ON THE NULL-SET OF A SOLUTION FOR THE EQUATION $\Delta u + k^2 u = 0$

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In the present paper we shall give a certain character of a null-set for the solution of the equation $\Delta u + k^2 u = 0$.

Before we formulate our main proposition we will show the following proposition I which leads us immediately to our main proposition II.

Proposition I. Let $m$ be a set of logarithmic mass zero which is contained in a domain $D_0$ with boundary $C$. If a function $u$, which is continuously differentiable twice and bounded in $D_0 - m$, satisfies an equation $\Delta u + k^2 u = 0$ in $D_0 - m$, $k$ being a constant, then $u$ satisfies necessarily the equation also for the points of the set $m$. Therefore, $u$ becomes analytic in the whole domain $D_0$ including even the set $m$.[1)]

Proof. Our method for the proof of the proposition I follows that of the Lindeberg's theorem[2)], suitably modified. Now, since we may suppose the set $m$, laid on the t-Plane, to be bounded, so we can cover it with a finite number of circles, $|t - a_\nu| < \rho_\nu$ $\nu = 1, 2, \cdots, n$, where, for any preassigned positive number $\varepsilon$, the $\rho$ 's satisfy a condition

$$(1) \quad \sum_{\nu=1}^{n} \frac{1}{|\log \rho_\nu|} < \varepsilon \quad .$$

Remove all these circles from the domain $D_0$ and denote by $D_\varepsilon$ the domain thus obtained. Let $C + C_\varepsilon$ be the boundary of the domain $D_\varepsilon$.

Now let us consider another function $v$ which satisfies the equation $\Delta v + k^2 v = 0$ everywhere in $D_0$ including the set $m$, and which on $C$ has the same boundary values as those of $u$. It is sufficient for the proof of our proposition to show that $u$ coincides with $v$ identically in $D_0 - m$. Now, let, by assumption, $|u| < k_1$ then the function $u - v$ has the following properties:

$$\Delta(u-v) + k^2(u-v) = 0 \quad \text{in} \quad D_\varepsilon$$

and

$$u - v = 0 \quad \text{on} \quad C \;,$$

$$|u - v| < k_1 + k_1' = K \quad \text{on} \quad C_\varepsilon \;,$$

since there exists a constant $k_1'$ such that $|v| < k_1'$ on $C_\varepsilon$.

If we define a function $W_\varepsilon$ by an equation

$$(2) \quad W_\varepsilon = K \sum_{\nu=1}^{n} \frac{Y_0(k|t - a_\nu|)}{\frac{1}{2}\log \rho_\nu} \;,$$

where $Y_0$ denotes the Neumann's cylindrical function, then it is obvious that $\Delta W_\varepsilon + k^2 W_\varepsilon = 0$ in $D_\varepsilon$. And $W_\varepsilon$ will behave as a majorant of $u - v$, that is, $W_\varepsilon > u - v$ in $D_\varepsilon$. In order to show this fact, we first investigate the boundary properties of the function $W_\varepsilon$. Let the distance between any point of $C$ and any point of $m$ be less than the number $\gamma_0/k$ where $\gamma_0$ denotes the smallest positive zero-point of $Y_0$ and let $\rho_\nu < 1$, then $W_\varepsilon > 0$ on $C$, since $Y_0(k|t - a_\nu|)$ becomes negative in $D_0$.

What we can next say about boundary property of $W_\varepsilon$ is that on $C_\varepsilon$ $W_\varepsilon > K$. In fact, there holds a limit equation

$$\lim_{x \to 0} \frac{Y_0(kx)}{\frac{2}{\pi}\log x} = 1 \;,$$

which implies

$$\frac{Y_0(kx)}{\frac{1}{2}\log x} > 1$$

for sufficiently small enough $|x|$. Hence we may suppose

$$\frac{Y_0(k\rho_\nu)}{\frac{1}{2}\log \rho_\nu} > 1$$

for points on the $\nu$ -th circle lying on $C_\varepsilon$, and further, remembering that $D_0$ is taken small enough,

$$\frac{Y_0(k|t - a_\mu|)}{\frac{1}{2}\log \rho_\mu} > 0 \qquad (\mu \neq \nu)$$

$t$ on the $\nu$-th circle.

