ON A METHOD OF CESÀRO SUMMATION FOR FOURIER SERIES

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1. Introduction. Let $f(t)$ be an L-integrable function with period 2π, and

$$f(t) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt),$$

$$\varphi(t) = \varphi_x(t) = f(x+t) + f(x-t) - 2f(x),$$

and

$$\Phi_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \varphi(u)\, du \qquad (\alpha > 0).$$

F. T. Wang [3] proved the following

THEOREM A. *If* $0 < \alpha < 1$ *and*

$$\Phi_1(t) = o(t^{1/\alpha}) \qquad (t \to +0),$$

then the Fourier series of $f(t)$ *is summable* (C, α) *to* $f(x)$ *at* $t = x$.

Later S. Izumi and G. Sunouchi [1] generalized the above theorem as follows:

THEOREM B. *If* $0 < \beta < \gamma$, $\beta \leqq (\gamma - \beta) + 1$ *and*

$$\Phi_\beta(t) = o(t^\gamma) \qquad (t \to +0),$$

then the Fourier series of $f(t)$ *is summable* (C, α) *to* $f(x)$ *at* $t = x$, *where* $\alpha = \beta/(\gamma - \beta + 1)$.

Moreover they conjectured that Theorem B would hold when $\alpha > 1$, and later G. Sunouchi [2] solved this problem by the method of Bessel summation.

In this paper we shall prove that in Theorems A and B o can be replaced by O, and then the latter theorem holds also when $\alpha > 1$ by means of modified Cesàro summation. Our theorems to be proved are as follows; above all the main purpose is to establish Theorem 4.

THEOREM 1. *If* $0 < \alpha < 1$ *and* $\Phi_1(t) = O(t^{1/\alpha})$, *then the Fourier series of* $f(t)$ *is summable* (C, α) *to* $f(x)$ *at* $t = x$.

THEOREM 2. *If* $0 < \beta < \gamma$, $\beta \leqq \gamma - \beta + 1$ *and* $\Phi_\beta(t) = O(t^\gamma)$, *then the Fou-*

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rier series of $f(t)$ *is summable* (C,α) *to* $f(x)$ *at* $t=x$, *where* $\alpha=\beta/(\gamma-\beta+1)$.

THEOREM 3. *Theorem 2 holds also when* $\alpha>1$.

THEOREM 4.[1] *If* $\alpha>-1$ *then there exist quasi Fejér kernels* $L_n^\alpha(t)$ *of order* α *such that* 1° *the Fourier series of* $f(t)$ *is summable* (C,α) *to* $f(x)$ *at* $t=x$ *if and only if*

$$\frac{1}{\pi}\int_0^\pi \varphi(t)L_n^\alpha(t)\,dt=o(1) \qquad (n\to\infty),$$

and that 2°

$$[L_n^\alpha(t)]^{(h)}=\left(\frac{d}{dt}\right)^h L_n^\alpha(t)=O(n^{h+1}) \qquad \textit{for all } t \textit{ and } h\geqq 0, \tag{1.1}$$

$$[L_n^\alpha(t)]^{(h)}=O(1/n^{\alpha-h}t^{\alpha+1}) \qquad \textit{for } nt\geqq 1 \textit{ and } h=0,1,\cdots,h_0, \tag{1.2}$$

and

$$\int_t^\pi L_n^\alpha(u)\,du=O(1/n^{\alpha+1}t^{\alpha+1}) \qquad \textit{for } nt\geqq 1, \tag{1.3}$$

where h_0 *may be as large as we wish.*

In this theorem we may take for $L_n^\alpha(t)$, e.g.

$$L_n^\alpha(t)=\alpha A_m^\alpha\sum_{k=0}^m(-1)^k\binom{m}{k}\frac{K_{n-k}^{\alpha+k}}{\alpha+k},$$

where $K_n^\gamma(t)$ are Fejér kernels defined by (2.2), and m is sufficiently large, h_0 depends on m, and

$$\frac{2}{\pi}\int_0^\pi L_n^\alpha(t)\,dt=1.$$

Theorem 4 gives a new modification of Cesàro summation for Fourier series corresponding to Bessel summation which is generally useful only for $\alpha>0$ where α is the order of summation.

