NOTES ON CONFORMAL MAPPINGS OF A RIEMANN SURFACE ONTO ITSELF

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It is well-known that a closed Riemann surface of genus $g \geqq 2$ admits only a finite number of conformal mappings onto itself. More precisely, A. Hurwitz [2] has shown that this number does not exceed $84(g-1)$ and this estimation is exact for $g = 3$[1]. On the other hand, a plane region of finite ($\geqq 3$) connectivity admits only a finite number of conformal mappings onto itself, and the estimation of this number has been determined completely by M. Heins [1]. In this paper, we shall treat a bordered Riemann surface and a closed Riemann surface with a finite number of distinguished points.

§ 1. General estimations.

1.1. Let W be a bordered Riemann surface (i. e. a compact subregion of a Riemann surface, the relative boundary of which consists of a finite number of closed analytic curves) and $\mathfrak{G}$ be the group of all conformal mappings of W onto itself. For given integers g ($\geqq 0$) and k ($\geqq 1$), we take the maximum of order of $\mathfrak{G}$ with respect to all W having genus g and k boundary components, and set

$$N(g, k) = \max (\text{ord.}\, \mathfrak{G}).$$

Next, on a closed Riemann surface W of genus g, we take k points $p_1, p_2, \cdots, p_k$ and consider the group $\mathfrak{G}$ of all conformal mappings of the region $W - \{p_1, \cdots, p_k\}$ onto itself. For given integers g and k, we take the maximum of order of $\mathfrak{G}$ with respect to all W of genus g and all sets of k points $p_1, \cdots, p_k \in W$, and set

$$N'(g, k) = \max (\text{ord.}\, \mathfrak{G}).$$

Concerning these numbers, we shall prove the following double inequality:

Theorem 1. *For* $2g + k - 1 \geqq 2 \quad (g \geqq 0,\ k \geqq 1)$,

$$N'(g, k) \leqq N(g, k) \leqq 12(g-1) + 6k^{*)}.$$

Obviously, for $g \geqq 2$, if k is large enough (i. e., $k \geqq 12(g-1)$), the esti-

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1) For $g = 2$, however, it is not exact. In this case, the surface is always hyperelliptic and this fact yields immediately that this number does not exceed 48. For $g \geqq 4$, it seems to remain still open.

*) *Added in proof.* We can really show that $N'(g, k) = N(g, k)$; the detail will be written elsewhere.

mation $N'(g, k) \leqq 12(g-1)+6k$ is worse than that of Hurwitz: $N'(g, k) \leqq 84(g-1)$.

1.2. In order to prove this theorem, we require a lemma:

LEMMA 1. *Let W be a closed Riemann surface of genus $g \geqq 2$, and $p' = \varphi(p)$ be a conformal mapping of W onto itself which is not an identity mapping and has a fixed point p_0. If this mapping is represented as $z' = \varphi(z)$ by a local parameter z about p_0 $(z = 0 \longleftrightarrow p_0)$, then*

$$\frac{d\varphi(0)}{dz} = e^{2\pi i \frac{m}{n}},$$

where m and n are integers and m/n is not an integer.

Proof. The Taylor expansion $\varphi(z) = \alpha z + \alpha' z^2 + \cdots$ yields the expansions $\varphi^2(z) \equiv \varphi \circ \varphi(z) = \alpha^2 z + \cdots$, $\varphi^3(z) \equiv \varphi \circ \varphi \circ \varphi(z) = \alpha^3 z + \cdots$, etc. Since $\mathfrak{G}$ is a finite group, there exists a number n such that $\varphi^n(z) = z$, so that $\alpha^n = 1$ and $\alpha = e^{2\pi i m/n}$. If m/n is an integer, the expansion is $\varphi(z) = z + \beta z^h + \cdots$ $(\beta \neq 0,\ h \geqq 2)$, since $\varphi(z) \neq z$. This implies $\varphi^2(z) = z + 2\beta z^h + \cdots$, $\varphi^3(z) = z + 3\beta z^h + \cdots$, etc. However, from $\varphi^n(z) = z$, we obtain $n\beta = 0$, which is a contradiction, so that m/n is not an integer.

