ON CERTAIN STRONG SUMMABILITY OF A FOURIER POWER SERIES

BY MASAKITI KINUKAWA

1. Suppose that

$$f(z) = \sum_{n=0}^{\infty} c_n z^n, \qquad (z = re^{i\varphi}),$$

is regular for $|z| = r < 1$ and its boundary function is $f(e^{i\theta})$. Let us put

$$M_p(r, f') = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f'(re^{i\varphi})|^p d\varphi\right)^{1/p}$$

$$\sigma_n^0(\theta) = s_n(\theta) = \sum_{\nu=0}^{n} c_\nu e^{i\nu\theta},$$

$$\sigma_n^\delta(\theta) = \frac{1}{A_n^\delta}\sum_{\nu=0}^{n} A_{n-\nu}^{\delta-1} s_\nu(\theta), \qquad \text{for } \delta > -1,$$

$$t_n(\theta) = nc_n e^{ni\theta}$$

and

$$\tau_n^\delta(\theta) = \frac{1}{A_n^\delta}\sum_{\nu=0}^{n} A_{n-\nu}^{\delta-1} t_\nu(\theta), \qquad \text{for } \delta > 0,$$

where

$$A_n^\delta = \binom{n+\delta}{n} \sim n^\delta/\Gamma(\delta+1).$$

Then we have

$$\tau_n^\delta(\theta) = \delta\{\sigma_n^{\delta-1}(\theta) - \sigma_n^\delta(\theta)\}. \tag{1.1}$$

Concerning the convergence of the series

$$\sum_{n=1}^{\infty} |\sigma_n^\delta(\theta) - f(e^{i\theta})|^k,$$

we have proved some results in [5]. In this paper we shall prove the following theorems. The method of the proof is due to H. C. Chow [2] and the author [5].

THEOREM 1. *If the integral*

$$\int_0^1 \Delta_1(r) M_p(r, f')\, dr$$

is finite, then the series

Received December 20, 1956.

$$\sum_{n=1}^{\infty} |\sigma_n^{\delta}(\theta) - f(e^{i\theta})|^q$$

converges for almost all θ, *where* $1 < p \leqq 2$, $q = p/(p-1)$, *and*

$$\Delta_1(r) = (1-r)^{1/p-1}, \qquad \text{or} \qquad = (1-r)^{1/p-1}\left(\log\frac{1}{1-r}\right)^{1/p},$$

according as

$$\delta > 1/p - 1, \qquad \text{or} \qquad \delta = 1/p - 1.$$

THEOREM 2. *If the integral*

$$\int_0^1 \Delta_2(r) M_p(r, f')\, dr$$

is finite, then the series

$$\sum_{n=1}^{\infty} |\sigma_n^{\delta}(\theta) - f(e^{i\theta})|^p$$

converges for almost all θ, *where* $1 < p \leqq 2$, *and*

$$\Delta_2(r) = (1-r)^{-1/p}, \qquad \text{or} \qquad = (1-r)^{-1/p}\left(\log\frac{1}{1-r}\right)^{1/p},$$

according as

$$\delta > 1/p - 1, \qquad \text{or} \qquad \delta = 1/p - 1.$$

2. Proof of Theorem 1.

LEMMA 1. *We have the following inequality,*[1] *for* $p \geqq 1$,

$$\begin{aligned}(2.1)\quad & \left(\int_{-\pi}^{\pi} |f(re^{i\varphi+i\theta}) - f(e^{i\theta})|^p d\theta\right)^{1/p} \\ & \leqq K\int_r^1 d\rho\left(\int_{-\pi}^{\pi} |f'(\rho e^{i\theta})|^p d\theta\right)^{1/p} + K|\varphi| \left(\int_{-\pi}^{\pi} |f'(re^{i\theta})|^p d\theta\right)^{1/p}.\end{aligned}$$

Proof is easy from the similar calculation used in [5].

LEMMA 2. *Under the assumption of Theorem* 1, *we have*

$$\sum_{n=1}^{\infty} |\sigma_n^{\alpha}(\theta) - f(e^{i\theta})|^q < \infty$$

for almost all θ, *where* $\alpha = \delta + 1$.

