NOTE ON SHIMODA'S THREE SPHERE THEOREM

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Let E and E' be two complex-Banach spaces and $x'=f(x)$ be an E'-valued analytic function defined on a domain D of E, i.e. $x'=f(x)$ is strongly continuous in D and admits a Gâteaux differential at each point of D [1].

If $f(x)$ is analytic on D, it may be expanded into the Taylor series

$$f(x)=f(x_0)+\sum_{n=1}^{\infty} f_n(x;x_0),\quad x_0\in D$$

in a sphere $S_\rho=\{x \mid \|x-x_0\|<\rho\}$ in D, where $f_n(x,x_0)$ is an E'-valued homogeneous polynomials of degree n given by

$$f_n(x,x_0)=\frac{1}{2\pi i}\int_C \frac{f(x_0+\alpha(x-x_0))}{\alpha^{n+1}}d\alpha,\quad x\in D$$

the integral taken in the positive sense on the circle C : $|\alpha|=1$.

The series converges absolutely and uniformly in the sphere $S_{\rho'}=\{x \mid \|x-x_0\|<\rho'\}$, where ρ' is a sufficiently small positive number[2].

According to Shimoda's Theorem, we may assume that D includes the origin[3].

Recently, Shimoda introduced the norm $M(r)$:

$$M(r)=\sup_{\|x\|=r}\|f(x)\|,$$

and proved Hadamard's Three Sphere Theorem for this norm[4].

In this Note, we are to introduce a norm $M_p(r)$:

$$M_p(r)=M_p(r,f)=\sup_{\|x\|=r} M_p(r,x,f),$$

$$M_p(r,x,f)=\frac{1}{2\pi}\int_0^{2\pi}\|f(e^{i\theta}x)\|^p d\theta$$

where $\|x\|=r$ and p is a positive integer, and prove Hadamard's Three Sphere Theorem for this norm.

Remark. Clearly,

$$\frac{1}{2\pi}\int_0^{2\pi}\|f(e^{i\theta}x)\|^p d\theta=\frac{1}{2\pi}\int_0^{2\pi}\|f(\alpha y)\|^p d\theta,\quad \alpha=re^{i\theta},\ \|y\|=1.$$

Lemma. Let $f_1(x)$, $f_2(x)$, ..., $f_n(x)$ be analytic in a closed domain D in E and not all norm-constant.[5] Put, for positive integer p

$$\phi(x)=\|f_1(x)\|^p+\|f_2(x)\|^p+\cdots+\|f_n(x)\|^p$$

Then, $\phi(x)$ is continuous in D and takes its maximum on the boundary of D.

Proof[6]. The continuity is immediate.

Let $k_p(x)$ be a p-power metric analytic function[7], which satisfies $\|k_p(x)\|=\|x\|^p$ and analytic on E'. Since $f_\nu(x)$ is analytic in D, $k_p(f_\nu(x))$ is analytic in D. Let x_0 be any interior point of D and choose r such that $x_0+\alpha(x-x_0)$ lie completely in D for $\|x-x_0\|\le r$. Then

$$k_p(f_\nu(x))=\frac{1}{2\pi}\int_0^{2\pi} k_p(f_\nu(x+re^{i\theta}(x-x_0)))d\theta,$$

and we have

$$\begin{aligned}\|f_\nu(x_0)\|^p&=\|k_p(f_\nu(x_0))\|\\ &\le\frac{1}{2\pi}\int_0^{2\pi}\|k_p(f_\nu(x_0+re^{i\theta}(x-x_0)))\|d\theta\\ &=\frac{1}{2\pi}\int_0^{2\pi}\|f_\nu(x_0+re^{i\theta}(x-x_0))\|^p d\theta.\end{aligned}$$

The equality can be established for all small r only when $f_\nu(x)$ is a norm-constant function in D. Since $f_1(x)$, ..., $f_n(x)$ are not all norm-constant, we have $\phi(x_0)<\frac{1}{2\pi}\int_0^{2\pi}\phi(x_0+re^{i\theta}(x-x_0))d\theta$, for suitable r which can be taken as small as we like. Therefore, $\phi(x_0)$ is not a maximum of $\phi(x)$.