1.3. *The proof of* THEOREM 1. It is not difficult to prove the first inequality $N'(g, k) \leqq N(g, k)$. Let W be a closed Riemann surface with distinguished points $p_1, \cdots, p_k$. If $g \geqq 2$, we take off from W sufficiently small k non-Euclidean discs with the same radius and having centers at $p_1, \cdots, p_k$ respectively. Then, any conformal mapping of the region $W - \{p_1, \cdots, p_k\}$ onto itself can be considered as a conformal mapping of the resulting bordered Riemann surface onto itself. For $g = 1$, we take off Euclidean discs and consider analogously. For $g = 0$ (then $k \geqq 3$), the above reduction is performed with the aid of elementary facts of linear transformations.

The proof of the second inequality $N(g, k) \leqq 12(g-1)+6k$ is essentially a mere modification of Hurwitz's one [2]. Consider the doubled Riemann surface $\hat{W}$ of the given bordered Riemann surface W. It is a closed Riemann surface of genus $\hat{g} = 2g + k - 1$, and any element of $\mathfrak{G}$ can be considered as a conformal mapping of $\hat{W}$ onto itself. Since $\hat{g} \geqq 2$, ord. $\mathfrak{G} = N$ is finite. What we want to show in the sequel is $N \leqq 6(\hat{g} - 1)$.

When we identify the points of $\hat{W}$ which are congruent by the transformation group $\mathfrak{G}$, we obtain a closed Riemann surface W_0, and $\hat{W}$ is an N-sheeted unbounded (but possibly ramified) covering surface of W_0; it is not difficult to verify this fact, because ramifying points are fixed points of the elements of $\mathfrak{G}$ (cf. LEMMA 1) and the number of them is finite. Denoting by g_0 the genus of W_0, we have the following equality from the well-known Hurwitz's formula:

$$2\hat{g} - 2 = N(2g_0 - 2) + \sum \text{(ramification index)}.$$

Now, with respect to a point $p \in W$, we collect all elements of $\mathfrak{G}$ which have a fixed point p, and denote this set by $\mathfrak{G}(p)$. This is a cyclic subgroup of $\mathfrak{G}$. For $p' = \varphi_0(p)$ ($\varphi_0 \in \mathfrak{G}$), we get immediately

$$\mathfrak{G}(p') = \varphi_0 \cdot \mathfrak{G}(p) \cdot \varphi_0^{-1},$$

which implies

$$\text{ord.}\, \mathfrak{G}(p) = \text{ord.}\, \mathfrak{G}(p').$$

Obviously, ord. $\mathfrak{G}(p) - 1$ is the ramification index of p with respect to W_0, so that, for any point $p^0 \in W_0$, the ramification indices of all points over p^0 are the same.

From the symmetricity of $\hat{W}$ and $\mathfrak{G}$, ramifying points are situated symmetrically on $\hat{W}$; furthermore LEMMA 1 shows that there is no such a point on the boundary of W. We project all ramifying points of $\hat{W}$ on W_0 and denote them symmetrically by $p_1^0, \tilde{p}_1^0, \cdots, p_r^0, \tilde{p}_r^0$ and the corresponding ramification indices by $\nu_1 - 1, \cdots, \nu_r - 1$, respectively. Then, since the numbers of points over $p_1^0, \tilde{p}_1^0, \cdots, p_r^0, \tilde{p}_r^0$ are equal to $N/\nu_1, \cdots, N/\nu_r$ respectively, we have

$$\sum (\text{ramification index}) = 2 \sum_{i=1}^{r} \frac{N}{\nu_i}(\nu_i - 1),$$

consequently we get

$$\frac{\hat{g} - 1}{N} = g_0 - 1 + \sum_{i=1}^{r}\left(1 - \frac{1}{\nu_i}\right). \tag{1}$$

(If W is an unramified covering surface of W_0, the $\sum$ term of (1) is absent.)

