ON THE GREATEST SEMILATTICE DECOMPOSITION OF A SEMIGROUP

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If there exists a homomorphism of a semigroup S onto a semigroup S^*, S is decomposed into the class sum of mutually disjoint subsets $\{S_{x^*}\}_{x^*\in S^*}$, each of which is a inverse image of some element x^* of S^*; i.e., $S = \cdot\sum_{x^*\in S^*} S_{x^*}$. ($\cdot\sum$ is meant the direct sum of sets) In this case, clearly $\{S_{x^*}\}_{x^*\in S^*}$ forms a factor algebraic system of S and is isomorphic to S^*. We call such a partition of S a decomposition of S to S^*, and each S_{x^*} a residue class of this decomposition. The decomposition to a semilattice is most important among others; i.e., $S = \cdot\sum S_\alpha$ where every S_α is a semigroup and for any S_β, S_γ, there exists a unique S_δ such that $S_\gamma S_\beta \subset S_\delta$ and $S_\beta S_\gamma \subset S_\delta$. Henceforward we will call the decomposition of S to a semilattice the semilattice decomposition of S.

Generally there exist many semilattice decompositions of a semigroup, but since it is proved that the collection of all semilattice decompositions of a semigroup forms a complete semilattice, there exists the greatest one.

In this paper, we shall determine the greatest semilattice decomposition of a semigroup. T. Tamura and N. Kimura showed that such the decomposition of a commutative semigroup is determined as the decomposition to the factor algebraic system under a congruence relation ($\sim$) introduced as follows (1). $a \sim b$ if $a^n = bx$ and $b^m = ay$ for some positive integers m, n and some elements x, y.

In this paper, the author deals with general case. To abbreviate the terminology, from now on, S denotes a general semigroup and the symbol $\exists$ denotes the word 'exist'. Hence if we describe as $\exists x$; ———, it means that there exists an element x which satisfies the relation ———.

§1 Semilattice decomposition

If we define $a \sim b$ between elements a, b of S to mean that a, b are contained in a same residue class of a semilattice decomposition of S, then the relation ($\sim$) is a congruence relation of S which satisfies the following two conditions:

(1) $a^2 \sim a$ for any $a \in S$.
(2) $ab \sim ba$ for any $a, b \in S$.

Moreover this converse also holds good; i.e.,

Lemma 1. If a congruence relation ($\sim$) which satisfies two conditions (1), (2) is defined on S, then the factor algebraic system of S under the relation ($\sim$) forms a semilattice.

Proof. Obvious by the definition of the congruence relation.

We turn our attention to a subsemigroup S' of S which has the following property (P):

(P) For any number of elements $a_1, a_2, \ldots\ldots, a_n \in S$, $S' \ni a_1 a_2 \ldots\ldots a_n$ implies $\xi_1 \xi_2 \ldots\ldots \xi_m \in S'$ for any number of elements $\xi_1, \xi_2, \ldots\ldots, \xi_m \in S$ which satisfy the relation $\{\xi_j\}_{j=1}^{m} = \{a_i\}_{i=1}^{n}$.

We call such a subsemigroup to be a P-subsemigroup of S.

Lemma 2. If S' is a P-subsemigroup of S, then
(1) $xy \in S'$ implies $yx \in S'$ for any $x, y \in S$.
(2) $x^k \in S'$ implies $x \in S'$ for any $x \in S$ and for any positive integer k.

Proof. Since S' has the property (P), if we set $m = 2$, $n = 2$, $a_1 = x$, $a_2 = y$, $\xi_1 = y$ and $\xi_2 = x$ in the abovementioned property (P) the first part of this Lemma is proved. Similarly if we set $n = k$, $m = 1$, $a_i = x$ ($i = 1, \ldots\ldots, k$) and $\xi_1 = x$ the second part follows.

We denote by Ω the collection of all P-subsemigroups of S, and by S_α, S_β, ... etc. elements of Ω, that is, P-subsemigroups of S. We introduce by a subcollection Γ of Ω the following relation ($\widetilde{\Gamma}$), which is closely related to one defined by Pierce (2).

