ON THE CESÀRO SUMMABILITY OF FOURIER SERIES

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(Comm. by T. Kawata)

1. Let $\varphi(t)$ be an even periodic function with Fourier series

$$(1.1) \qquad \varphi(t) \sim \sum_{n=0}^{\infty} a_n \cos nt, \quad a_0 = 0$$

The α-th integral of $\varphi(t)$ is defined by

$$\Phi_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \varphi(t)(t-u)^{\alpha-1} du, \quad (\alpha > 0)$$

and the β-th Cesáro sum of (1.1) at $t = 0$ is denoted by s_n^β $(\beta > -1)$.

C.Loo [3] proved the following theorem.

THEOREM 1. If $\alpha > 0$ and

$$(1.2) \qquad s_n^\alpha = o(n^\alpha / \log n) \quad \text{as} \quad n \to \infty$$

then

$$(1.3) \qquad \Phi_{\alpha+1}(t) = o(t^{\alpha+1}) \quad \text{as} \quad t \to 0.$$

He proved this Theorem using the Young function. We shall prove, in this paper, this Theorem in another way.

On the other hand, G.H.Hardy and J.E.Littlewood [2] proved that Theorem 1 holds for $\alpha = 0$, under the additional condition

$$a_n = O(n^{-\delta}) \quad \text{as} \quad n \to \infty,$$

for $0 < \delta < 1$. Later, O.Szász [5] proved this Theorem, under some weaker condition. (See Corollary in §3) Concerning this theorem, we shall prove, in this paper, the following theorem.

THEOREM 2. If (1.2) holds for $-1 < \alpha < 0$ and

$$(1.4) \qquad \sum_{\nu=n}^{\infty} |a_\nu| / \nu = O(n^{-\delta})$$

for $0 < \delta < 1$ and $\alpha + \delta > 0$, then we have (1.3).

For the proof, we need the following Lemmas due to G.Sunouchi [4].

LEMMA 1. If $\alpha \geqq 1$ and $\beta \geqq 0$, then

$$(1.5) \qquad \int_0^t u^\beta (t^2 - u^2)^{\alpha-1} \cos nu \, du = O(n^{-\alpha} t^{\alpha+\beta-1}).$$

LEMMA 2. If $2 \geqq \alpha \geqq 0$ and $\beta \geqq 0$, then

$$(1.6) \qquad \int_0^t u^\beta (t-u)^{\alpha-1} \cos nu \, du = O(n^{-\alpha} t^\beta).$$

2. PROOF OF THEOREM 1. Let us write

$$\Phi^*_{\alpha+1}(t) = \sum_{n=0}^{\infty} a_n \int_0^t (t^2 - u^2)^\alpha \cos nu \, du$$

$$= \left(\sum_{n=0}^{M} + \sum_{n=M+1}^{\infty} \right) = I + J,$$

where $M = [t^{-r}]$ and $r > \max(\alpha + 1, (\alpha+1)/\alpha)$.

Since $a_n = o(1)$ as $n \to \infty$, we have, by (1.5),

$$J = O\left(\sum_{n=M+1}^{\infty} n^{-(\alpha+1)} t^\alpha \right) = O(M^{-\alpha} t^\alpha)$$

$$= O(t^{\alpha r + \alpha}) = o(t^{2\alpha+1}),$$

for $\alpha r > \alpha + 1$. Using the well-known formula

$$(2.1) \qquad a_n = \sum_{\nu=0}^{n} (-1)^{n-\nu} \binom{\alpha+1}{n-\nu} s_\nu^\alpha,$$

we have

$$I = \sum_{n=0}^{M} a_n \int_0^t (t^2-u^2)^\alpha \cos nu\, du$$

$$= \sum_{\nu=0}^{M} s_\nu^\alpha \int_0^t \left\{ \sum_{n=\nu}^{M} (-1)^{n-\nu} \binom{\alpha+1}{n-\nu} \cos nu \right\} (t^2-u^2)^\alpha du$$

$$= \sum_{\nu=0}^{M} s_\nu^\alpha \int_0^t \Big[2^{\alpha+1} \left(\sin \tfrac{u}{2}\right)^{\alpha+1} \cos\left\{ \left(\tfrac{\alpha+1}{2}+\nu\right)u + \tfrac{\alpha+1}{2}\pi \right\}$$

$$- \sum_{m=M-\nu+1}^{\infty} (-1)^m \binom{\alpha+1}{m} \cos(m+\nu)u \Big] (t^2-u^2)^\alpha du$$