Now, if $g_0 \geqq 2$, (1) shows $(\hat{g} - 1)/N \geqq g_0 - \geqq 1$ and we have $N \leqq \hat{g} - 1$. If $g_0 = 1$, then $r \geqq 1$ since $\hat{g} \geqq 2$, and from (1), we get $(\hat{g} - 1)/N \geqq 1 - 1/\nu_1 \geqq 1 - 1/2 = 1/2$ and $N \leqq 2(\hat{g} - 1)$. If $g_0 = 0$, then $r \geqq 2$ by the same reason as above. In the case of $g_0 = 0$, $r \geqq 3$, (1) implies $(\hat{g} - 1)/N \geqq r/2 - 1 \geqq 1/2$ and $N \leqq 2(\hat{g} - 1)$. In the case of $g_0 = 0$, $r = 2$, the equalily (1) is

$$\frac{\hat{g} - 1}{N} = 1 - \left(\frac{1}{\nu_1} + \frac{1}{\nu_2}\right).$$

If $\nu_1 \geqq 3$, $\nu_2 \geqq 3$, we get $(\hat{g} - 1)/N \geqq 1 - 2/3 = 1/3$ and $N \leqq 3(\hat{g} - 1)$. If $\nu_1 \geqq 3$, $\nu_2 = 2$, we get $(\hat{g} - 1)/N \geqq 1 - 1/2 - 1/3 = 1/6$ and $N \leqq 6(\hat{g} - 1)$. The case $\nu_1 = \nu_2 = 2$ does not occur. Consequently, in any case, we have $N \leqq 6(\hat{g} - 1) = 12(g - 1) + 6k$.

§ 2. A special case.

2.1. Naturally we ask whether the estimation of THEOREM 1 is exact or not. For the case of $g = 0$, M. Heins [1] has determined numbers N and N', namely he has proved

$$N(0,\ k) = N'(0,\ k) \qquad \text{for} \quad k \geqq 3$$

and

$$N'(0,\ k) = 2k \qquad \text{for} \quad k \neq 4,\ 6,\ 8,\ 12,\ 20,\ k \geqq 3,$$

$$N'(0,\ 4) = 12,\ N'(0,\ 6) = N'(0,\ 8) = 24,\ N'(0,\ 12) = N'(0,\ 20) = 60.$$

We shall determine the number N' for $g=1$.

THEOREM 2. *For $k \geqq 1$,*

$$N'(1,\ k) = \begin{cases} 6k & \text{for } k = m^2 + 3n^2 \\ 4k & \text{for } k = m^2 + n^2, \text{ but not be representable} \\ & \qquad \text{as } k = m^2 + 3n^2, \\ 3k & \text{for } k = 2(m^2 + 3n^2), \text{ but not be representable} \\ & \qquad \text{as } k = m^2 + n^2,^{2)} \\ 2k & \text{for other } k, \end{cases}$$

where $m,\ n = 0,\ 1,\ 2,\ \cdots$.

The conditions for representability of k in above types are obtained by the prime number decompositions: An integer k is representable in the form $k = m^2 + n^2$, if and only if the prime number decomposition of k is $2^{\gamma}\Pi_i p_i^{\alpha_i}\Pi_j q_j^{2\beta_j}$, where α, β and γ are non–negative integers, p and q are prime numbers such that $p \equiv 1$ (mod. 4) and $q \equiv 3$ (mod. 4); similarly, the condition for $k = m^2 + 3n^2$ is $k = 2^{2\gamma}3^{\delta}\Pi_i p_i^{\alpha_i}\Pi_j q_j^{2\beta_l}$ where $p \equiv 1$ (mod. 3) and $q \equiv 2$ (mod. 3) and $q \neq 2$.

Using them, we can compute $N'(1,\ k)$, for instance, as follows:

$$N'(1,\ 1) = 6k = 6, \qquad N'(1,2) = 4k = 8,$$

$$N'(1,\ 6) = 3k = 18, \qquad N'(1,\ 11) = 2k = 22.$$

2.2 In order to prove THEOREM 2 we use the elementary properties of lattices. We consider a lattice in the complex ζ–plane and denote the principal lattice points by ω, ω'. In order that the lattice may be determined uniquely by ω, ω', we must assume that they satisfy the following conditions:

$$(2) \qquad \begin{gathered} -\frac{1}{2} \leqq \Re\frac{\omega'}{\omega} < \frac{1}{2}, \qquad \Im\frac{\omega'}{\omega} > 0, \\ \left|\frac{\omega'}{\omega}\right| \geqq 1 \quad \text{if} \quad -\frac{1}{2} \leqq \Re\frac{\omega'}{\omega} \leqq 0, \\ \left|\frac{\omega'}{\omega}\right| > 1 \quad \text{if} \quad 0 < \Re\frac{\omega'}{\omega} < \frac{1}{2}. \end{gathered}$$

2) In this case, the prime number decomposition shows that k can not be represented as $k = m^2 + 3n^2$.