Theorem 1. Let $f(x)$ be an analytic function of non-norm-constant in a ring domain D : $R_1\le\|x\|\le R_2$. Then, $M_p(r,x,f)$ is continuous in the interval $R_1\le\|x\|\le R_2$ and takes its maximum at one of the end points.

Proof. The continuity is evident.

Put $\omega_\nu = e^{\frac{2\pi\nu i}{n}}$, $(\nu = 1, 2, \dots, n)$. Then $f(\omega_\nu x)$ is analytic in D. Therefore, for all n,

$$g_n(x) = \frac{1}{n}\sum_{\nu=1}^{n} \| f(\omega_\nu x)\|^p, \quad \|x\| = r$$

takes its maximum on the boundary by Lemma. Then

$$\lim_{n\to\infty} g_n(x) = \frac{1}{2\pi}\int_0^{2\pi} \|f(e^{i\theta}x)\|^p d\theta = M_p(r, x, f),$$
$$\|x\| = r.$$

This proves Theorem 1.

Theorem 2. If $f(x)$ is analytic and non-norm-constant in $D: R_1 \le \|x\| \le R_2$, then $\log M_p(r)$ is a convex function of $\log r$ in the interval $R_1 \le \|x\| \le R_2$.

Proof.

Put $g(\alpha x) = \| f(\alpha x)\| \cdot f(\frac{r^2}{\bar\alpha} x)$,

where $R_1 \le \frac{r}{h} \le \|\alpha x\| \le rh \le R_2$, $h > 1$ and $\bar\alpha$ denotes the conjugate complex number of α. Then

$$M_{\frac{p}{2}}(\tfrac{r}{h}, x, g) = M_{\frac{p}{2}}(rh, x, g)$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \|f(\tfrac{r}{h}e^{i\theta}x)\|^{\frac{p}{2}} \|f(rhe^{i\theta}x)\|^{\frac{p}{2}} d\theta$$

and

$$M_{\frac{p}{2}}(r, x, g) = \frac{1}{2\pi}\int_0^{2\pi} \|f(re^{i\theta}x)\|^p d\theta$$
$$= M_p(r, x, f).$$

Since, for the interval $\frac{r}{h} \le r \le rh$, $M_{\frac{p}{2}}(r, x, g)$ takes its maximum at one of the end points,

$$M_{\frac{p}{2}}(r, x, g) \le \frac{1}{2\pi}\int_0^{2\pi} \|f(\tfrac{r}{h}e^{i\theta}x)\|^{\frac{p}{2}} \|f(rhe^{i\theta}x)\|^{\frac{p}{2}} d\theta$$

and, by Schwarz' inequality,

$$\le \left\{\frac{1}{2\pi}\int_0^{2\pi} \|f(\tfrac{r}{h}e^{i\theta}x)\|^p d\theta\right\}^{\frac{1}{2}} \left\{\frac{1}{2\pi}\int_0^{2\pi} \|f(rhe^{i\theta}x)\|^p d\theta\right\}^{\frac{1}{2}}.$$

Accordingly, for all $R_1 \le \frac{r}{h} < rh \le R_2$

$$M_p(r, x, f)^2 \le M_p(\tfrac{r}{h}, x, f)\, M_p(rh, x, f).$$

Therefore

(1) $$M_p(r, f)^2 \le M_p(\tfrac{r}{h}, f)\, M_p(rh, f).$$

This inequality and the continuity of $\log M_p(r)$ completes the proof.

If $f(x)$ is a function of norm-constant, in $R_1 \le \|x\| \le R_2$, we have

$$M_p(r) = M_p(\tfrac{r}{h}) = M_p(rh)$$

and the inequality (1) holds also in this case. Therefore, we have

Theorem 2'. Let $f(x)$ be analytic in a domain $D: R_1 \le \|x\| \le R_2$. Then, $\log M_p(r)$ is a convex function of $\log r$.