2. Proof of Theorems 1, 2 and 3. We require a lemma.

LEMMA 1. *If* $\alpha>0$ *and*

$$L_n^\alpha(t)=O(1/n^\alpha t^{\alpha+1}) \qquad \textit{for } nt\geqq 1,$$

then the integrability of $\varphi(t)$ *in Lebesgue sense implies*

$$\int_{H^{-1}n^{-\alpha/(\alpha+1)}}^\pi \varphi(t)L_n^\alpha(t)\,dt=o(1) \qquad (n\to\infty),$$

where $H\geqq 1$ *is a constant.*

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1) Theorem 4 may be applied to Fourier series of functions integrable in Cesàro-Lebesgue sense.

Indeed divide the left hand side integral into two parts,

$$\int_{H^{-1}n^{-\alpha/(\alpha+1)}}^{n^{-\alpha/2(\alpha+1)}} + \int_{n^{-\alpha/2(\alpha+1)}}^{\pi} = I_1 + I_2,$$

say. Then

$$I_1 = O\left(\int_{H^{-1}n^{-\alpha/(\alpha+1)}}^{n^{-\alpha/2(\alpha+1)}} \frac{|\varphi(t)|}{n^\alpha t^{\alpha+1}}\, dt\right)$$

$$= O\left(H^{\alpha+1}\int_{H^{-1}n^{-\alpha/(\alpha+1)}}^{n^{-\alpha/2(\alpha+1)}} |\varphi(t)|\, dt\right) = o(1) \qquad (n\to\infty),$$

since the last integral tends to zero with $1/n$. And

$$I_2 = O\left(\int_{n^{-\alpha/2(\alpha+1)}}^{\pi} \frac{|\varphi(t)|}{n^\alpha t^{\alpha+1}}\, dt\right)$$

$$= O\left(n^{-\alpha/2}\int_0^\pi |\varphi(t)|\, dt\right) = o(1) \qquad (n\to\infty).$$

Thus we have the desired result.

For the proof of Theorems 1 and 2 it is sufficient to replace the division

$$\int_0^\pi = \int_0^{\psi n^{-\alpha/(\alpha+1)}} + \int_{\psi n^{-\alpha/(\alpha+1)}}^{\pi}$$

in the second proof of Izumi-Sunouchi [1] by

$$\int_0^\pi = \int_0^{\psi^{-1}n^{-\alpha/(\alpha+1)}} + \int_{\psi^{-1}n^{-\alpha/(\alpha+1)}}^{\pi} = J_1 + J_2,$$

where $\psi = \psi(n)$ tends to infinity with n sufficiently slowly, and plays the rôle of H in the above lemma. In fact $J_2 = o(1)$ by Lemma 1, and we see easily that under our assumption J_1 tends to zero with $1/n$ by the same arguments as in the cited paper [1].

We prove Theorem 1 for the sake of completeness. Let $\sigma_n^\alpha(x)$ be the n th Cesàro mean of order α of the Fourier series of $f(t)$ at $t = x$, and $K_n^\alpha(t)$ be the Fejér kernel, i. e.

$$\sigma_n^\alpha(x) = \frac{1}{2}a_0 + \sum_{\nu=1}^{n} A_{n-\nu}^\alpha (a_\nu \cos \nu x + b_\nu \sin \nu x)/A_n^\alpha, \tag{2.1}$$

and

$$K_n^\alpha(t) = \frac{1}{2} + \sum_{\nu=1}^{n} A_{n-\nu}^\alpha \cos \nu t/A_n^\alpha, \tag{2.2}$$

where A_n^α are Andersen's notation defined by

$$\frac{1}{(1-x)^{\alpha+1}} = \sum_{n=0}^{\infty} A_n^\alpha x^n \qquad (|x|<1).$$