For, we have

$$\sum_{n=1}^{\infty} A_n^{\alpha}\{\sigma_n^{\alpha}(\theta) - f(e^{i\theta})\}z^n = \frac{f(ze^{i\theta}) - f(e^{i\theta})}{(1-z)^{1+\alpha}} = F(z;\theta).$$

For $\beta > 0$ and $|z| = r < 1$, we have

$$F_\beta(z;\theta) = \sum_{n=0}^{\infty} (A_{n+1}^{\beta})^{-1} A_n^{\alpha}\{\sigma_n^{\alpha}(\theta) - f(e^{i\theta})\}z^n$$

1) We denote by K an absolute constant. In what follows, the value of K may be different from one occurrence to another.

$$= \beta z^{-\beta} \int_0^z (z-u)^{\beta-1} F(u;\theta)\, du,$$

where z^β and $(z-u)^{\beta-1}$ assume their principal values. Taking $\beta = \alpha > 0$ and using the Hausdorff-Young theorem, we get

$$\Big(\sum_{n=1}^\infty | \sigma_n^\alpha(\theta) - f(e^{i\theta}) |^q r^{nq} \Big)^{1/q} \leqq K \Big(\int_{-\pi}^{\pi} | F_\alpha (re^{i\varphi};\theta) |^p d\varphi \Big)^{1/p}.$$

Since

$$F_\alpha (re^{i\varphi};\theta) = \alpha r^{-\alpha} \int_0^r (r-\rho)^{\alpha-1} F(\rho e^{i\varphi};\theta)\, d\rho$$

$$= \alpha \int_0^1 (1-\rho)^{\alpha-1} F(r\rho e^{i\varphi};\theta)\, d\rho,$$

we get

$$\Big(\sum_{n=1}^\infty | \sigma_n^\alpha(\theta) - f(e^{i\theta}) |^q r^{nq} \Big)^{1/q} \leqq K \Big(\int_{-\pi}^{\pi} \Big| \int_0^1 (1-\rho)^{\alpha-1} F(r\rho e^{i\varphi};\theta)\, d\rho \Big|^p d\varphi \Big)^{1/p}.$$

$$\leqq K \int_0^1 d\rho \Big(\int_{-\pi}^{\pi} |(1-\rho)^{\alpha-1} F(r\rho e^{i\varphi};\theta)|^p d\varphi \Big)^{1/p}$$

$$= K \int_0^1 (1-\rho)^{\alpha-1} d\rho \Big(\int_{-\pi}^{\pi} | F(r\rho e^{i\varphi};\theta) |^p d\varphi \Big)^{1/p}.$$

Thus we have

$$\int_{-\pi}^{\pi} \Big(\sum_{n=1}^\infty | \sigma_n^\alpha(\theta) - f(e^{i\theta}) |^q \Big)^{1/q} d\theta$$

$$\leqq K \int_{-\pi}^{\pi} d\theta \int_0^1 (1-\rho)^{\alpha-1} d\rho \Big(\int_{-\pi}^{\pi} | F(\rho e^{i\varphi};\theta) |^p d\varphi \Big)^{1/p}$$

$$= K \int_{-\pi}^{\pi} d\theta \int_0^1 (1-\rho)^{\alpha-1} d\rho \Big(\int_{-\pi}^{\pi} \frac{|f(\rho e^{i\varphi+i\theta}) - f(e^{i\theta})|^p}{|1-\rho e^{i\varphi}|^{p(1+\alpha)}} d\varphi \Big)^{1/p}$$

$$\leqq K \int_0^1 (1-\rho)^{\alpha-1} d\rho \Big(\int_{-\pi}^{\pi} \frac{d\varphi}{|1-\rho e^{i\varphi}|^{p(1+\alpha)}} \int_{-\pi}^{\pi} |f(\rho e^{i\varphi+i\theta}) - f(e^{i\theta})|^p d\theta \Big)^{1/p}.$$