If p increases to infinity, $M_p(r)$ tends to $M(r)$. Therefore $\log M(r)$, as the limit of $\log M_p(r)$, is also convex function of $\log r$. Therefore, for $r_1 \le r_2 \le r_3$, we have

$$M(r_2) \le M(r_1)^\theta\, M(r_3)^{1-\theta},$$

where θ is any number between 0 and 1. This is the Shimoda's Theorem:

Put $\theta = \dfrac{\log r_3 - \log r_2}{\log r_3 - \log r_1}$,

$$M(r_2) \le M(r_1)^{\frac{\log r_3 - \log r_2}{\log r_3 - \log r_1}} . M(r_3)^{\frac{\log r_2 - \log r_1}{\log r_3 - \log r_1}}$$

Theorem 3. $M_p(r)$ is an increasing function of r.

Proof.

Since $f(x)$ is analytic in $D: \|x\| < R$, $h_p(f(x))$ is also analytic in D. So

$$h_p(f(0)) = \frac{1}{2\pi i}\int_C \frac{h_p(f(\alpha x))}{\alpha} d\alpha,$$

where C is the circle $|\alpha| = 1$. Put $\alpha = e^{i\theta}$ ($0 \le \theta \le 2\pi$), we have

$$M_p(0) = \|f(0)\|^p \le \frac{1}{2\pi}\int_0^{2\pi} \|f(e^{i\theta}x)\|^p d\theta = M_p(r, x, f).$$

By Theorem 1, $M_p(r, x, f)$ attains its maximum at an end point of the interval, and r increases from 0 to R. So $M_p(0) \le M_p(r, x, f)$ and at last $M_p(0) \le M_p(r)$. This shows the increaseness of $M_p(r)$.

Theorem 4. Let $f(x)$ be analytic in $\|x\| < R$ and put $g(x) = f(x) - f(0)$. If $M_p(r_1, g) = M_p(r_2, g)$ for $r_1 < r_2 < R$, $f(x)$ is constant on $\|x\| \le r_2$.

Proof.

For $0 < r \le r_1$, by Theorem 2', the convexity of $\log M_p(r, g)$ shows the inequality

$$M_p(r_1, g) \leq M_p(r, g)^{\theta} \; M_p(r_2, g)^{1-\theta}$$

where $\theta = \dfrac{\log r_2 - \log r_1}{\log r_2 - \log r}$. By our assumption $M_p(r_1, g) = M_p(r_2, g)$,

$$M_p(r_1, g)^{\theta} \leq M_p(r, g)^{\theta}.$$

On the other hand, by Theorem 3,

$$M_p(r_1, g) \geq M_p(r, g)$$

Therefore $M_p(r, g) = M_p(r_1, g)$. Then we have

$$M_p(r, g) = M_p(r_1, g) = M_p(r_2, g)$$

$$(0 < r \leq r_1).$$

By the same discussion, we have

$$M_p(r, g) = M_p(r_1, g) = M_p(r_2, g)$$

$$\text{for } r_1 \leq r \leq r_2 .$$

Therefore, for any $0 < r \leq r_2$, $M_p(r, g)$ must be constant.

Now, by the strong continuity of $g(x)$ in $\|x\| < r_2$, for any positive constant ε there exists a positive number δ such that

$$\|g(x)\|^p < \varepsilon, \quad \|x\| = r < \delta,$$

Therefore $M_p(r, g) \leq \varepsilon$.

Since ε is arbitrary positive number and the increasing function $M_p(r, g)$ of r is constant, we have

$$0 = M_p(r, g) = M_p(0) = \|g(0)\|^p, \quad 0 < r \leq r_2 .$$

Therefore $g(x) \equiv 0$, i.e. $f(x) \equiv f(0)$.

Corollary. An analytic function which is constant in a sphere is identically constant in its domain of analyticity.

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1) See, A. E. Taylor, On the properties of analytic functions in abstract spaces, Math. Ann. 115(1938) and E. Hille, Functional Analysis and Semi-Groups, Amer. Math. Soc. Coll. Publ. (1948)

2) Loc. cit.

3) I. Shimoda, On Analytic Functions in Abstract Spaces, Proc. Imp. Acad. Tokyo, vol. XIX(1943).

4) I. Shimoda, Note on General Analysis, (III): On the Norm of Analytic Functions, Journal of Gakugei, Tokushima Univ., vol. 4(1954).

5) See, I. Shimoda, loc. cit.

6) I owe this proof to Mr. Shimoda.

7) I. Shimoda, On Isometric Analytic Function in Abstract Spaces. Proc. of Japan Acad. vol. 30(1954), No. 8.

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