Then it is well-known that

$$K_n^\alpha(t) = O(n), \qquad [K_n^\alpha(t)]' = O(n^2) \qquad \text{for all} \quad t,$$

and

$$K_n^\alpha(t) = O(1/n^\alpha t^{\alpha+1}), \qquad [K_n^\alpha(t)]' = O(1/n^{\alpha-1}t^{\alpha+1})$$

for $nt \geqq 1$ and $-1 < \alpha \leqq 1$. Now denoting $\Phi_1(t)$ by $\Phi(t)$ we have

$$\begin{aligned}
\pi[\sigma_n^\alpha(x) - f(x)] &= \int_0^\pi \Phi(t) K_n^\alpha(t)\,dt \\
&= -\left[\int_0^{n^{-1}} + \int_{n^{-1}}^{H^{-1}n^{-\alpha/(\alpha+1)}}\right] \Phi(t)[K_n^\alpha(t)]'dt \\
&\quad + \Big[\Phi(t)\, K_n^\alpha(t)\Big]_{t=0}^{H^{-1}n^{-\alpha/(\alpha+1)}} + \int_{H^{-1}n^{-\alpha/(\alpha+1)}}^{\pi} \varphi(t)\, K_n^\alpha(t)\, dt \\
&= I_1 + I_2 + I_3 + I_4,
\end{aligned}$$

say. Then

$$\begin{aligned}
I_1 &= O\left(n^2 \int_0^{n^{-1}} t^{1/\alpha}dt\right) = O(1/n^{1/\alpha-1}) = o(1), \\
I_2 &= O\left(\int_{n^{-1}}^{H^{-1}n^{-\alpha/(\alpha+1)}} n^{1-\alpha}t^{1/\alpha-\alpha-1}dt\right) \\
&= O(1/H^{1/\alpha-\alpha}) + O(1/n^{1/\alpha-1}) \\
&= O(1/H^{1/\alpha-\alpha}) + o(1), \\
I_3 &= \Big[O(t^{1/\alpha}/n^\alpha t^{\alpha+1})\Big]_{t=H^{-1}n^{-\alpha/(\alpha+1)}} = O(H^{\alpha+1-1/\alpha}/n^{1/(\alpha+1)}) \\
&= o(1),
\end{aligned}$$

and by Lemma 1 $I_4 = o(1)$. Therefore tending $n \to \infty$ and then $H \to \infty$ we have $\sigma_n^\alpha(x) - f(x) = o(1)$, which proves our theorem.

Using Theorem 4 and Lemma 1 we can easily prove Theorem 3 by the method of the cited Izumi-Sunouchi [1]. The proof is omitted.

3. Lemma 2. For the proof of Theorem 4 we require some lemmas. We suppose that $\alpha > -1$ and $\alpha \neq 0$. The Fejér kernel $K_n^\alpha(t)$ defined by (2.2) equals to the real part of the expression

$$\frac{e^{int}}{A_n^\alpha(1-e^{-it})^{\alpha+1}} - \sum_{j=1}^m \frac{A_n^{\alpha-j}}{A_n^\alpha} \cdot \frac{e^{-it}}{(1-e^{-it})^{j+1}} - R_n^\alpha(t),$$

where

$$R_n^\alpha(t) = \frac{1}{A_n^\alpha(1-e^{-it})^{m+1}} \sum_{\nu=n+1}^\infty A_\nu^{\alpha-m-1} e^{i(n-\nu)t},$$

and m is an arbitrary integer greater than $\alpha - 1$ (K. Yano [5]). Clearly if α is an integer then $A_n^{\alpha-j}$ vanish for all j and n such that $j > \alpha$ and $n \geqq j - \alpha$, in particular $R_n^\alpha(t)$ vanish for all sufficiently large n, and the number of terms of the sum $\sum_{j=1}^{m}$ remains constant α. Here let

$$(3.1)\qquad \Lambda_n^\alpha(t) = \Re\left(\frac{e^{int}}{A_n^\alpha(1-e^{-it})^{\alpha+1}}\right) = \frac{\cos(nt+(\alpha+1)(t-\pi)/2)}{A_n^\alpha(2\sin t/2)^{\alpha+1}}.$$

For simplicity we denote the above expressions in imaginary forms by $K_n^\alpha(t)$ and $\Lambda_n^\alpha(t)$ themselves respectively for a moment. Now

$$\frac{A_n^{\alpha-j}}{A_n^\alpha} = \frac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{(\alpha+n)(\alpha+n-1)\cdots(\alpha+n-j+1)}.$$