If we replace by (2.1) the last integral in the right side of the above, then the above integral becomes

$$\int_0^1 (1-\rho)^{\alpha-1} d\rho \Big\{ \int_{-\pi}^{\pi} \frac{d\varphi}{|1-\rho e^{i\varphi}|^{p(1+\alpha)}} \Big[\int_\rho^1 dr \Big(\int_{-\pi}^{\pi} |f'(re^{i\theta})|^p d\theta \Big)^{1/p} \Big]^p$$

$$+ \int_{-\pi}^{\pi} \frac{|\varphi|^p}{|1-\rho e^{i\varphi}|^{p(1+\alpha)}} d\varphi \int_{-\pi}^{\pi} |f'(\rho e^{i\theta})|^p d\theta \Big\}^{1/p}$$

$$\leqq K \int_0^1 (1-\rho)^{\alpha-1} d\rho \Big(\int_{-\pi}^{\pi} \frac{d\varphi}{|1-\rho e^{i\varphi}|^{p(1+\alpha)}} \Big)^{1/p} \int_\rho^1 dr \Big(\int_{-\pi}^{\pi} |f'(re^{i\theta})|^p d\theta \Big)^{1/p}$$

$$+ K \int_0^1 (1-\rho)^{\alpha-1} d\rho \Big(\int_{-\pi}^{\pi} \frac{|\varphi|^p}{|1-\rho e^{i\varphi}|^{p(1+\alpha)}} d\varphi \Big)^{1/p} \Big(\int_{-\pi}^{\pi} |f'(\rho e^{i\theta})|^p d\theta \Big)^{1/p}$$

$= J_1 + J_2$, say. We have

$$\int_{-\pi}^{\pi} \frac{|\varphi|^p}{|1-\rho e^{i\varphi}|^{p(1+\alpha)}} d\varphi \begin{cases} \leqq K(1-\rho)^{1-\alpha p}, & \text{for } \alpha p > 1, \\ \leqq K\Big(\log \dfrac{1}{1-\rho}\Big), & \text{for } \alpha p = 1. \end{cases}$$

Thus we get

$$J_2 \leqq K\int_0^1 \Delta_1(\rho) M_p(\rho, f') d\rho.$$

For the remaining part J_1, we have

$$\int_{-\pi}^{\pi} \frac{d\varphi}{|1-\rho e^{i\varphi}|^{p(1+\alpha)}} \leqq K(1-\rho)^{1-p(1+\alpha)},$$

since $\alpha > 0$. Thus we get

$$J_1 \leqq K\int_0^1 (1-\rho)^{\alpha-1+1/p-(1+\alpha)} d\rho \int_\rho^1 M_p(r, f') dr$$

$$= K\int_0^1 M_p(r, f') dr \int_0^r (1-\rho)^{-2+1/p} d\rho$$

$$\leqq K\int_0^1 (1-r)^{-1+1/p} M_p(r, f') dr,$$

which is dominated by J_2. Thus we get Lemma 2.

LEMMA 3. *Under the assumption of Theorem* 1, *we have*

$$\sum_{n=1}^{\infty} |\tau_n^\alpha(\theta)|^q < \infty,$$

for almost all θ, *where* $\alpha = 1 + \delta$.

For, since

$$\sum_{n=0}^{\infty} A_n^\alpha \tau_n^\alpha(\theta) z^n = \frac{z e^{i\theta} f'(z e^{i\theta})}{(1-z)^\alpha} = G(z;\theta),$$

we have

$$G_\beta(z;\theta) = \sum_{n=0}^{\infty} (A_{n+1}^\beta)^{-1} A_n^\alpha \tau_n^\alpha(\theta) z^n = \beta z^{-\beta} \int_0^z (z-u)^{\beta-1} G(u;\theta) du.$$

Using the same method used in Lemma 2, we get

$$\int_{-\pi}^{\pi} \Big(\sum_{n=0}^{\infty} |\tau_n^\alpha(\theta)|^q\Big)^{1/q} d\theta \leqq K\int_{-\pi}^{\pi} d\theta \int_0^1 (1-\rho)^{\alpha-1} d\rho \Big(\int_{-\pi}^{\pi} |G(\rho e^{i\varphi};\theta)|^p d\varphi\Big)^{1/p}$$

$$\leqq K\int_0^1 (1-\rho)^{\alpha-1} d\rho \Big(\int_{-\pi}^{\pi} \frac{d\varphi}{|1-\rho e^{i\varphi}|^{\alpha p}}\Big)^{1/p} \Big(\int_{-\pi}^{\pi} |f'(\rho e^{i\theta})|^p d\theta\Big)^{1/p}$$

$$\leqq K\int_0^1 \Delta_1(\rho) M_p(\rho, f') d\rho.$$

Thus we get the required result.

Combining Lemmas 2, 3 and (1.1), we get Theorem 1.

