d\tau(\varphi)\right\}+a_0,$$

whence follows the desired representation (4).

The explicit representation for the Laurent coefficients having been established, the estimation from above and below can be easily performed. It gives an alternative proof of Komatu's result [2] which may be re-stated as in the following theorem.

THEOREM 3. *For any function* $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ *of the class* $\mathfrak{F}$, *there holds a set of inequalities*

$$
(7)\qquad \begin{aligned}
-\frac{n}{1-q^{2n}}\left(\frac{\kappa_n}{n}(a_1-a_{-1})+(a_1q-a_{-1}q^{-1})\,q^n\right) &\leqq a_n \\
\leqq \frac{n}{1-q^{2n}}\left(a_1-a_{-1}+\frac{\kappa_n}{n}(a_1q-a_{-1}q^{-1})\,q^n\right) & \qquad (n=\pm 1, \pm 2, \cdots)
\end{aligned}
$$

where κ_n *is defined by*

$$
(8)\qquad \kappa_{\pm 1} = \pm 1, \qquad \kappa_n = -\operatorname*{Min}_{0\leqq\varphi\leqq\pi} \frac{\sin n\varphi}{\sin\varphi}, \qquad \kappa_{-n} = -\kappa_n \qquad (n=2,3,\cdots).
$$

Proof. By means of the representation (4), any coefficient with a positive suffix is readily estimated by

$$
(9)\qquad \begin{aligned}
\frac{1}{1-q^{2n}}\left\{\operatorname*{Min}_{0\leqq\varphi\leqq\pi}\frac{\sin n\varphi}{\sin\varphi}\int_0^\pi d\rho(\varphi) - q^n \operatorname*{Max}_{0\leqq\varphi\leqq\pi}\frac{\sin n\varphi}{\sin\varphi}\int_0^\pi d\tau(\varphi)\right\} &\leqq a_n \\
\leqq \frac{1}{1-q^{2n}}\left\{\operatorname*{Max}_{0\leqq\varphi\leqq\pi}\frac{\sin n\varphi}{\sin\varphi}\int_0^\pi d\rho(\varphi) - q^n \operatorname*{Min}_{0\leqq\varphi\leqq\pi}\frac{\sin n\varphi}{\sin\varphi}\int_0^\pi d\tau(\varphi)\right\}& \\
(n>0),&
\end{aligned}
$$

since $\rho(\varphi)$ and $\tau(\varphi)$ satisfy the condition (3) and $1/(1-q^{2n}) > 0$ for any $n>0$. To obtain a corresponding inequality for coefficient with a negative suffix, we remark an identity

$$
\frac{1}{1-q^{2n}}\,\frac{\sin n\varphi}{\sin\varphi} = q^{-2n}\frac{1}{1-q^{-2n}}\,\frac{\sin(-n\varphi)}{\sin\varphi}.
$$

The representation (4) then leads to

$$
(10)\qquad \begin{aligned}
\frac{q^{-2n}}{1-q^{-2n}}\left\{\operatorname*{Min}_{0\leqq\varphi\leqq\pi}\frac{\sin(-n\varphi)}{\sin\varphi}\int_0^\pi d\rho(\varphi) - q^n \operatorname*{Max}_{0\leqq\varphi\leqq\pi}\frac{\sin(-n\varphi)}{\sin\varphi}\int_0^\pi d\tau(\varphi)\right\} &\leqq a_n \\
\leqq \frac{q^{-2n}}{1-q^{-2n}}\left\{\operatorname*{Max}_{0\leqq\varphi\leqq\pi}\frac{\sin(-n\varphi)}{\sin\varphi}\int_0^\pi d\rho(\varphi) - q^n \operatorname*{Min}_{0\leqq\varphi\leqq\pi}\frac{\sin(-n\varphi)}{\sin\varphi}\int_0^\pi d\tau(\varphi)\right\}& \\
(n<0).&
\end{aligned}
$$

Thus, (9) and (10) show, in view of the definition (8), that the set of inequalities (7) holds good.

We are now in position to state the main result of the present paper, showing that the whole of extremal functions for our coefficient problem can be completely determined.

THEOREM 4. *The estimation established in theorem* 3 *is precise. For every* *fixed n with* $|n| > 1$, *possible extremal functions for upper bound are exhausted by*

$$f^*(z;\psi_n,\lambda,\mu) = \sum_{m=-\infty}^{\infty}{}' \frac{1}{1-q^{2m}}\Big(m(\lambda+(1-\lambda)(-1)^{m-1})(a_1-a_{-1})$$
$$-\frac{\sin m\psi_n}{\sin\psi_n}(\mu+(1-\mu)(-1)^{m-1})(a_1q-a_{-1}q^{-1})\,q^m\Big)z^m + a_0$$

and those fer lower bound by

$$f_*(z;\psi_n,\lambda,\mu) = -f^*\left(\frac{q}{z};\psi_n,\lambda,\mu\right)$$
$$= \sum_{m=-\infty}^{\infty}{}' \frac{1}{1-q^{2m}}\Big(\frac{\sin m\psi_n}{\sin\psi_n}(\mu+(1-\mu)(-1)^{m-1})(a_1-a_{-1})$$
$$-m(\lambda+(1-\lambda)(-1)^{m-1})(a_1q-a_{-1}q^{-1})\,q^m\Big)z^m + a_0.$$