If $\{(x,y)\mid xay\in S_\alpha\}=\{(x,y)\mid xby\in S_\alpha\}$ for every element $S_\alpha\in\Gamma$, then $a\underset{\Gamma}{\sim}b$ in S.

It is easy to see $(\tilde{\Gamma})$ to be an equivalence relation of S.

Lemma 3. $(\tilde{\Gamma})$ is congruence relation, and the factor algebraic system of S under $(\tilde{\Gamma})$ forms a semilattice. Therefore, $(\tilde{\Gamma})$ gives one of semilattice decompositions of S.

Proof. We first show that $a\underset{\Gamma}{\sim}b$ implies $ac\underset{\Gamma}{\sim}bc$ as well as $ca\underset{\Gamma}{\sim}cb$ for any $c\in S$. Let S_α be any element of Γ. Then $xa(cy)=xacy\in S_\alpha$ implies $xb(cy)=xbcy\in S_\alpha$. Conversely, $xb(cy)=xbcy\in S_\alpha$ implies $xa(cy)=xacy\in S_\alpha$. Hence $\{(x,y)\mid xacy\in S_\alpha\}=\{(x,y)\mid xbcy\in S_\alpha\}$ for any $S_\alpha\in\Gamma$, and this implies $ac\underset{\Gamma}{\sim}bc$. Similarly $ca\underset{\Gamma}{\sim}cb$ is easy to prove. Next if $a\underset{\Gamma}{\sim}b$ and $c\underset{\Gamma}{\sim}d$ are assumed, then $ac\underset{\Gamma}{\sim}bc$, $bc\underset{\Gamma}{\sim}bd$ follow from the above. Hence $ac\underset{\Gamma}{\sim}bd$ by transitivity. This implies $(\tilde{\Gamma})$ to be a congruence relation. Since S_α is a P-subsemigroup of S, $xa^2y\in S_\alpha$ or $xaby\in S_\alpha$ is equivalent to $xay\in S_\alpha$ or $xbay\in S_\alpha$ respectively. Therefore, $a\underset{\Gamma}{\sim}a^2$ and $ab\underset{\Gamma}{\sim}ba$. Accordingly, the remainder of our Lemma follows from Lemma 1.

Lemma 4. Any semilattice decomposition of S is the decomposition to the factor algebraic system of S under a congruence relation $(\tilde{\Gamma})$ introduced by some subcollection Γ of Ω.

Proof. Let $S=\sum_\alpha D_\alpha$ be a semilattice decomposition of S. Since it is not hard to verify that each D_α is a P-subsemigroup of S, $\Gamma\subset\Omega$ if we set $\Gamma=\{D_\alpha\}_\alpha$. Take up any two elements a, b contained in a same residue class D_α. Then since $xay\in D_\beta$ is equivalent to $xby\in D_\beta$ for any $D_\beta\in\Gamma$ and any $x,y\in S$. Therefore $a\underset{\Gamma}{\sim}b$ follows from the definition of $(\tilde{\Gamma})$. On the other hand if $a\underset{\Gamma}{\sim}b$, then $aba\in D_\alpha$ and $bab\in D_\beta$ hold good because of $aaa\in D_\alpha$ and $bbb\in D_\beta$, where D_α or D_β is a residue class such that it contains an element a or b respectively. As D_α and D_β are P-subsemigroups of S, ab and ba are contained in both D_α and D_β. Hence $a\underset{\Gamma}{\sim}b$ implies $D_\alpha=D_\beta$, that is, a, b are contained in a same residue class of the decomposition.

Summarizing the above-mentioned results, we obtain the following Theorem which will play an important part in the next paragraph.

Theorem 1. Any semilattice decomposition of S is the decomposition to the factor algebraic system of S under a congruence relation $(\tilde{\Gamma})$ introduced by some subcollection Γ of Ω. Conversely, the decomposition to the factor algebraic system of S under a congruence relation $(\tilde{\Gamma})$ introduced by any subcollection Γ of Ω is a semilattice decomposition of S.