Finally let us consider a function $\overline{W}_\varepsilon$ defined by $\overline{W}_\varepsilon = W_\varepsilon - (u - v)$. It is readily seen that $\Delta \overline{W}_\varepsilon + k^2 \overline{W}_\varepsilon = 0$ in $D_\varepsilon$ and $\overline{W}_\varepsilon > 0$ on the boundary of $D_\varepsilon$, that is on $C + C_\varepsilon$. With help of a character of the first eigen-value as a domain function, we can conclude further that the boundary property for $W_\varepsilon$ remains to hold inside of $D_\varepsilon$ as well, that is, $W_\varepsilon > 0$ in $D_\varepsilon$. We will show a reasoning for the above conclusion in the following way.

Let the domain $D_0$ be included entirely in a circle $E$ with radius $\lambda_0/k$, where $\lambda_0$ denotes the smallest positive zero-point of $J_0$. The first eigen-value of the equation $\Delta u + c^2 u = 0$ for $E$ under the boundary condition $u = 0$, is given by $\lambda_0 / (\lambda_0/k) = k$.

Let $k_0$ be the first eigen-value of the same equation for $D_0$ under the boundary condition $u = 0$, then $k_0 \geq k$, since $D_0 \subseteq E$ and the first eigen-value is a monotone decreasing domain function in the strict sense. Now suppose that a domain $G$ for which $W_\varepsilon < 0$ appeared interior of $D_0$. In this case our $k$ could be regarded as the first eigen-value of the equation $\Delta u + k^2 u = 0$ for the domain $G$ under the boundary condition $u = 0$. As $G \subsetneqq D_0$, $k > k_0$. The last inequality leads to a contradiction. Therefore $W_\varepsilon > 0$ in $D_\varepsilon$. Hence $w_\varepsilon > u - v$ in $D_\varepsilon$.

Finally we will show that $\lim_{\varepsilon \to 0} w_\varepsilon = 0$ for every point of $D_0 - m$. In fact, if $t$ is any fixed point belonging to $D_0$ outside $C_\varepsilon$ and put $B = \min_\nu |t - a_\nu|$, then

$$|Y_0(k|t - a_\nu|)| \leqq |Y_0(kB)|,$$

so that we get an inequality

$$0 < w_\varepsilon < 2K |Y_0(kB)| \sum_{\nu=1}^{n} \frac{1}{|\log \rho_\nu|}$$

As the set $m$ is supposed to be of logarithmic mass zero, $w_\varepsilon$ tends to zero for every point of $D_0 - m$. After all $u - v \leqq 0$ in $D_0 - m$, and similarly we are also able to get the opposite inequality $u - v \geqq 0$ by considering a function $W'_\varepsilon = w_\varepsilon - (v - u)$.

Thus we have shown that $u - v \equiv 0$ for the set $D_0 - m$ so that the proof is completed.

We are now in the position to formulate our main proposition II. Here we will use the same notations as those in the previous proposition.

Proposition II. If a bounded positive function $u$ satisfies the equation $\Delta u + k^2 u = 0$ at every point in $D_0$ excluding a set $m$ of logarithmic mass zero, then the function $u$ never could become zero only for the point of the set $m$.

Proof. According to the proposition I, the function $u$ becomes analytic throughout in the whole domain $D_0$. Therefore the set of points for which $u = 0$, is an analytic curve. In general, an analytic curve may have a branch which consists of one isolated point. But, in our case, such a circumstance never appears.

In fact, otherwise, let $p$ be such an isolated point, and take $p$ as the origin. Write a circle with $p$ as center and let its radius $r$ be small enough such that $J_0(kr) > 0$. This is possible since $J_0(0) = 1$. On the other hand, we have the mean value theorem for the equation $\Delta u + k^2 u = 0$:

$$u_p J_0(kr) = \frac{1}{2\pi} \int_0^{2\pi} u \, d\theta$$

But the left hand member becomes zero while the right hand is positive, and hence our assumption causes a contradiction. Thus our proposition II has been established.

Finally we note that all our results obtained for the two dimensional case can be extended to that for the three dimensional, by replacing the logarithmic mass by the Newtonian. In the proof the function $\cos kr/r$ may then be availed instead of the function $Y_0(kr)$.

## Reference


1) E. Picard, Lecons sur quelques types simples d'equations aux derivees partielles. Paris, 1950.
2) P.J.Myrberg, Uber die Existenz der Greenschen Funktionen auf einer Riemannsehen Flache. Acta Mathematica, 61 (1933), 39-79; 7.



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