AN INVERSION FORMULA FOR CONVOLUTION TRANSFORMS

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1. Introduction.

In the present paper we shall study the inversion theory for the class of convolution transforms

$$(1) \qquad f(x)=\int_{-\infty}^{\infty} G(x-t)\varphi(t)dt$$

for which the kernel $G(t)$ is of the form

$$(2) \qquad G(t)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{F(s)}e^{st}ds.$$

Here

$$(3) \qquad F(s)=\frac{e^{bs}\prod_{k=1}^{\infty}(1-s/a_k)e^{s/a_k}}{\prod_{k=1}^{\infty}(1-s/c_k)e^{s/c_k}}$$

where b, $\{a_k\}_1^\infty$, $\{c_k\}_1^\infty$ are real constants such that

$$(4) \qquad \begin{gathered} a_k c_k>0, \quad |a_k|\leqq|c_k|, \quad k=1,2,\cdots, \\ \sum_{k=1}^{\infty}a_k^{-2}<\infty, \quad \sum_{k=1}^{\infty}c_k^{-2}<\infty. \end{gathered}$$

The integral transform

$$(5) \qquad F(y)=\int_0^\infty e^{-\frac{1}{2}uy}W_{k+1/2,m}(uy)(uy)^{-k-1/2}\Phi(u)du,$$

$W_{k+1/2,m}(uy)$ being a Whittaker's function, is an example. After an exponential change of variables, (5) becomes putting $f(x)=F(e^x)e^x$, $\varphi(t)=\Phi(e^{-t})$,

$$f(x)=\int_{-\infty}^{\infty}e^{-(k-1/2)(x-t)}e^{-\frac{1}{2}e^{x-t}}W_{k+1/2,m}[e^{x-t}]\varphi(t)dt,$$

we may verify that

$$e^{-(k-1/2)t}e^{-\frac{1}{2}e^t}W_{k+1/2,m}[e^t]=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(1/2+m-k-s)\Gamma(1/2-m-k-s)}{\Gamma(-2k+1/2-s)}e^{st}ds.$$

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This transform has been studied by Meijer [4].

2. Infinite convolutions of distribution functions.

We must define here certain "elementary functions"

$$(1)\qquad \begin{aligned} g_k(t) &= \begin{cases} a_k e^{a_k t-1}, & (-\infty < t < 1/a_k) \\ 0, & (1/a_k \leqq t < \infty) \end{cases} \quad \text{if} \quad a_k > 0, \\ g_k(t) &= \begin{cases} 0, & (-\infty < t < 1/a_k) \\ -a_k e^{a_k t-1}, & (1/a_k \leqq t < \infty) \end{cases} \quad \text{if} \quad a_k < 0, \end{aligned}$$

and

$$(2)\qquad h_k(t) = \int_{-\infty}^{t} \left(1 - \frac{a_k}{c_k}\right) g_k\left(u + \frac{1}{c_k}\right) du + \frac{a_k}{c_k}\, j\left(t - \frac{c_k - a_k}{a_k c_k}\right),$$

where $j(t)$ is the standard jump function

$$j(t) = \begin{cases} 0 & (t < 0), \\ 1/2 & (t = 0), \\ 1 & (t > 0). \end{cases}$$

LEMMA 1. *If $h_k(t)$ is defined as in* (2), *then $h_k(t)$ is a distribution function. The mean of $h_k(t)$ is zero and its variance is $a_k^{-2} - c_k^{-2}$. The characteristic function of $h_k(t)$ is*

$$\frac{(1 + i\tau/c_k)e^{-i\tau/c_k}}{(1 + i\tau/a_k)e^{-i\tau/a_k}}.$$

All the requisite properties may be verified by the straightforward computations starting from the definitions of the variance and the characteristic function of a distribution function.

LEMMA 2. *If $H_m(t)$ is defined by the equation*

$$(3)\qquad H_m(t) = \lim_{r \to \infty} h_{m+1} * h_{m+2} * \cdots * h_r(t),$$

then it is a distribution function with mean zero and variance

$$\sum_{m+1}^{\infty} (a_k^{-2} - c_k^{-2}).$$

The characteristic function of $H_m(t)$ is given by

$$\prod_{m+1}^{\infty} \frac{(1 + i\tau/c_k)e^{-i\tau/c_k}}{(1 + i\tau/a_k)e^{-i\tau/a_k}}.$$

These properties are well known from the theory of probability.

3. Bilateral Laplace transform of non-decreasing functions.

The following lemma is implied in [6; 58—59].