3. Proof of Theorem 2.

LEMMA 4. *Under the assumption of Theorem* 2, *we have*

$$\sum_{n=1}^{\infty} |\sigma_n^\alpha(\theta) - f(e^{i\theta})|^p < \infty,$$

for almost all θ, where $\alpha = \delta + 1$.

As in the proof of Lemma 2, we have

$$\sum_{n=0}^{\infty} (A_{n+1}^{\beta})^{-1} A_n^{\alpha}\{\sigma_n^{\alpha}(\theta) - f(e^{i\theta})\} z^n = \beta z^{-\beta}\int_0^z (z-u)^{\beta-1} F(u;\theta)\,du$$

$$= F_\beta(z;\theta).$$

By the Hardy–Littlewood theorem, we get, for $1 < p \leqq 2$,

$$\sum_{n=1}^{\infty} n^{p-2}(A_n^{\alpha}/A_n^{\beta})^p\,|\sigma_n^{\alpha}(\theta) - f(e^{i\theta})|^p r^{np} \leqq K\int_{-\pi}^{\pi} |F_\beta(z;\theta)|^p d\varphi,$$

where

$$\left(\int_{-\pi}^{\pi} |F_\beta(z;\theta)|^p d\varphi\right)^{1/p} \leqq K\left\{\int_{-\pi}^{\pi}\left|\int_0^1 (1-\rho)^{\beta-1}F(r\rho e^{i\varphi};\theta)\,d\rho\right|^p d\varphi\right\}^{1/p}$$

$$\leqq K\int_0^1 (1-\rho)^{\beta-1}d\rho\left(\int_{-\pi}^{\pi}|F(r\rho e^{i\varphi};\theta)|^p d\varphi\right)^{1/p}.$$

Since $p - 2 + p(\alpha - \beta) = 0$, that is, $\beta = \alpha - 2/p + 1 > 0$,

$$\left[\int_{-\pi}^{\pi}\left\{\sum_{n=1}^{\infty}|\sigma_n^{\alpha}(\theta) - f(e^{i\theta})|^p\right\}d\theta\right]^{1/p}$$

$$\leqq K\left[\int_{-\pi}^{\pi}d\theta\left\{\int_0^1(1-\rho)^{\beta-1}d\rho\left(\int_{-\pi}^{\pi}|F(\rho e^{i\varphi};\theta)|^p d\varphi\right)^{1/p}\right\}^p\right]^{1/p}$$

$$\leqq K\int_0^1(1-\rho)^{\beta-1}d\rho\left(\int_{-\pi}^{\pi}d\theta\int_{-\pi}^{\pi}|F(\rho e^{i\varphi};\theta)|^p d\varphi\right)^{1/p}$$

$$\leqq K\int_0^1(1-\rho)^{\beta-1}d\rho\left(\int_{-\pi}^{\pi}\frac{d\varphi}{|1-\rho e^{i\varphi}|^{p(1+\alpha)}}\int_{-\pi}^{\pi}|f(\rho e^{i\varphi+i\theta}) - f(e^{i\theta})|^p d\theta\right)^{1/p}.$$

Thus, by the similar way used in the proof of Lemma 2, the above integral is majorized by

$$K\int_0^1(1-\rho)^{\beta-1}d\rho\left(\int_{-\pi}^{\pi}\frac{d\varphi}{|1-\rho e^{i\varphi}|^{p(1+\alpha)}}\right)^{1/p}\int_\rho^1 dr\left(\int_{-\pi}^{\pi}|f'(re^{i\theta})|^p d\theta\right)^{1/p}$$

$$+ K\int_0^1(1-\rho)^{\beta-1}d\rho\left(\int_{-\pi}^{\pi}\frac{|\varphi|^p}{|1-\rho e^{i\varphi}|^{p(1+\alpha)}}d\varphi\right)^{1/p}\left(\int_{-\pi}^{\pi}|f'(\rho e^{i\theta})|^p d\theta\right)^{1/p}$$