A point ζ which is representable in the form $\zeta = m\omega + n\omega'$ (m and n are integers) is called a lattice point, and the set of all of them is denoted by

$$L(\omega, \omega') = \{m\omega + n\omega';\ m,\ n = 0,\ \pm 1,\ \pm 2,\ \cdots\}.$$

LEMMA 2. *Let $P = \{s\Omega + t\Omega';\ 0 \leqq s < 1,\ 0 \leqq t < 1\}$ be a parallelogram, four vertices of which are lattice points (i. e. $\Omega,\ \Omega' \in L(\omega, \omega')$). Then, the number of lattice points which belong to P is equal to the ratio of areas, namely it is equal to*

$$|\mathfrak{J}\Omega'\bar{\Omega}|/\mathfrak{J}\omega'\bar{\omega}.$$

This is a famous property of lattice and we omit the proof. The following lemma also seems to be well-known, and it can be proved very easily:

LEMMA 3. *Suppose there exists a linear transformation $\zeta' = \alpha\zeta + \beta$ which gives a one-to-one mapping of $L(\omega, \omega')$ onto itself.*

(i) *If $\omega' = i\omega$, then α must be one among the numbers $\pm 1,\ \pm i$.*

(ii) *If $\omega' = \varepsilon^2\omega\,(\varepsilon = e^{i\pi/3})$, then α must be one among the numbers ± 1, $\pm \varepsilon,\ \pm \varepsilon^2$.*

(iii) *In the other cases, α must be either of ± 1.*

Conversely, in each case, there exist transformations $\zeta' = \alpha\zeta$ with such α.

2.3 For the purpose of preparation, let us consider the group $\mathfrak{G}^*$ of all conformal mappings of a closed Riemann surface W of genus 1 onto itself.

As is known, the group Γ of covering transformations of the universal covering surface $|z| < \infty$ of W consists of linear transformations

$$z' = z + \zeta, \qquad \zeta \in L(\omega, \omega').$$

We may assume that ω'/ω satisfies the condition (2). Then the surface W is determined uniquely by ω'/ω; the surface W with $\omega' = i\omega$ will be denoted by W_i with $\omega' = \varepsilon^2\omega$ ($\varepsilon = e^{i\pi/3}$) will be denoted by W_ε.

An element of $\mathfrak{G}^*$ induces in the well-known manner a linear transformation $S(z) = az + b$ of the universal covering surface $|z| < \infty$ onto itself. It satisfies a relation

$$(3) \qquad z' = z + \zeta \quad \longleftrightarrow \quad S(z') = S(z) + \zeta', \qquad |z| < \infty,$$

where $\zeta,\ \zeta' \in L(\omega, \omega')$ and the correspondence $\zeta \longleftrightarrow \zeta'$ does not depend on the choice of z; in other words, S determines an automorphism of Γ. Conversely, any $S(z) = az + b$ satisfying the relation (3) determines an element of $\mathfrak{G}^*$. So that, denoting by G^* the group of all $S(z) = az + b$ with the condition (3), we get immediately

$$G^*/\Gamma \cong \mathfrak{G}^*.$$

For any W, linear transformations $z' = z + b$ and $z' = -z + b$ are contained in G^* for arbitrary b. As regards $S(z) = az$ $(a \neq \pm 1)$, however, LEMMA 3 shows that $S(z) \notin G^*$ for $W \neq W_i$, W_ε; for W_i, G^* contains $z' = \pm iz$ and only them; for W_ε, G^* contains $z' = \pm \varepsilon z$, $z' = \pm \varepsilon^2 z$ and only them. Now, let G_0^* be a set of all linear transformations $S(z) = z + b$. It is evidently a normal subgroup of G^*, and the above consideration shows

$$(4) \quad G^*/G_0^* \cong \begin{cases} \{I, U\} & \text{for } W \neq W_i,\ W_\varepsilon, \\ & \quad \text{where } I(z) = z,\ U(z) = -z, \\ \{I, V, V^2, V^3\} & \text{for } W_i, \text{ where } V(z) = iz, \\ \{I, T, T^2, T^3, T^4, T^5\} & \text{for } W_\varepsilon, \text{ where } T(z) = \varepsilon z. \end{cases}$$