Here λ *and* μ *are parameters which are both equal to* 1 *for even n, while they may be any non-negative numbers not exceeding* 1 *for odd n. On the other hand, for even n,* ψ_n *is equal to* π *while, for odd n, it is determined by*

$$\kappa_n = -\frac{\sin n\psi_n}{\sin\psi_n}, \qquad \frac{\pi}{|n|} < \psi_n < \frac{2\pi}{|n|}.$$

Remark. The dependence of f^* as well as f_* on the parameters λ and μ is apparent when n is even, while these parameters generate really a family of extremal functions when n is odd. On the other hand, for even n, ψ_n is equal to π so that $\sin m\psi_n/\sin\psi_n$ must be understood to be equal to $(-1)^{m-1}m$. Consequently, the extremal functions for *even n* are unique except additive constants and really given by

$$f^*(z;\pi,1,1) = \sum_{m=-\infty}^{\infty}{}' \frac{m}{1-q^{2m}}(a_1-a_{-1}+(-1)^m(a_1q-a_{-1}q^{-1})\,q^m)\,z^m + a_0$$

and

$$f_*(z;\pi,1,1) = \sum_{m=-\infty}^{\infty}{}' \frac{m}{1-q^{2m}}((-1)^{m-1}(a_1-a_{-1})-(a_1q-a_{-1}q^{-1})\,q^m)\,z^m + a_0,$$

respectively. In particular, these are also independent of n.

Proof. In order to determine the extremal function for upper bound, we distinguish two cases according to the parity of n. When n is even, the equality sign in the inequality

$$\frac{1}{1-q^{2n}}\frac{\sin n\varphi}{\sin\varphi} \leqq \frac{n}{1-q^{2n}} \qquad (0 \leqq \varphi \leqq \pi)$$

holds only for $\varphi = 0$. Thus, since

$$d\rho(\varphi) \geqq 0 \quad \text{for} \quad 0 \leqq \varphi \leqq \pi \qquad \text{and} \qquad \int_0^\pi d\rho(\varphi) = a_1 - a_{-1},$$

the equality

$$\frac{1}{1-q^{2n}}\int_0^\pi \frac{\sin n\varphi}{\sin\varphi}\,d\rho(\varphi)=\frac{n}{1-q^{2n}}(a_1-a_{-1})$$

holds if and only if $d\rho(\varphi)=0$ except at $\varphi=0$ where $\rho(\varphi)$ possesses a jump with the height a_1-a_{-1}. Similarly, the equality

$$-\frac{q^n}{1-q^{2n}}\int_0^\pi \frac{\sin n\varphi}{\sin\varphi}\,d\tau(\varphi)=\frac{nq^n}{1-q^{2n}}(a_1q-a_{-1}q^{-1})$$

holds if and only if $d\tau(\varphi)=0$ except at $\varphi=\pi$ where $\tau(\varphi)$ possesses a jump with the height $a_1q-a_{-1}q^{-1}$. Hence the extremal function is given by

$$\begin{aligned}f^*(z)&=\left[\frac{\zeta(i\lg z+\varphi)-\zeta(i\lg z-\varphi)}{2\sin\varphi}\right]^{\varphi=0}(a_1-a_{-1})\\&\quad-\left[\frac{\zeta_3(i\lg z+\varphi)-\zeta_3(i\lg z-\varphi)}{2\sin\varphi}\right]^{\varphi=\pi}(a_1q-a_{-1}q^{-1})\\&=-\wp(i\lg z)(a_1-a_{-1})-\wp(i\lg z+\pi)(a_1q-a_{-1}q^{-1})\end{aligned}$$

which, expanded in Laurent series, coincides with $f^*(z;\pi,1,1)$. On the other hand, when n is odd, the equality sign in

$$\frac{1}{1-q^{2n}}\frac{\sin n\varphi}{\sin\varphi}\leqq\frac{n}{1-q^{2n}}\qquad(0\leqq\varphi\leqq\pi)$$

holds only for $\varphi=0$ and $\varphi=\pi$. Hence the equality

$$\frac{1}{1-q^{2n}}\int_0^\pi \frac{\sin n\varphi}{\sin\varphi}\,d\rho(\varphi)=\frac{n}{1-q^{2n}}(a_1-a_{-1})$$

holds if and only if $d\rho(\varphi)=0$ except at $\varphi=0$ and $\varphi=\pi$ where $\rho(\varphi)$ possesses jumps with the heights $\lambda(a_1-a_{-1})$ and $(1-\lambda)(a_1-a_{-1})$, respectively, λ being any constant with $0\leqq\lambda\leqq 1$. In the same way, it is shown that the equality

$$\frac{q^n}{1-q^{2n}}\int_0^\pi \frac{\sin n\varphi}{\sin\varphi}\,d\tau(\varphi)=\frac{\kappa_nq^n}{1-q^{2n}}(a_1q-a_{-1}q^{-1})$$