$= J_1 + J_2$, say. Since

$$\left(\int_{-\pi}^{\pi}\frac{|\varphi|^p}{|1-\rho e^{i\varphi}|^{p(1+\alpha)}}d\varphi\right)^{1/p} \leqq K\begin{cases}(1-\rho)^{1/p-\alpha}, & \text{for } \alpha p > 1,\\ \left(\log\dfrac{1}{1-\rho}\right)^{1/p}, & \text{for } \alpha p = 1,\end{cases}$$

we get

$$J_2 \leqq K\int_0^1 \Delta_2(\rho)M_p(\rho, f')\,d\rho,$$

and, since $\beta - 1 + 1/p - \alpha = -1/p < 0$, we get

$$J_1 \leqq K\int_0^1(1-\rho)^{\beta-1+1/p-(1+\alpha)}d\rho\int_\rho^1 M_p(r, f')\,dr$$

$$= K\int_0^1 M_p(r,f')\,dr\int_0^r (1-\rho)^{\beta+1/p-2-\delta}d\rho$$

$$\leqq K\int_0^1 (1-r)^{-1/p}M_p(r,f')\,dr,$$

which completes the proof of Lemma 4.

LEMMA 5. *Under the assumption of Theorem* 2, *we have*

$$\sum_{n=1}^{\infty} |\tau_n^\alpha(\theta)|^p < \infty,$$

for almost all θ, *where* $\alpha = \delta + 1$.

For, we have

$$\sum_{n=0}^{\infty}(A_{n+1}^{\beta})^{-1}A_n^\alpha\tau_n^\alpha(\theta)z^n = \beta z^{-\beta}\int_0^z (z-u)^{\beta-1}G(u;\theta)\,du = G_\beta(z;\theta),$$

where $\beta = \alpha + 1 - 2/p > 0$. Hence we get

$$\sum_{n=1}^{\infty} n^{p-2}(A_n^\alpha/A_n^\beta)^p|\tau_n^\alpha(\theta)|^p r^{np} \leqq K\int_{-\pi}^{\pi}|G_\beta(z;\theta)|^p d\varphi.$$

By the same argument as in Lemma 4, we get

$$\left[\int_{-\pi}^{\pi}\left(\sum_{n=1}^{\infty}|\tau_n^\alpha(\theta)|^p\right)d\theta\right]^{1/p} \leqq K\int_0^1 (1-\rho)^{\beta-1}\left(\int_{-\pi}^{\pi}\frac{d\varphi}{|1-\rho e^{i\varphi}|^{\alpha p}}\right)^{1/p} M_p(\rho,f')\,d\rho$$

$$\leqq K\int_0^1 \Delta_2(\rho)M_p(\rho,f')\,d\rho,$$

which completes the proof of Lemma 5.

Using Lemmas 4 and 5, we get Theorem 2.

4. If we use Chow's theorem [2, Theorem 2], we may get easily by the method used above the following

THEOREM 3. *If the integral*

$$\int_0^1 M_p^p(r,f')\,dr \tag{4.1}$$

is finite, then the series

$$\sum_{n=1}^{\infty}|\sigma_n^\delta(\theta) - f(e^{i\theta})|^p/n$$

converges for almost all θ, *where* $1 < p < 2$, *and* $\delta = 2(1/p - 1)$.

THEOREM 4. *If the integral* (4.1) *is finite, then the series*

$$\sum_{n=1}^{\infty}|\sigma_n^\delta(\theta) - f(e^{i\theta})|^q/n$$

converges for almost all θ, *where* $1 < p \leqq 2$, $q = p/(p-1)$ *and* $\delta \geqq 1/p - 1$.

5. Finally we shall prove the following theorem, which is analogous to T. Tsuchikura's theorem ([6], [7]).

THEOREM 5. *If, for a point θ,*

$$\int_0^t |f(e^{i\varphi+i\theta}) - f(e^{i\theta})|^p d\varphi = O\{t(\log 1/t)\}^{-\beta}\},$$

then the series

$$\sum_{n=1}^{\infty} |\sigma_n^\delta(\theta) - f(e^{i\theta})|^k/n$$

converges at the point θ, where $2 \geqq p > 1$, $1/p + 1/p' = 1$, $p' \geqq k > 0$, $\beta > p/k$ and $\delta > 1/p - 1$.

The method of proof is due to Hardy–Littlewood [4] and Chow [1]. For the purpose, we need the following lemma.

LEMMA 6. *If $u(\theta)$ is integrable L,*

$$u(r, \theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(\psi) \frac{1-r^2}{1-2r\cos(\psi-\theta)+r^2} d\psi$$

and

$$\int_0^x |u(t)| dt = O\{|x|/(\log 1/|x|)^\beta\}, \qquad (\beta \geqq 0),$$

then

$$\int_0^x |u(r, t)| dt = O\{|x|/(\log 1/|x|)^\beta\}, \quad \textit{uniformly for } 1 - r \leqq |x|,$$

or $$= O\{|x|/(\log 1/|x|)^\beta + |(1-r)x^{1-\varDelta}\},$$

where $1 > \varDelta > 0$.