LEMMA 3. *If the bilateral Laplace-Stieltjes transform*

$$(1) \qquad \chi(s) = \int_{-\infty}^{\infty} e^{-st} dh(t),$$

where $h(t)$ is non-decreasing, is defined on any line $\Re s = c$, then the region of convergence of (1) *is the largest vertical strip containing the line $\Re s = c$ in which $\chi(s)$ is regular when continued from the line $\sigma = c$ into the complex plane.*

Let us define the constants

$$(2) \qquad \alpha_1 \equiv \max_{a_k<0} (a_k, -\infty), \quad \alpha_2 \equiv \min_{a_k>0} (a_k, +\infty).$$

Further we define

$$(3) \qquad E_1(s) \equiv e^{bs} \prod_{k=1}^{\infty} (1 - s/a_k) e^{s/a_k},$$

$$(4) \qquad E_{1,m}(s) \equiv \prod_{k=m+1}^{\infty} (1 - s/a_k) e^{s/a_k}, \qquad m = 0, 1, 2, \cdots,$$

$$(5) \qquad E_2(s) \equiv \prod_{k=1}^{\infty} (1 - s/c_k) e^{s/c_k},$$

$$(6) \qquad E_{2,m}(s) \equiv \prod_{k=m+1}^{\infty} (1 - s/c_k) e^{s/c_k}, \qquad m = 0, 1, 2, \cdots.$$

LEMMA 4. *If $H_m(t)$ is the function defined in* (3) *of Lemma* 2, *then the bilateral Laplace-Stieltjes transform*

$$\int_{-\infty}^{\infty} e^{-st} dH_m(t) = \frac{E_{2,m}(s)}{E_{1,m}(s)}$$

converges (*absolutely*) *in the strip $\alpha_1 < \Re s < \alpha_2$ to the function indicated.*

Using Lemmas 2 and 3, and properties of the function of § 2 considered as a function of a complex variable, the result follows immediately.

LEMMA 5. *If $H_m(t)$ is defined by* (3) *of Lemma* 2, *then*

$$\frac{d}{dt} H_m(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{E_{2,m}(s)}{E_{1,m}(s)} e^{st} ds \qquad (-\infty < t < \infty).$$

This is an immediate consequence of the complex inversion formula for the bilateral Laplace transform. See [6; 241—243].

4. Admissible sequence.

The sequence a_k/c_k is said to be *admissible* with respect to the kernel $G(t)$ if

$$(1) \qquad \lim_{|\tau|\to\infty} \left| \frac{1}{F(\sigma + i\tau)} (\sigma + i\tau) \right| = 0$$

uniformly for σ in any finite interval, and if the integral

$$(2) \qquad \int_{-\infty}^{\infty}\left|\frac{1}{F(\sigma+i\tau)}(\sigma+i\tau)\right|d\tau$$

converges for every value of σ for which $E_1(\sigma)\neq 0$.

The convergence of the integral (2) implies that the integral

$$(3) \qquad \int_{-\infty}^{\infty}\frac{1}{F(i\tau)}d\tau=\int_{-\infty}^{\infty}\frac{E_2(i\tau)}{E_1(i\tau)}d\tau$$

is absolutely convergent.

It is trivial to establish that the integrals

$$(4) \qquad \int_{-\infty}^{\infty}\frac{E_{2,m}(i\tau)}{E_{1,m}(i\tau)}d\tau \qquad (m=0,\ 1,\ 2,\ \cdots)$$

are convergent absolutely as well.

The absolute convergence of the integrals (4) implies that $H_m(t)$ have continuous first derivatives, so that we may write

$$(5) \qquad \frac{d}{dt}H_m(t)=G_m(t) \qquad (m=0,\ 1,\ 2,\ \cdots).$$

LEMMA 6. *If the sequence a_k/c_k is admissible with respect to the kernel $G(t)$ and if $G_m(t)$ are defined by* (5), *then $G_m(t)$ are frequency functions with mean equal to zero and with variances equal to $\sum_{k=m+1}^{\infty}(a_k^{-2}-c_k^{-2})$. Further the Laplace transforms*

$$\int_{-\infty}^{\infty}e^{-st}G_m(t)dt=\frac{E_{2,m}(s)}{E_{1,m}(s)}$$

converge absolutely for $\alpha_1<\Re s<\alpha_2$.

This follows from Lemmas 2 and 4.

LEMMA 7. *If $G_m(t)$ are defined by* (5), *then*

$$(7) \qquad G_m(t)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{E_{2,m}(s)}{E_{1,m}(s)}e^{st}ds.$$

This follows from the well-known inversion formula for the bilateral Laplace transform and the convergence of the integral (4).