§2 Greatest semilattice decomposition

Let $\varphi;\ S=\sum_\alpha S_\alpha$ and $\psi;\ S=\sum_{\alpha'} S_{\alpha'}$ be any two semilattice decompositions of S. We define an ordering $\varphi\geq\psi$ between φ and ψ to mean that for any S_α there exists $S_{\beta'}$ such that $S_\alpha\subset S_{\beta'}$.

Then the collection D of all semilattice decompositions of S forms not only a partial ordered set but also a complete semilattice (1), and therefore there exists the greatest element, that is, the greatest semilattice decomposition of S. Our first purpose of this paragraph is to show that the greatest semilattice decomposition of S is the decomposition to the factor algebraic system of S under the congruence relation $(\tilde{\Omega})$, and the second is to obtain a necessary and sufficient condition for each redisue class of the greatest semilattice decomposition to be either a nonpotent semigroup or a unipotent semigroup (3), (4).

Theorem 2. The greatest semilattice decomposition of S is the decomposition to the factor algebraic system of S under the congruence relation $(\tilde{\Omega})$.

Proof. Let $\varphi;\ S=\sum S_\alpha$ be the greatest semilattice decomposition of S. Then $\Gamma\subset\Omega$ if we set $\Gamma=\{S_\alpha\}_\alpha$, because each residue class of any semilattice decomposition of S is a P-subsemigroup of S. Hence for any $a,b\in S$, $a\underset{\Omega}{\sim}b$ implies $a\underset{\Gamma}{\sim}b$. On the one hand the decomposition to the factor algebraic system under the congruence relation $(\tilde{\Gamma})$ is the greatest semilattice decomposition of S as is seen in the proof of Lemma 4, and on the other hand, by Lemma 3, the decomposition to the factor algebraic system under the congruence relation $(\tilde{\Omega})$ is a semilattice decomposition of S. Therefore $(\tilde{\Gamma})=(\tilde{\Omega})$, i.e., $(\tilde{\Omega})$ gives the greatest semilattice decomposition of S.

Corollary 1. The greatest semilattice decomposition of a commutative semigroup S is the decomposition to the factor algebraic system of S under a congruence relation ($\approx$) introduced as follows (1).

$a \approx b$ if $a^m = bx$ and $b^m = ay$ are satisfied for some positive integers m, n and some elements $x, y \in S$.

Proof. First of all, since S is a commutative semigroup, a relation $\{(x, y) \mid xay \in S_\alpha\} = \{(x, y) \mid xby \in S_\alpha\}$ is equivalent to a relation $\{x \mid xa \in S_\alpha\} = \{x \mid xb \in S_\alpha\}$ for any $S_\alpha \in \Omega$. We first show that $a \underset{\Omega}{\sim} b$ implies $a \approx b$ for any elements $a, b \in S$. By the definition, $a \underset{\Omega}{\sim} b$ means $\{x \mid xa \in S_\alpha\} = \{x \mid xb \in S_\alpha\}$ to be satisfied for any $S_\alpha \in \Omega$. If we set $S' = \{t \mid \exists$ positive integers m, n $\exists$ elements x, y ; $a^m = tx, t^n = ay\}$, S' is clearly a P-subsemigroup which contain the element a. Therefore $\{x \mid xa \in S'\} = \{x \mid xb \in S'\}$. Since $a \in S'$, the following results follow in order; i.e., $a^2 \in S'$, $ab \in S'$, $b^2 \in S'$ and consequently $b \in S'$.
Therefore, there exist positive integers m, n and elements x, y such that

$$a^m = bx \quad \text{and} \quad b^n = ay$$

Hence $a \approx b$. Next, $a \approx b$ implies $a \underset{\Omega}{\sim} b$ for any elements $a, b \in S$.
If $a \approx b$, there exist positive integers m, n and elements x, y such that

$$a^m = bx \quad \text{and} \quad b^n = ay$$

Take up any $S_\alpha \in \Omega$. Then if $ta \in S_\alpha$, the following relation are satisfied in order; i.e., $ta^{mn} \in S_\alpha$, $tayx^n \in S_\alpha$, $ta^{m+1}yx^n \in S_\alpha$, $tay^2bx^{n+1} \in S_\alpha$, $taya^m \in S_\alpha$, $tb^n \in S_\alpha$ and consequently $tb \in S_\alpha$. Hence $ta \in S_\alpha$ implies $tb \in S_\alpha$ for any element $t \in S$. Similarly for any $t \in S$, $tb \in S_\alpha$ implies $ta \in S_\alpha$. Thus $a \underset{\Omega}{\sim} b$. Therefore $(\approx) = (\underset{\Omega}{\sim})$. This completes the proof of this corollary.