Proof of Lemma 6 is quite similar to the Hardy–Littlewood lemma [4]. Hence it will be sufficient to sketch the proof.

We may suppose $1 > x > 0$. We have

$$J(x, h) = \int_0^x |u(r, t)| dt \leqq K\int_{-\pi}^{\pi} |u(\varphi)| \left\{\int_0^x \frac{h}{h^2 + (\varphi - t)^2} dt\right\} d\varphi$$

$$= K\int_{-\pi}^{\pi} |u(\varphi)| \chi(\varphi, x, h) d\varphi, \qquad (h = 1 - r).$$

Using the assumption

$$U(\varphi) = \int_0^\varphi |u(t)| dt = O\{|\varphi|/(\log 1/|\varphi|)^\beta\}$$

and integrating by parts, we have

$$\int_0^\pi |u(\varphi)| \chi(\varphi, x, h) d\varphi = U(\pi)\chi(\pi, x, h) - \int_0^\pi U(\varphi) \frac{\partial \chi}{\partial \varphi} d\varphi$$

where

$$U(\pi)\chi(\pi, x, h) = O(hx). \tag{5.1}$$

Since $\partial X/\partial\varphi = h/(h^2+\varphi^2) - h/(h^2+(\varphi-x)^2)$, we have

$$\left|\int_0^\pi U(\varphi)\frac{\partial X}{\partial\varphi}d\varphi\right| \leqq K\left\{\int_0^\pi U(\varphi)\frac{hx\varphi}{(h^2+\varphi^2)\{h^2+(\varphi-x)^2\}}d\varphi + \int_0^\pi U(\varphi)\frac{hx^2}{(h^2+\varphi^2)\{h^2+(\varphi-x)^2\}}d\varphi\right\}$$
$$= K(J_1+J_2).$$

Let us put $0<\varDelta<1$, and let us split up the first integral into following two parts,

$$J_1 = \int_0^{x^\varDelta} + \int_{x^\varDelta}^\pi = J_{11}+J_{12},$$

then we have

$$(5.2)\qquad J_{11} = O\left\{hx\int_0^{x^\varDelta}\frac{\varphi^2}{(h^2+\varphi^2)\{h^2+(\varphi-x)^2\}\cdot(\log 1/\varphi)^\beta}d\varphi\right\}$$
$$= O\left\{\frac{x}{(\log 1/x)^\beta}\int_0^\infty\frac{h}{h^2+(\varphi-x)^2}d\varphi\right\} = O\{x/(\log 1/x)^\beta\}$$

and

$$(5.3)\qquad J_{12} = O\left\{hx\int_{x^\varDelta}^\pi\frac{d\varphi}{h^2+(\varphi-x)^2}\right\} = O(hx^{1-\varDelta}).$$

By (5.2) and (5.3), we have

$$(5.4)\qquad J_1 = O\{x/(\log 1/x)^\beta + (1-r)x^{1-\varDelta}\}.$$

Next we consider the integral J_2. We have

$$J_2 = \int_0^\pi U(\varphi)\frac{hx^2}{(h^2+\varphi^2)\{h^2+(\varphi-x)^2\}}d\varphi$$
$$= \int_0^{x/2} + \int_{x/2}^{2x} + \int_{2x}^{x^\varDelta} + \int_{x^\varDelta}^\pi = J_{21}+J_{22}+J_{23}+J_{24},$$

where

$$(5.5)\qquad J_{21} = O\left\{\frac{x}{(\log 1/x)^\beta}\int_0^{x/2}\frac{h}{h^2+\varphi^2}d\varphi\right\} = O\{x/(\log 1/x)^\beta\},$$

$$(5.6)\qquad J_{22} = O\left\{\frac{x}{(\log 1/x)^\beta}\int_{x/2}^{2x}\frac{h}{h^2+(\varphi-x)^2}d\varphi\right\} = O\{x/(\log 1/x)^\beta\},$$

$$(5.7)\qquad J_{23} = O\left\{\frac{x}{(\log 1/x)^\beta}\int_{2x}^{x^\varDelta}\frac{h\varphi^2}{(h^2+\varphi^2)^2}d\varphi\right\} = O\{x/(\log 1/x)^\beta\}$$

and

$$(5.8)\qquad J_{24} = O\left\{hx^2\int_{x^\varDelta}^\pi\frac{d\varphi}{\varphi^3}\right\} = O(hx^{2-2\varDelta}).$$