On the basic surface W, denoting by $\mathfrak{G}_0^*$ the set of all elements of $\mathfrak{G}^*$ which have no fixed point on W, we see immediately that $G_0^*/\Gamma \cong \mathfrak{G}_0^*$. So that $\mathfrak{G}_0^*$ is a normal subgroup of $\mathfrak{G}^*$ and $\mathfrak{G}^*/\mathfrak{G}_0^* \cong G^*/G_0$, and we can see the structure of $\mathfrak{G}^*/\mathfrak{G}_0^*$ immediately from (4).

2.4. *Proof of* THEOREM 2. Let W be a closed Riemann surface of genus 1 with distinguished points $p_1, \cdots, p_k$, and $\mathfrak{G}$ be the group of all conformal mappings of the region $W - \{p_1, \cdots, p_k\}$ onto itself. All elements of $\mathfrak{G}$ are considered as conformal mappings of W onto itself, i.e. $\mathfrak{G} \subset \mathfrak{G}^*$. We denote by $\mathfrak{G}_0$ the set of all elements of $\mathfrak{G}$ which have no fixed point on W. Since $\mathfrak{G}_0 = \mathfrak{G} \cap \mathfrak{G}_0^*$, $\mathfrak{G}_0$ is a normal subgroup of $\mathfrak{G}$ and

$$(5) \qquad \mathfrak{G}/\mathfrak{G}_0 \subset \mathfrak{G}^*/\mathfrak{G}_0^* \cong G^*/G_0^*.$$

It is now not difficult to construct an example such that ord. $\mathfrak{G}_0 = k$, ord. $(\mathfrak{G}/\mathfrak{G}_0) = 2$, concerning any $k \geqq 1$; hence we have

$$N'(1, k) \geqq 2k, \qquad \text{for} \quad k \geqq 1.$$

From the definition of the group $\mathfrak{G}_0$, we can easily see that ord. $\mathfrak{G}_0$ is equal to one of the numbers k, $k/2$, $k/3$, $\cdots$. So that we conclude from (4) and (5) that the possibility ord. $\mathfrak{G} > 2k$ occurs only in the following cases: For W_i, ord. $\mathfrak{G}_0 = k$ and ord. $(\mathfrak{G}/\mathfrak{G}_0) = 4$; for W_ε, ord. $\mathfrak{G}_0 = k$ or $k/2$ and ord. $(\mathfrak{G}/\mathfrak{G}_0) = 3$ or 6; for $\neq W_i$, W_ε, it is impossible. Consequently, for the purpose of determining $N'(1, k)$, it suffices to consider the following four cases:

Case I: On W_i, the distinguished points $p_1, \cdots, p_k$ are congruent to each other by $\mathfrak{G}_0$ and $\mathfrak{G}$ contains an element which corresponds to $V(z) = iz + b$. In this case, ord. $\mathfrak{G} = 4k$.

Case II: On W_ε, $p_1, \cdots, p_k$ are congruent to each other by $\mathfrak{G}_0$, and $\mathfrak{G}$ contains an element corresponding to $T(z) = \varepsilon z + b$. In this case, ord. $\mathfrak{G} = 6k$.

Case III: On W_ε, $p_1, \cdots, p_k$ are congruent to each other by $\mathfrak{G}_0$ and $\mathfrak{G}$ contains an element corresponding to $T_1(z) = \varepsilon^2 z + b$. In this case, ord. $\mathfrak{G} = 3k$.

Case IV: k is even. On W_ε, only $p_1, \cdots, p_{k/2}$ are congruent to each other by $\mathfrak{G}_0$, and $\mathfrak{G}$ contains an element corresponding to $T(z) = \varepsilon z + b$. In this case, ord. $\mathfrak{G} = 3k$.