5. Operational calculus.

Denote by D the operation of differentiation and by $e^{D/a}$ the operation of translation through a distance $1/a$. Suppose that we seek a solution of the differential equation

$$f(x)-\frac{1}{a}f'(x)=\varphi(x),$$

by the operational method. Using the symbol D, we have

$$(1-D/a)f(x)=\varphi(x), \qquad f(x)=1/(1-D/a)\varphi(x),$$

but it remains to interpret the operation $1/(1-D/a)$. Now

$$\frac{1}{1-x/a}=\int_{-\infty}^{\infty}e^{-xy/a}h(y)dy, \qquad 1<x/a,$$

where

$$h(y) = \begin{cases} e^y & (-\infty, 0), \\ 0 & (0, \infty). \end{cases}$$

This is easily verified by direct integration. Hence

$$\frac{1}{1 - D/a}\varphi(x) = \int_{-\infty}^{\infty} e^{-yD/a}\varphi(x)h(y)dy.$$

Thus we may interpret the following operations [5].

$$F(D) \equiv \frac{e^{bD}\prod_{k=1}^{\infty}(1 - D/a_k)e^{D/a_k}}{\prod_{k=1}^{\infty}(1 - D/c_k)e^{D/c_k}},$$

$$F_m(D) \equiv \frac{e^{bD}\prod_{k=1}^{m}(1 - D/a_k)e^{D/a_k}}{\prod_{k=1}^{m}(1 - D/c_k)e^{D/c_k}}.$$

6. Inversion formula.

THEOREM. *If*

(1) $G(t)$ *is defined in* (2) *of* §1,

(2) a_k/c_k *is admissible with respect to* $G(t)$,

(3) $\varphi(x)$ *is bounded and continuous on* $(-\infty, \infty)$,

(4) $F(D)$ *and* $F_m(D)$ *are defined as in* §5,

(5) $f(x) = \int_{-\infty}^{\infty} G(x-t)\varphi(t)dt$,

then

$$F(D)f(x) = \lim_{m\to\infty} F_m(D)f(x) = \varphi(x) \qquad (-\infty < x < \infty).$$

Proof. Our desired result is equivalent to

$$\lim_{m\to\infty}\int_{-\infty}^{\infty} K_m(x-t)\varphi(t)dt = \varphi(x), \qquad \text{where } K_m(x) \equiv F_m(D)G(x).$$

We have applied the operator $F_m(D)$ under the integral sign (5), a step easily justified with present hypotheses.

Again applying $F_m(D)$ to $G(x)$, and using Lemma 7,

$$K_m(x) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{e^{sx}F_m(s)}{F(s)}ds = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{E_{2,m}(s)}{E_{1,m}(s)}e^{sx}ds = G_m(x).$$

From the Lemma 6, we may set

$$I_m(x) = \int_{-\infty}^{\infty} G_m(x-t)\varphi(t)dt - \varphi(x) = \int_{-\infty}^{\infty} G_m(t)[\varphi(x-t) - \varphi(x)]dt.$$

For a fixed x and $\delta > 0$ write the integral $I_m(x)$ as the sum of two integrals $I_m'(x)$ and $I_m''(x)$ corresponding to the ranges of integration $|t| \leq \delta$ and $|t|$

$> \delta$ respectively. Then

$$|I_m'(x)| \leqq \max_{|t| \leqq \delta} |\varphi(x-t) - \varphi(x)|,$$

also

$$|I_m''(x)| \leqq 2 \sup_{-\infty<t<\infty} |\varphi(t)| \int_{|t|>\delta} \frac{t^2}{\delta^2} G_m(t)dt$$

$$\leqq 2/\delta^2 \sup_{-\infty<t<\infty} |\varphi(t)| \sum_{k=m+1}^{\infty} (a_k^{-2} - c_k^{-2})$$

by the properties of Lemma 6. It is thus clear that $I_m'(x)$ can be made small by choice of δ, $I_m''(x)$ by choice of m, so that $I_m(x) \to 0$ when $m \to \infty$, as desired.

REMARK. Hirschman and Widder had studied the inversion and representation theory for the class of convolution transforms (1) of § 1, for which the kernel $G(t)$ has a representation of the form (2) of § 1, where

$$F(s) = e^{bs} \prod_{k=1}^{\infty} (1 - s/a_k) e^{s/a_k}.$$

See [1], [2], [3], [5].

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