Summing up (5.5), (5.6), (5.7) and (5.8), we get

$$(5.9)\qquad J_2 = O\{x/(\log 1/x)^\beta + hx^{2-2\varDelta}\}.$$

Collecting (5.4) and (5.9), we get Lemma 6.

LEMMA 7. *If*

$$\int_0^t |f(e^{i\varphi+i\theta}) - f(e^{i\theta})|^p d\varphi = O\{|t|/(\log 1/|t|)^\beta\},$$

then we have

$$I(\theta) = \int_{-\pi}^{\pi} \frac{|f(re^{i\varphi+i\theta}) - f(e^{i\theta})|^p}{\{(1-r)^2+\varphi^2\}^{\mu/2}} d\varphi = O\left\{(1-r)^{1-\mu}\Big/\left(\log \frac{1}{1-r}\right)^\beta\right\},$$

where $p > 1$, $\mu > 1$, $\beta > 0$.

Proof. Let

$$F(z) = F(re^{i\varphi}) = f(ze^{i\theta}) - f(e^{i\theta}) = f(re^{i\varphi+i\theta}) - f(e^{i\theta}).$$

Then $F(z)$ is regular for $|z| < 1$, and belongs to H^p and

$$(5.10) \qquad \begin{aligned} \int_0^t |F(e^{i\varphi})|^p d\varphi &= \int_0^t |f(e^{i\varphi+i\theta}) - f(e^{i\theta})|^p d\varphi \\ &= O\{t/(\log 1/t)^\beta\}, \end{aligned}$$

by the assumption. Since

$$(5.11) \qquad \begin{aligned} |F(re^{i\varphi})|^p &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{i\psi}) \frac{1-r^2}{1-2r\cos(\psi-\varphi)+r^2} d\psi\right|^p \\ &\leqq K\int_{-\pi}^{\pi} |F(e^{i\varphi})|^p \frac{1-r^2}{1-2r\cos(\psi-\varphi)+r^2} d\psi, \end{aligned}$$

it follows from (5.10), (5.11) and Lemma 6 that

$$G(r,t) = \int_0^t |F(re^{i\varphi})|^p d\varphi = O\{|t|/(\log 1/|t|)^\beta\},$$

uniformly for $1-r \leqq |t|$,

and $\qquad = O\{|t|/(\log 1/|t|) + (1-r)t^{1-\Delta}\}.$

Using the above and integrating by parts, we have

$$\begin{aligned} I(\theta) &= \int_{-\pi}^{\pi} \frac{|F(re^{i\varphi})|^p}{\{(1-r)^2+\varphi^2\}^{\mu/2}} d\varphi \\ &= O(1) + K\left\{\int_0^{\pi} + \int_{-\pi}^{0}\right\} \frac{\varphi G(r,\varphi)}{(h^2+\varphi^2)^{\mu/2+1}} d\varphi, \qquad (h = 1-r), \\ &= O(1) + K\{I_1 + I_2\}, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \int_0^h + \int_h^\gamma + \int_\gamma^\pi, \qquad (\gamma = h^\xi,\ 0 < \xi < 1), \\ &= I_{11} + I_{12} + I_{13}, \end{aligned}$$

say. We get

$$I_{11} = O\left\{\int_0^h \frac{\varphi}{h^{\mu+2}}\left[\frac{\varphi}{(\log 1/\varphi)^\beta} + h\varphi^{1-\Delta}\right] d\varphi\right\} = O\left\{h^{1-\mu}\Big/\left(\log \frac{1}{h}\right)^\beta\right\},$$

$$I_{12} = O\left\{\int_h^\gamma \frac{\varphi^2}{(\log 1/\varphi)^\beta} \frac{1}{(h^2+\varphi^2)^{\mu/2+1}} d\varphi\right\} = O\{h^{1-\mu}/(\log 1/h)^\beta\}$$

and

$$I_{13} = O\left\{\int_\gamma^\pi \varphi^{2-\mu-2} d\varphi\right\} = O(\gamma^{1-\mu}) = O\{h^{1-\mu}/\log 1/h)^\beta\}.$$

For I_2, we have also the same estimation. Thus we get Lemma 7.