Now, on the universal covering surface $|z| < \infty$ of W, the groups G, G_0 of linear transformations correspond to $\mathfrak{G}$, $\mathfrak{G}_0$ respectively. Since $G_0/\Gamma \cong \mathfrak{G}_0$ and ord. $\mathfrak{G}_0 \leqq k < \infty$, the group G_0 consists of the transformations of the following forms:

$$z' = z + m\mu + n\mu', \qquad m, n = 0, \pm 1, \pm 2, \cdots,$$

where we may assume thet μ'/μ satisfies the condition (2). The fact $G_0 \supset \Gamma$ implies $L(\mu, \mu') \supset L(\omega, \omega')$.

Case I: Take the point $z = 0$ over a distinguished point p_1. Then the set of all points z that are congruent to $z = 0$ by G_0, namely the set $L(\mu, \mu')$, coincides with the set of all points z situated over $p_1, \cdots, p_k$. Consequently, the principal parallelogram of the lattice (ω, ω'), which is a fundamental region of the group Γ, contains k points of $L(\mu, \mu')$. Then LEMMA 2 shows that k is equal to the ratio of areas of principal parallelograms of lattices (ω, ω') and (μ, μ').

On the other hand, G contains an element $V(z) = iz + b$, which gives a one-to-one transformation of $L(\mu, \mu')$ onto itself, since any element of $\mathfrak{G}$ preserves the set $\{p_1, \cdots, p_k\}$. So that, when we apply LEMMA 3 to $L(\mu, \mu')$, we have $i\mu = \mu'$, which means that lattices (ω, ω') and (μ, μ') are similar. Supposing now $\omega = m\mu + n\mu'$, the side of principal parallelogram of (ω, ω') is $\sqrt{m^2 + n^2} \cdot |\mu|$, and then the ratio of areas is equal to $m^2 + n^2$. Consequently we obtain

$$k = m^2 + n^2.$$

Conversely, if $k = m^2 + n^2$, the above consideration shows that we can easily find points $p_1, \cdots, p_k$ on W_ι so that Case I may occur.

Case II, *Case III* are analogous to the above case. We can see $\mu' = \varepsilon^2\mu$. If $\omega = m\mu + n\mu'$, the side of principal parallelogram of lattice (ω, ω') is $\sqrt{m_2 + n^2 - mn} \cdot |\mu|$, and consequently we have $k = m^2 + n^2 - mn$. It is not difficult to see that k is representable as $k = m^2 + n^2 - mn$ if and only if k is representable as $k = m^2 + 3n^2$.

Case IV: Take $z = 0$ over p_1. Since only $p_1, \cdots, p_{k/2}$ are congruent to each other by $\mathfrak{G}_0$, the number of lattice points of $L(\mu, \mu')$ that are contained in the principal parallelogram of the lattice (ω, ω') is equal to $k/2$. By the assumption of Case IV, G contains a linear transformation of the form $T(z) = \varepsilon z + b$, to which corresponds an element $p' = \varphi(p)$ in $\mathfrak{G}$.

If $\varphi(\{p_1, \cdots, p_{k/2}\}) = \{p_1, \cdots, p_{k/2}\}$, the situation is similar to the case II. We obtain analogously $k/2 = m^2 + n^2 - mn$. (To tell the truth, this case does not occur. We omit the proof of it, since it has no effect on the proof of our theorem.)

If φ does not satisfy the condition above, we can show easily

$$\varphi(\{p_1, \cdots, p_{k/2}\}) = \{p_{k/2+1}, \cdots, p_k\},$$

$$\varphi \circ \varphi(\{p_1, \cdots, p_{k/2}\}) = \{p_1, \cdots, p_{k/2}\}.$$

So that, repeating the same argument as in Case III with respect to $\varphi \circ \varphi$, we get $k/2 = m^2 + n^2 - mn$. Conversely, if $k = 2(m^2 + n^2 - mn)$, we can easily find points $p_1, \cdots, p_k$ on $W_\mathcal{E}$ so that Case IV may occur.

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