We set

$$B_j^\alpha = \alpha(\alpha-1)\cdots(\alpha-j+1) \qquad (j=1,2,\cdots,m),$$

then

$$K_n^\alpha(t) - \Lambda_n^\alpha(t) + R_n^\alpha(t)$$
$$= -\sum_{j=1}^{m}[B_j^\alpha/(\alpha+n)(\alpha+n-1)\cdots(\alpha+n-j+1)]\frac{e^{-it}}{(1-e^{-it})^{j+1}}.$$

Similarly we have

$$K_{n-k}^{\alpha+k}(t) - \Lambda_{n-k}^{\alpha+k}(t) + R_{n-k}^{\alpha+k}(t)$$
$$= -\sum_{j=1}^{m}[B_j^{\alpha+k}/(\alpha+n)(\alpha+n-1)\cdots(\alpha+n-j+1)]\frac{e^{-it}}{(1-e^{-it})^{j+1}},$$

where

$$(3.2)\qquad B_j^{\alpha+k} = (\alpha+k)(\alpha+k-1)\cdots(\alpha+k-j+1) \qquad \begin{pmatrix} j=1,2,\cdots,m \\ k=0,1,\cdots,m \end{pmatrix}.$$

We determine m unknown constants $\lambda_1, \lambda_2, \cdots, \lambda_m$ by

$$(3.3)\qquad \sum_{k=1}^{m} B_j^{\alpha+k}\lambda_k = B_j^\alpha \qquad (j=1,2,\cdots,m),$$

which will be solved later. Then we have for such λ_k

$$K_n^\alpha(t) - \Lambda_n^\alpha(t) + R_n^\alpha(t)$$
$$= \sum_{k=1}^{m}\lambda_k[K_{n-k}^{\alpha+k}(t) - \Lambda_{n-k}^{\alpha+k}(t) + R_{n-k}^{\alpha+k}(t)],$$

i.e.

$$(3.4)\qquad L_n^\alpha(t) = M_n^\alpha(t) - N_n^\alpha(t),$$

where

$$(3.5)\qquad L_n^\alpha(t) = K_n^\alpha(t) - \sum_{k=1}^{m}\lambda_k K_{n-k}^{\alpha+k}(t),$$

and

$$(3.6)\qquad \begin{cases} M_n^\alpha(t) = \Lambda_n^\alpha(t) - \sum_{k=1}^{m}\lambda_k\Lambda_{n-k}^{\alpha+k}(t), \\ N_n^\alpha(t) = R_n^\alpha(t) - \sum_{k=1}^{m}\lambda_k R_{n-k}^{\alpha+k}(t). \end{cases}$$

Hence we have the following

LEMMA 2. *If* $\alpha > -1$ *then the functions* $L_n^\alpha(t)$ *defined by* (3.5) *possess all the properties* (1.1), (1.2) *and* (1.3) *in Theorem* 4.

Indeed the relation (1.1) follows immediately from (3.5). The relation (1.2) follows from (3.4), (3.6) and the definitions of $R_n^\alpha(t)$ and $\Lambda_n^\alpha(t)$, since

$$[L_n^\alpha(t)]^{(h)} = O\Big(\sum_{k=0}^{m} 1/n^{\alpha+k-h}t^{\alpha+k+1} + \sum_{i=0}^{h} 1/n^{m+i-h}t^{m+i+1}\Big),$$

which equals to $O(1/n^{\alpha-h}t^{\alpha+1})$ for $nt \geqq 1$ and $h = 0, 1, \cdots, h_0$ provided that m be taken sufficiently large corresponding to h_0. It is analogous for the relation (1.3). Thus we have the desired results.