Theorem 3. In the greatest semilattice decomposition of S, each of residue classes is either a nonpotent semigroup or a unipotent semigroup if and only if, for each pair of mutually different idempotent elements e_1, e_2, there exists a P-subsemigroup S' of S such that either $S' \ni e_1$ but $S' \not\ni e_2$ or $S' \ni e_2$ but $S' \not\ni e_1$.

Proof. Since necessity of the condition is obvious, we may prove only sufficiency. We assume that there exists a pair of mutually different idempotent elements e_1, e_2 such that $e_1 \underset{\Omega}{\sim} e_2$. By the hypothesis, there exists a P-subsemigroup S' of S such that either $S' \ni e_1$ but $S' \not\ni e_2$ or $S' \ni e_2$ but $S' \not\ni e_1$. Without loss of generality, we may assume $S' \ni e_1$ but $S' \not\ni e_2$. Since $e_1 \underset{\Omega}{\sim} e_2$ and $e_1 e_1 e_1 \in S'$, $e_1 e_2 e_1 \in S'$. Hence $e_1 e_2 \in S'$ because S' is a P-subsemigroup of S. Therefore $e_2 e_1 e_2 \in S'$; hence $e_2 e_2 e_2 \in S'$; hence $e_2 \in S'$. This is contradictory to $e_2 \notin S'$. Thus, there exist no pairs of mutually different idempotent elements e_1, e_2 such that $e_1 \underset{\Omega}{\sim} e_2$.

Remark. In the greatest semilattice decomposition of a general semigroup, each of residue classes is not necessarily a nonpotent or unipotent semigroup. This is obtained by a simple example as follows.

Example. Let S be a right singular semigroup (4) consisting of two or more elements. Since S contains no P-subsemigroups of S except S own, residue classes of the greatest semilattice decomposition of S are S alone. However, S is neither a nonpotent semigroup nor a unipotent semigroup.

Corollary 2. In the greatest semilattice decomposition of a commutative semigroup, each of residue classes is either a nonpotent semigroup or a unipotent semigroup.

Proof. Let S be a commutative semigroup and e_1, e_2 be two mutually different idempotent elements of S. If we set $S_1 = \{a \mid \exists x, y \in S, \exists$ positive integer n ; $e_1 = ax$, $a^n = e_1 y\}$ then S_1 is a P-subsemigroup of S. It is obvious that $e_1 \in S_1$. If e_2 is also contained in S_1, then there exist elements x, y such that $e_1 x = e_2$ and $e_2 y = e_1$.
Therefore $e_1 e_2 = e_1 e_1 x = e_1 x = e_2$ and $e_2 e_1 = e_2 e_2 y = e_2 y = e_1$, and consequently $e_1 = e_2$. Hence $e_2 \notin S_1$. By Theorem 3, this completes the proof of this Corollary.

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Differentialgeometrie, I. Von P. K. Strubecker. Slg. Goschen Bd. 1113-1113a, 18 Fig. 150 S. 1955 geh. DM 4.80.

This book deals with the elementary differential geometry of curves and surfaces in two- and three-euclidean spaces.