$$= I_1 - I_2,$$

say. Further, we write

$$I_1 = \Big(\sum_{\nu=0}^{N} + \sum_{\nu=N+1}^{M} \Big) = I_1' + I_1'',$$

where $N = [t^{-1}]$. Then, by

$$\int_0^t u^{\alpha+1} \cos nu\, (t^2-u^2)^\alpha du = O(t^{3\alpha+2}),$$

we have

$$I_1' = o\Big(\sum_{\nu=2}^{N} \frac{\nu^\alpha}{\log \nu} \cdot t^{3\alpha+2} \Big)$$

$$= o\Big(t^{3\alpha+2} \cdot \frac{N^{\alpha+1}}{\log N} \Big) = o(t^{2\alpha+1}),$$

and, by (1.6),

$$I_1'' = o\Big(\sum_{\nu=2}^{N} \frac{\nu^\alpha}{\log \nu} \cdot \frac{t^{2\alpha+1}}{\nu^{\alpha+1}} \Big)$$

$$= o(t^{2\alpha+1} \log r) = o(t^{2\alpha+1}).$$

Concerning I_2, we have

$$I_2 = o\Big(\sum_{\nu=2}^{M} \frac{\nu^\alpha}{\log \nu} \sum_{m=M-\nu+1} \frac{1}{m^{\alpha+2}} \cdot \frac{t^\alpha}{(m+\nu)^{\alpha+1}} \Big)$$

$$= o\Big(\frac{t^\alpha}{M^{\alpha+1}} \sum_{\nu=2}^{M} \frac{\nu^\alpha}{(M-\nu+1)^{\alpha+1}} \Big)$$

$$= o\Big(\frac{t^\alpha}{M} \sum_{\nu=1}^{M} \frac{1}{(M-\nu+1)^{\alpha+1}} \Big)$$

$$= o(M^{-1} t^\alpha) = o(t^{\alpha+r}) = o(t^{2\alpha+1}),$$

for $r > \alpha+1$.

Thus, we have

$$\Phi_{\alpha+1}^*(t) = o(t^{2\alpha+1}).$$

Then, by K.Chandrasekharan and O.Szász's Theorem [1], we have

$$\Phi_{\alpha+1}(t) = o(t^{\alpha+1}),$$

which is the required.

3. PROOF OF THEOREM 2. The case $\alpha = 0$ is due to O.Szasz [5]. For $\alpha < 0$, we write

$$\Gamma(\alpha+1)\, \Phi_{\alpha+1}(t)$$

$$= \sum_{n=0}^{\infty} a_n \int_0^t \cos nu\, (t-u)^\alpha du$$

$$\doteq \Big(\sum_{n=0}^{M} + \sum_{n=M+1}^{\infty} \Big) = I + J$$

where $M = [t^{-r}]$ and $r > (\alpha+1)/(\alpha+\delta)$. Let us write

$$p_n = \sum_{\nu=n}^{\infty} |a_\nu|/\nu,$$

then $|a_n| = n(p_n - p_{n+1})$ and for $\alpha+1 > \varepsilon > 1-\delta$,

$$\sum_{\nu=m}^{n} |a_\nu|/\nu^\varepsilon = \sum_{\nu=m}^{n} \nu^{1-\varepsilon} (p_\nu - p_{\nu+1})$$

$$= O(m^{-\varepsilon-\delta+1}) + O\Big(\sum_{\nu=m}^{n} \nu^{-\varepsilon-\delta} \Big)$$

$$= O(m^{-\varepsilon-\delta+1}).$$

Thus, we have, by (1.4) and (1.5),

$$J = O\Big(\sum_{\nu=M+1}^{\infty} |a_\nu|/\nu^{\alpha+1} \Big)$$

$$= O\Big(\sum_{\nu=M}^{\infty} \frac{|a_\nu|}{\nu^\varepsilon} \cdot \nu^{\varepsilon-\alpha-1} \Big)$$

$$= O(M^{\varepsilon-\alpha-1} M^{\varepsilon-\delta+1})$$

$$= O(M^{-\alpha-\delta}) = O(t^{(\alpha+\delta)r})$$

$$= O(t^{\alpha+1})$$

for $(\alpha+\delta)r - (\alpha+1) > 0$.