4. Proof of Theorem 4. Using $\sigma_n^\alpha(x)$ defined by (2.1) we have from (3.5)

$$\frac{1}{\pi}\int_0^\pi \varphi(t)\, L_n^\alpha(t)\, dt = [\sigma_n^\alpha(x) - f(x)] - \sum_{k=1}^{m} \lambda_k [\sigma_{n-k}^{\alpha+k}(x) - f(x)].$$

On the other hand we can easily solve the simultaneous equations (3.3) with respect to $\lambda_1, \lambda_2, \cdots, \lambda_m$, the solution is

$$\lambda_k = \frac{\Delta(p_1, p_2, \cdots, p_{k-1}, \alpha, p_{k+1}, \cdots, p_m)}{\Delta(p_1, p_2, \cdots, p_{k-1}, p_k, p_{k+1}, \cdots, p_m)},$$

where $p_k = \alpha + k (k = 1, 2, \cdots, m)$, and

$$\Delta(p_1, p_2, \cdots, p_m) = p_1 p_2 \cdots p_m (p_m - p_1)(p_m - p_2)\cdots(p_m - p_{m-1})$$
$$\cdot (p_{m-1} - p_1)(p_{m-1} - p_2)\cdots(p_{m-1} - p_{m-2})\cdots(p_3 - p_1)(p_3 - p_2)(p_2 - p_1).$$

Thus by easy calculation we have

$$\lambda_k = (-1)^{k-1}\binom{m}{k}\frac{\alpha}{\alpha + k} \qquad (k = 1, 2, \cdots, m).$$

Therefore the above identity is written as

$$(4.1) \qquad \begin{aligned} \frac{1}{\pi}\int_0^\pi \varphi(t)\, L_n^\alpha(t)\, dt &= \alpha \sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{\sigma_{n-k}^{\alpha+k}(x) - f(x)}{\alpha + k} \\ &= \alpha \sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{\sigma_{n-k}^{\alpha+k}(x)}{\alpha + k} - \frac{f(x)}{A_m^\alpha}, \end{aligned}$$

since $\alpha \sum_{k=0}^{m} (-1)^k \binom{m}{k}/(\alpha + k) = 1/A_m^\alpha$, and by (3.5)

$$(4.2) \qquad L_n^\alpha(t) = \alpha \sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{K_{n-k}^{\alpha+k}(t)}{\alpha + k}.$$

Now we prove Theorem 4. The quasi kernels are given by (4.2), and the property 1° follows from the next Lemma 3, and 2° from Lemma 2 immediately.

LEMMA 3. *If* $\alpha > -1$ *then a necessery and sufficient condition that a sequence* $s_0, s_1, \cdots, s_n, \cdots$ *is summable* (C, α) *to* s *is*

$$\alpha \sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{\sigma_{n-k}^{\alpha+k}}{\alpha+k} = \frac{s}{A_m^\alpha} + o(1) \qquad (n \to \infty), \tag{4.3}$$

where σ_n^γ denote the Cesàro means of order γ of the sequence $\{s_n\}$, and m is an arbitrary non-negative integer.

This is a generalization of the Mercer's theorem [6]. We require a lemma.

LEMMA 4. *If $\beta > \alpha$ where $\alpha \neq 0$ is arbitrary, then*

$$(n+p)s_n - \alpha s_{n-1}^1 = o(n^\beta) \qquad (n \to \infty)$$

implies $s_n = o(n^{\beta-1})$, where $p \gtreqless 0$ is arbitrary and $s_n^1 = s_0 + s_1 + \cdots + s_n$.

This lemma is false when $\beta \leqq \alpha$.

Lemma 4 can be proved quite analogously as the Mercer's theorem, the proof is omitted.

For the proof of Lemma 3, the necessity is evident, and so we prove the sufficiency. We may suppose that $\alpha \neq 0$ and $s = 0$. Then the equation (4.3) is written as

$$\sum_{k=0}^{m-1} (-1)^k \binom{m-1}{k} \left(\frac{\sigma_{n-k}^{\alpha+k}}{\alpha+k} - \frac{\sigma_{n-k-1}^{\alpha+k+1}}{\alpha+k+1} \right) = o(1) \qquad (n \to \infty). \tag{4.4}$$