holds if and only if $d\tau(\varphi)=0$ except at $\varphi=\psi_n$ and $\varphi=\pi-\psi_n$ where $\tau(\varphi)$ possesses jumps with the heights $\mu(a_1q-a_{-1}q^{-1})$ and $(1-\mu)(a_1q-a_{-1}q^{-1})$, respectively, μ being any constant with $0\leqq\mu\leqq 1$. Consequently, the extremal function is given by

$$\begin{aligned}&f^*(z;\psi_n,\lambda,\mu)\\&=\Bigg(\lambda\left[\frac{\zeta(i\lg z+\varphi)-\zeta(i\lg z-\varphi)}{2\sin\varphi}\right]^{\varphi=0}\\&\quad+(1-\lambda)\left[\frac{\zeta(i\lg z+\varphi)-\zeta(i\lg z-\varphi)}{2\sin\varphi}\right]^{\varphi=\pi}\Bigg)(a_1-a_{-1})\\&\quad-\Bigg(\mu\left[\frac{\zeta_3(i\lg z+\varphi)-\zeta_3(i\lg z-\varphi)}{2\sin\varphi}\right]^{\varphi=\psi_n}\end{aligned}$$

$$+ (1-\mu)\left[\frac{\zeta_3(i \lg z+\varphi)-\zeta_3(i \lg z-\varphi)}{2 \sin \varphi}\right]^{\varphi=\pi-\psi_n}\Big)(a_1 q - a_{-1}q^{-1})$$

$$= -(\lambda \wp(i \lg z) + (1-\lambda)\wp(i \lg z+\pi))(a_1 - a_{-1})$$

$$-\Big(\mu \frac{\zeta_3(i \lg z+\psi_n)-\zeta_3(i \lg z-\psi_n)}{2 \sin \psi_n}$$

$$+ (1-\mu)\frac{\zeta_3(i \lg z+\pi-\psi_n)-\zeta_3(i \lg z-\pi+\psi_n)}{2 \sin \psi_n}\Big)(a_1 q - a_{-1}q^{-1}).$$

Next, so as to determine the extremal function for lower bound, we see that, if $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ belongs to the class $\mathfrak{F}$, then the function defined by

$$-f\left(\frac{q}{z}\right) = \sum_{n=-\infty}^{\infty} (-a_{-n})\, q^{-n} z^n$$

also does, and conversely. Hence, for any fixed n, the coefficient a_n of z^n for $f(z)$ is minimized simultaneously when the coefficient $-a_n q^n$ of z^{-n} for $-f(q/z)$ is maximized. Consequently, the minimizing functions for a_n are given by

$$f_*(z; \psi_n, \lambda, \mu) = -f^*\left(\frac{q}{z}; \psi_{-n}, \lambda, \mu\right) = -f^*\left(\frac{q}{z}; \psi_n, \lambda, \mu\right).$$

The proof is thus completed.

The last theorem determines the complete form of extremal functions, so that those given in the former papers must be, of couse, regarded as its particular form. For instance, the extremal functions given in the paper [2] may be expressed, in terms of the zeta-function with quasi-periods $2\omega_1 = 2\pi$ and $2\omega_3 = -2i \lg q$, in the form

$$f_0(z; \varphi) = \frac{\zeta(i \lg z+\varphi)-\zeta(i \lg z-\varphi)}{2 \sin \varphi}$$

with $\varphi = 0$ and $\varphi = \psi_n$ for upper and lower bounds, respectively. It is evident that they are contained as particular cases of our general family. In fact, the general form given in our theorem can be constructed, conversely, by mean of this particular form. We have namely, except an unessential additive constant,

$$f^*(z; \psi_n, \lambda, \mu) = \{\lambda f_0(z; 0) + (1-\lambda) f_0(z; \pi)\}(a_1 - a_{-1})$$
$$- \left\{\mu f_0\left(\frac{q}{z}; \psi_n\right) + (1-\mu) f_0\left(\frac{q}{z}; \pi-\psi_n\right)\right\}(a_1 q - a_{-1}q^{-1})$$

and

$$f_*(z; \psi_n, \lambda, \mu) = \{\mu f_0(z; \psi_n) + (1-\mu) f_0(z; \pi-\psi_n)\}(a_1 - a_{-1})$$
$$- \left\{\lambda f_0\left(\frac{q}{z}; 0\right) + (1-\lambda) f_0\left(\frac{q}{z}; \pi\right)\right\}(a_1 q - a_{-1}q^{-1}).$$

REFERENCES

[1] KOMATU Y., Darstellungen der in einem Kreisringe analytischen Funktionen nebst den Anwendungen auf konforme Abbildung. Jap. Journ. Math. **19** (1945), 203—215.
[2] KOMATU Y., On the coefficients of typically-real Laurent series. Kōdai Math. Sem. Rep. **9** (1957), 42—48.
[3] NEHARI Z., AND B. SCHWARZ, On the coefficients of univalent Laurent series. Proc. Amer. Math. Soc. **5** (1954), 212—217.

SHIBAURA INSTITUTE OF TECHNOLOGY.