We are now in position to prove Theorem 5. By the Hausdorff-Young theorem, we have

$$\left\{\sum_{n=1}^\infty |A_n^\delta\{\sigma_n^\delta(\theta) - f(e^{i\theta})\}r^n|^{p'}\right\}^{p/p'} \leqq K\int_{-\pi}^\pi \left|\frac{f(ze^{i\theta}) - f(e^{i\theta})}{(1-z)^{1+\delta}}\right|^p d\varphi$$

$$\leqq K\int_{-\pi}^\pi \frac{|f(re^{i\varphi+i\theta}) - f(e^{i\theta})|^p}{\{(1-r)^2+\varphi^2\}^{(1+\delta)p/2}} d\varphi, \qquad (z = re^{i\varphi}).$$

Since $\mu = (1+\delta)p > 1$, by Lemma 6, the above integral is

$$O\left\{(1-r)^{1-\mu}\Big/\left(\log \frac{1}{1-r}\right)^\beta\right\}.$$

Let us put $1 - r = \pi/2^{\lambda+1}$, then we have

$$\left\{\sum_{n=2^{\lambda-1}+1}^{2^\lambda} |A_n^\delta\{\sigma_n^\delta(\theta) - f(e^{i\theta})\}r^n|^{p'}\right\}^{p/p'} = O\{2^{\lambda(\mu-1)}\lambda^{-\beta}\}.$$

Hence

$$\left\{\sum_{n=2^{\lambda-1}+1}^{2^\lambda} |\sigma_n^\delta(\theta) - f(e^{i\theta})|^{p'}\right\}^{p/p'} = O\{2^{\lambda(\mu-1-\delta p)}\lambda^{-\beta}\} = O\{2^{\lambda(p-1)}\lambda^{-\beta}\}.$$

Let $kq' = p'$, $1/q + 1/q' = 1$ and $q' > 1$, then by the Hölder inequality,

$$\sum_{n=2^{\lambda-1}+1}^{2^\lambda} |\sigma_n^\delta(\theta) - f(e^{i\theta})|^k/n \leqq \left(\sum_{n=2^{\lambda-1}+1}^{2^\lambda} 1/n^q\right)^{1/q}\left(\sum_{n=2^{\lambda-1}+1}^{2^\lambda} |\sigma_n^\delta(\theta) - f(e^{i\theta})|^{kq'}\right)^{1/q'}$$

$$\leqq 2^{\lambda(1-q)/q}\left(\sum_{n=2^{\lambda-1}+1}^{2^\lambda} |\sigma_n^\delta(\theta) - f(e^{i\theta})|^{p'}\right)^{k/p'}$$

$$= O\{2^{\lambda(1-q)/q}(2^\lambda\lambda^{-\beta p'/p})^{k/p'} = O(\lambda^{-\beta k/p}).$$

Thus

$$\sum_{n=2}^\infty |\sigma_n^\delta(\theta) - f(e^{i\theta})|^k/n = O\left\{\sum_{\lambda=1}^\infty \lambda^{-k\beta/p}\right\} = O(1),$$

which completes the proof of Theorem 5.

Finally, the author is thankful to Prof. S. Izumi and Miss M. Satô for their kind suggestions.

References

[1] H. C. Chow, Theorems on power series and Fourier series. Proc. London Math. Soc. **1** (1951), 206—216.

[2] ——————, A further note on the summability of a power series on its circle of convergence. Ann. of Acad. Sinica, **1** (1954), 559—567.

[3] G. H. Hardy and J. E. Littlewood, Some new properties of Fourier constant. Math. Annalen **97** (1926), 159—209.

[4] ——————, The strong summability of Fourier series. Fund. Math. **25** (1935), 162—189.

[5] M. Kinukawa, Some strong summability of Fourier series II. Proc. Japan Acad. **32** (1956), 377—382.

[6] T. Tsuchikura, Convergence character of Fourier series at a point. Mathematica Japonicae **1** (1949), 135—139.

[7] ——————, Absolute Cesàro summability of orthogonal series. Tôhoku Math. Journ. **5** (1953), 52—66.

Natural Science Division,
International Christian University, Tokyo.