Here we denote by s_n^γ the Cesàro sums of order γ of the sequence $\{s_n\}$, then

$$\begin{aligned} \frac{\sigma_n^\alpha}{\alpha} - \frac{\sigma_{n-1}^{\alpha+1}}{\alpha+1} &= \frac{s_n^\alpha}{\alpha A_n^\alpha} - \frac{s_{n-1}^{\alpha+1}}{(\alpha+1)A_{n-1}^{\alpha+1}} \\ &= \frac{n s_n^\alpha - \alpha s_{n-1}^{\alpha+1}}{\alpha A_n^\alpha n} = \frac{n s_n^\alpha - \alpha s_{n-1}^{\alpha+1}}{\alpha A_{n-1}^\alpha (\alpha+n)}. \end{aligned}$$

Generally we have

$$\frac{\sigma_{n-k}^{\alpha+k}}{\alpha+k} - \frac{\sigma_{n-k-1}^{\alpha+k+1}}{\alpha+k+1} = \frac{(n-k)s_{n-k}^{\alpha+k} - (\alpha+k)\, s_{n-k-1}^{\alpha+k+1}}{(\alpha+k)A_{n-k-1}^{\alpha+k}(\alpha+n)} \tag{4.5}$$

for $k = 0, 1, \cdots$. Now letting

$$t_n = t_n^0 = n s_n^\alpha - \alpha s_{n-1}^{\alpha+1},$$

and t_n^γ be the Cesàro sums of $t_0, t_1, \cdots$ of order γ, we have

$$t_{n-k}^k = (n-k)s_{n-k}^{\alpha+k} - (\alpha+k)s_{n-k-1}^{\alpha+k+1} \qquad (k = 1, 2, \cdots).$$

Hence by (4.5), the equation (4.4) is equivalent to

$$\sum_{k=0}^{m-1} (-1)^k \binom{m-1}{k} \frac{t_{n-k}^k}{(\alpha+k)A_{n-k-1}^{\alpha+k}} = o(n) \qquad (n \to \infty). \tag{4.6}$$

Similarly letting

$$u_n = u_n^0 = (n-1)t_n - \alpha t_{n-1}^1,$$

and u_n^γ be the Cesàro sums of $u_0, u_1, \cdots$ of order γ, we see that (4.6) is equivalent to

$$\sum_{k=0}^{m-2}(-1)^k\binom{m-2}{k}\frac{u_{n-k}^k}{(\alpha+k)A_{n-k-2}^{\alpha+k}}=o(n^2)\qquad(n\to\infty),\tag{4.7}$$

which is again equivalent to

$$\sum_{k=0}^{m-3}(-1)^k\binom{m-3}{k}\frac{v_{n-k}^k}{(\alpha+k)A_{n-k-3}^{\alpha+k}}=o(n^3)\qquad(n\to\infty),\tag{4.8}$$

where

$$v_n=v_n^0=(n-2)u_n-\alpha u_{n-1}^1,$$

and v_n^γ denote the Cesàro sums of $v_0, v_1, \cdots$ of order γ, and so on.

We prove our lemma for $m=3$, since the proof is analogous for arbitrary m. Then (4.8) gives

$$v_n/\alpha A_{n-3}^\alpha=o(n^3),$$

i. e.

$$(n-2)u_n-\alpha u_{n-1}^1=v_n=o(n^{\alpha+3}),$$

from which we have in turn by Lemma 4

$$(n-1)t_n-\alpha t_{n-1}^1=u_n=o(n^{\alpha+2}),$$

$$ns_n^\alpha-\alpha s_{n-1}^{\alpha+1}=t_n=o(n^{\alpha+1}),$$

and finally $s_n^\alpha=o(n^\alpha)$. Hence $\sigma_n^\alpha=o(1)$, and our lemma is established.

Theorem 4 may be expressed in alternate form as follows:

THEOREM 4′. *Let $\alpha>-1$ and $L_n^\alpha(t)$ be defined by* (4.2). *Then the Fourier series of $f(t)$ is summable (C,α) to $f(x)$ at $t=x$ if and only if*

$$\int_0^\pi \varphi(t)L_n^\alpha(t)\,dt=o(1)\qquad(n\to\infty),$$

and for $nt\geqq 1$ $L_n^\alpha(t)$ are written as

$$L_n^\alpha(t)=\alpha\sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{A_{n-k}^{\alpha+k}}{\alpha+k}-N_n^\alpha(t),$$

where $A_n^\gamma(t)$ are defined by (3.1), *$N_n^\alpha(t)$ by* (3.6),

$$[N_n^\alpha(t)]^{(h)}=O(1/n^{m-h}t^{m+1})\qquad(h=0,1,\cdots,[m-\alpha]),$$

and m is an arbitrary integer such that $m\geqq\alpha$.