Next, we shall prove $I = o(t^{\alpha+1})$. By the formula (2.1), we have

$$I = \sum_{n=0}^{M} a_n \int_0^t \cos nu\, (t-u)^\alpha du$$

$$= \sum_{\nu=0}^{M} s_\nu^\alpha \int_0^t \Big[2^{\alpha+1} \left(\sin\frac{u}{2}\right)^{\alpha+1} \cos\left\{\left(\frac{\alpha+1}{2}+\nu\right)u + \frac{\alpha+1}{2}\pi\right\} - \sum_{m=M-\nu+1}^{\infty} (-1)^m \binom{\alpha+1}{m} \cos(m+\nu)u \Big] (t-u)^\alpha du$$

$$= I_1 - I_2 ,$$

say. Further, we write

$$I_1 = \Big(\sum_{\nu=0}^{N} + \sum_{\nu=N+1}^{M} \Big) = I_1' + I_1'' ,$$

where $N = [t^{-(\alpha+1)}]$. Then, by (1.2) and (1.6),

$$I_1'' = o\Big(\sum_{\nu=N+1}^{M} \frac{\nu^\alpha}{\log \nu} \cdot \frac{t}{\nu^{\alpha+1}} \Big)$$

$$= o\Big(\sum_{\nu=N}^{M} \frac{t^{\alpha+1}}{\nu \log \nu} \Big)$$

$$= o\Big(t^{\alpha+1} \log \frac{r}{\alpha+1} \Big) = o(t^{\alpha+1}).$$

Since, by the second mean value theorem,

$$\int_0^t \left(\sin\frac{u}{2}\right)^{\alpha+1} \cos\left\{\left(\frac{\alpha+1}{2}+\nu\right)u + \frac{\alpha+1}{2}\pi\right\} (t-u)^\alpha du$$

$$= O(t^{2\alpha+2}),$$

we have

$$I_1' = o\Big(\sum_{\nu=2}^{N} \frac{\nu^\alpha}{\log \nu} \cdot t^{2\alpha+2} \Big)$$

$$= o\Big(\frac{N}{\log N} \cdot t^{2\alpha+2} \Big) = o(t^{\alpha+1}).$$

By the following estimation:

$$\sum_{\nu=1}^{M} \frac{\nu^\alpha}{(M-\nu+1)^{\alpha+1}}$$

$$= \sum_{\nu=1}^{M/2} \frac{\nu^\alpha}{(M-\nu+1)^{\alpha+1}} + \sum_{\nu=M/2+1}^{M} \frac{\nu^\alpha}{(M-\nu+1)^{\alpha+1}}$$

$$= O\Big(M^{-\alpha-1} \sum_{\nu=1}^{M/2} \nu^\alpha \Big) + O\Big(M^\alpha \sum_{\nu=1}^{M} \frac{1}{(M-\nu+1)^{\alpha+1}} \Big)$$

$$= O(M^{-\alpha-1} M^{\alpha+1}) = O(M^\alpha M^{-\alpha})$$

$$= O(1),$$

we have

$$I_2 = o\Big(\sum_{\nu=2}^{M} \frac{\nu^\alpha}{\log \nu} \sum_{\nu=M-\nu+1}^{\infty} \frac{1}{m^{\alpha+2}} \cdot \frac{1}{(m+\nu)^{\alpha+1}} \Big)$$

$$= o\Big(M^{-(\alpha+1)} \sum_{\nu=1}^{M} \frac{\nu^\alpha}{(M-\nu+1)^{\alpha+1}} \Big)$$

$$= o(M^{-(\alpha+1)}) = o(t^{(\alpha+1)r}) = o(t^{\alpha+1}).$$

Thus, we have

$$\Phi_{\alpha+1}(t) = o(t^{\alpha+1}),$$

which is the required.

We end this paper by proving

COROLLARY. If (1.2) holds and

$$\sum_{\nu=n}^{2n} (|a_\nu| - a_\nu) = O(n^{1-\delta}) \quad \text{as } n \to \infty,$$

for $0 < \alpha+\delta \leq 1$ and $0 < \delta < 1$, then we have (1.3).

PROOF. The case $\alpha = 0$ is due to O.Szász [5].

From (1.2), we have, by the well-known theorem,

$$s_n^0 = s_n = o(n^\alpha) = o(n^{1-\delta}).$$

Then, using Szász's argument [5], we have

$$\sum_{\nu=n}^{\infty} |a_\nu|/\nu = O(n^{-\delta}),$$

which is (1.4).

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(*) Received October 10, 1955.