The contents of this book which are divided into two parts, may be sketched as follows. The first is the theory of plane curves, and the second that of space curves. The former contains: § 1. Vector calculus in plain. § 2-5. Representations of curves, Tangential formulas, etc. § 6-7. Arc length, and its geometrical meaning. § 11-14. Natural equations of curves, Canonical representation, Osculation of higher orders, Circle of curvature. § 15-16. Evolute, Involute. § 17-18. Special curves, etc. The latter contains: § 1. Vector calculus in the plain. § 2-6. Representations of curves, Arc length, Osculating circles, Principal normal, Binormal. Formulas of Frenet. § 7. Metric classification of space curves by E. Study. § 8-10. Three spheric images of curves and their examples. § 11-13. Canonical development, Natural equations. etc. § 14. Osculation of higher orders, Osculating circle, Sphere, Spherical curves. § 15-16. Families of surfaces, etc. § 17-18. Various sorts of torsions, etc. § 19-20. Evolute surface, Involute surface. § 21. The theory of isotropic space curves.

The book covers the whole field of the elementary differential geometry, the vector notation being adopted throughout. Concise and clear explanations can be found passim. Both relevant remarks and rich examples in the book will help the reader in getting the ideal which the author wants to tell in the book. We may say, at the close of this short comments, the book is very handy for students.

(A. Kuribayasi)

<u>Fünfstellige Tafeln der Kreis- und Hyperbelfunktionen. Neudruck.</u> By Keiichi Hayashi. Walter de Gruyer & Co. Berlin. 1955. 182 pp. DM 12.00.

This is the "Neudruck" of the table published first in 1921. This interesting and useful table gives five figure values of trigonometric and hyperbolic functions: cos x, sin x, tan x, cosh x, sinh x, tanh x, six figure values of e^x and seven figure values of e^{-x} for

$x = 0\ (.0001)\ 0.1$
$x = 0.1\ (.001)\ 3.0$
$x = 3.0\ (.01)\ 6.3$
$x = 6.3\ (.1)\ 10.0$

and $x = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3}{2}\pi, \pi, \frac{5}{4}\pi, \frac{3}{2}\pi, \frac{7}{4}\pi, 2\pi, \frac{9}{4}\pi, \frac{5}{2}\pi, \frac{11}{4}\pi, 3\pi.$

The argument x is measured in radian and its value in degree (noted φ in the table) is also given for each above mentioned value of x to two decimals in second. In six final pages, there are also well chosen lists of formulas relevant to the functions tabulated and a page of "conversion table of radians (x) into degrees (φ)".

The values of all functions are justaposed in two successive pages, so that the users of this table get the facilities of finding the values of trigonometric, hyperbolic and exponential functions of the same argument x at a time.

(Kazumichi Hayashi, Tokyo Institute of Technology.)

Differential- und Integralrechnung, unter besonderer Berücksichtigung neuerer Ergebnisse. (Göschens Lehrbücherei, I. Gruppe : Reine und angewandte Mathematik Bd. 26.) III. Band: Integralrechnung. Zweite, völlig neubearbeitete Auflage. Von Otto Haupt, Georg Aumann und Christian Y. Pauc. Walter de Gruyter & Co., Berlin, 1955. xii+319 Seiten. DM 28.00.

The present work is formally the second edition of a book with the same title written by the first two of the authors and published in 1938. However, its contents are, compared with the former edition, so substantially revised throughout that it seems to be quite another new book. Attempting to make the reader familiar with new formulations and methods in the theory of integrals from classical as well as modern view-points, this book takes an intermediate situation. For instance, on the one hand, measures and integrals are dealt with in usual sense while, on the other hand, the theory of linear functionals is developed as an extension of Lebesgue integral.

The titles of contents listed in the following lines will well explain an extensive and profound character of this book:

First part. Contents, measures and their extensions. I. Introduction to the theory of Boolean lattices. II. General theorems on contents and measures. III. Extension of contents and measures.

Second part. Integrals by subdivision and σ-additive functions. Linear functionals. IV. Integral by subdivision belonging to a measure. V. Additive functions with arbitrary sign. VI. Linear continuous functionals. VII. Measures and integrals in product spaces. Multiple integrals.

Third part. Measures and integrals in topological spaces. VIII. Measures and contents adaptive to a topology. Integrals belonging to them.

Fourth part. Primitive functions. Indefinite integral. IX. σ-additive function as a primitive function. X. Additive function as a primitive function.

Fifth part. Some Applications. XI. Functions and surfaces of bounded dilatation in E_n.

Literature.

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