Particularly if α is an integer, then $m>0$ is arbitrary, and the last relation is replaced by

$$[N_n^\alpha(t)]^{(h)}=O(1/n^{m+1}t^{m+2+h})\qquad(h=0,1,\cdots).$$

5. Application. Let $f(x)$ belong to Lip 1 class in $(0, 2\pi)$, i.e.

$$f(x+h) - f(x) = O(h)$$

holds uniformly for x and $x+h$ in $(0, 2\pi)$. Then Bernstein's classical approximation theorem reads as follows:

THEOREM C. *If $f \in \mathrm{Lip}\,1$ in $(0, 2\pi)$ then*

$$\sigma_n^1(x) - f(x) = O(\log n/n) \qquad (n \to \infty),$$

uniformly in x, where $\sigma_n^\alpha(x)$ are defined by (2.1).

This theorem is best possible, i.e. O cannot be replaced by o (see O. Szász [4]). But we can prove the following theorem.

THEOREM 5. *If $f \in \mathrm{Lip}\,1$ then*

$$(5.1) \qquad [\sigma_n^1(x) - f(x)] - \frac{1}{2}[\sigma_{n-1}^2(x) - f(x)] = O(1/n) \qquad (n \to \infty),$$

more generally

$$(5.2) \qquad \sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{\sigma_{n-k}^{1+k}(x) - f(x)}{1+k} = O(1/n) \qquad (n \to \infty)$$

$$(m = 1, 2, \cdots),$$

uniformly in x.

This theorem holds also when σ_{n-k}^{1+k} are replaced by σ_n^{1+k}.

Indeed if we take for $L_n(t) = L_n^1(t)$ in Theorem 4

$$L_n(t) = K_n^1(t) - \frac{1}{2} K_{n-1}^2(t),$$

then

$$L_n(t) = -\frac{\cos(n+1)t}{(n+1)(2\sin t/2)^2} + \frac{\sin(n+1/2)t}{n(n+1)(2\sin t/2)^3} - \frac{1}{2n(n+1)(2\sin t/2)^2},$$

from which we have

$$L_n(t) = O(n) \qquad \text{for all} \quad t,$$

and

$$\int_t^\pi L_n(u)\,du = O(1/n^2t^2) \qquad \text{for all} \quad nt \geqq 1.$$

On the other hand by the assumption

$$\varphi(t) = O(t), \qquad D\varphi(t) = O(1) \qquad \text{for} \quad 0 < t \leqq \pi,$$

uniformly in x, where $D\varphi(t)$ denotes one of the derivatives of $\varphi(t)$. Now the left hand side of (5.1) multiplied by π equals to

$$\int_0^\pi \varphi(t)\, L_n(t)\, dt = \int_0^{1/n} + \int_{1/n}^\pi = I_1 + I_2,$$

say, then

$$I_1 = \int_0^{1/n} \varphi(t)\, L_n(t)\, dt = O\left(\int_0^{1/n} nt\, dt\right) = O(1/n),$$

and integrating by parts

$$\begin{aligned} I_2 &= \int_{1/n}^\pi \varphi(t)\, L_n(t)\, dt \\ &= \varphi(1/n) \int_{1/n}^\pi L_n(u)\, du + \int_{1/n}^\pi D\varphi(t)\, dt \int_t^\pi L_n(u)\, du \\ &= O(1/n) + O\left(\int_{1/n}^\pi (1/n^2t^2)\, dt\right) = O(1/n). \end{aligned}$$

Therefore

$$\int_0^\pi \varphi(t)\, L_n(t)\, dt = O(1/n) \qquad (n \to \infty)$$

uniformly in x, which proves (5.1). The proof of (5.2) is analogous.

References

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