ON THE STOCHASTIC PROCESS OF RANDOM NOISE

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1. Suppose that events occur in accordance with a Poisson process with a parameter c, each event has a certain intensity U and has an after-effect $U\cdot\Phi(t)$ after t time units. Let intensities U_1, U_2, ... at occurrences of events be mutually independent. The sum of after-effects at time t can be represented as

$$(1.1)\qquad X_1(t)=\int_{-\infty}^{\infty}\Phi(t-s)\,dy(s) = \int_{-\infty}^{t}\Phi(t-s)\,dy(s),$$

where $y(t), -\infty<t<\infty$, is a stochastic process whose sample functions are constant between the events and increase by the corresponding intensity U_i at each event.

Suppose that $\Phi(t)$ be defined in $[0,\infty]$,

$$(1.2)\qquad \Phi(t)\in L_1(0,\infty) \text{ and } \in L_2(0,\infty),$$

and

$$(1.3)\qquad \int_0^{\infty}\Phi(t)\,dt=a,\quad \int_0^{\infty}\Phi^2(t)\,dt=b,$$

and, moreover,

$$(1.4)\qquad E(U_i)=\alpha,\quad E(U_i^2)=\beta.$$

If events are arrivals of electrons at the anode of vacuum tube, (1.1) represents the noise current. The formulation (1.1) is due to J. L. Doob.(1) Formally, the extensive analysis of random noise was done by S. O. Rice, in the case $X_1(t)$ is a trigonometric polynomial. The object of this paper is to prove some results, in connection with Rice theory, with $X_1(t)$ defined by (1.1) in a rather rigorous way from mathematical view points. The proofs of some known results are contained, because of completeness.

2. Under the condition (1.2), the integral defining $X_1(t)$ in (1.1) exists(1). $X_1(t)$ is strictly stationary and consequently stationary in wide sense.

Campbell's theorem shows

$$(2.1)\qquad E[X_1(t)]=c\alpha a,\qquad E[(X_1(t)-c\alpha a)^2]=c\beta b,$$

where $E[X]$ denotes the expectation of a random variable $X(t)$ and, for $s>0$,

$$Pr\{y(t+s)-y(t)\leqq\xi\}$$
$$(2.2)\qquad \begin{aligned} &=0, && \xi<0\\ &=e^{-cs} && \xi=0\\ &=e^{-cs}+\sum_{k=1}^{\infty}e^{-cs}\frac{(cs)^k}{k!}Pr\Big(\sum_{i=1}^{k}U_i\leqq\xi\Big), && \xi>0.\end{aligned}$$

(2.1) was proved by J. L. Doob

Put

$$(2.3)\qquad \begin{aligned} Y(t)&=y(t)-c\alpha t,\\ \Delta_s Y(t)&=\Delta_s y(t)-c\alpha s\\ &=y(t+s)-y(t)-c\alpha s.\end{aligned}$$

Let the characteristic functions of U_i and $\Delta_s Y(t)$ be $g(u)$ and $h(u;s)$ respectively. It is easily verified that

$$(2.4)\qquad h(u;s)=\exp.\{cs(g(u)-1-i\alpha u)\}.$$

$$(2.5)\qquad E[\Delta_s Y(t)]=0,\quad E[\{\Delta_s Y(t)\}^2]=\beta cs$$

are easily obtained.

We assume throughout this paper, that $\Phi(t)$ is a real valued function defined on $(-\infty,\infty)$ and to be not necessarily zero in $(-\infty,0)$ for certain reasons, (see later) and suppose that $\Phi(t)\geqq 0$,

$$(2.6)\qquad \int_{-\infty}^{\infty}\Phi(t)\,dt=a,\qquad \int_{-\infty}^{\infty}\Phi^2(t)\,dt=b.$$

Further suppose

$$(2.7)\qquad E[U_i^2]<\infty.$$

Let

$$(2.8)\qquad X(t)=\int_{-\infty}^{\infty}\Phi(t-s)\,dY(s).$$

<u>Theorem 1.</u> <u>The characteristic function $f(u)$ of $X(t)$ is given</u>

(2.9) $f(u)=$
$$\exp\left[c\int_{-\infty}^{\infty}\{g(\Phi(s)u)-1-i\alpha\,\Phi(s)u\}\,ds\right].$$

This is perhaps known, but for the completeness' sake we shall prove it.

Let $G(t)$ be a function of $L_2(-\infty,\infty)$. We take a sequence of step-functions $\{G_m(t)\}$,

$$\begin{aligned}G_m(t)&=0, & t&<a_1\\ &=c_j, & a_{j-1}&\leqq t<a_j,\ j\leqq n\\ &=0, & t&\geqq a_j,\end{aligned}$$

$a_j=a_j(n)$, $c_j=c_j(n)$, such that

(2.10) $$\int_{-\infty}^{\infty}|G(t)-G_m(t)|^2dt\to 0,\quad (n\to\infty).$$

Then by definition,

(2.11) $$\int_{-\infty}^{\infty}G(t)\,dY(t)$$
$$=\underset{n\to\infty}{l.i.m.}\int_{-\infty}^{\infty}G(t)\,dY(t)$$

where $l.i.m.$ means limit in variance, that is (2.11) means

$$E\left[\left\{\int_{-\infty}^{\infty}G_m(t)dY(t)-\int_{-\infty}^{\infty}G(t)dY(t)\right\}^2\right]$$
$$\to 0,\quad (m\to\infty).$$

We, then, have

$$E\left\{\exp\left(iu\int_{-\infty}^{\infty}G(t)\,dY(t)\right)\right\}$$
$$=\lim_{n\to\infty}E\left\{\exp\left(iu\int_{-\infty}^{\infty}G_m(t)\,dY(t)\right)\right\}$$
$$=\lim_{n\to\infty}\left[\exp\left\{iu\sum_{j=2}^{n}c_j\left(Y(a_j-0)-Y(a_{j-1}-0)\right)\right\}\right].$$

Since $Y(a_j-0)-Y(a_{j-1}-0)$, $j=2,\dots,n$, are mutually independent, by using (2.4), the above expression is equal to

$$\lim_{n\to\infty}\prod_{j=2}^{n}E\left[\exp\{iuc_j(Y(a_j-0)-Y(a_{j-1}-0))\}\right]$$
$$=\lim_{n\to\infty}\prod_{j=2}^{n}\exp\left\{c(a_j-a_{j-1})(g(c_ju)-1-i\alpha c_ju)\right\}$$
$$=\exp\left\{\lim_{n\to\infty}\sum_{j=2}^{n}(g(c_ju)-1-i\alpha c_ju)(a_j-a_{j-1})\right\}$$

(2.12) $$=\exp\left[c\int_{-\infty}^{\infty}\{g(G(t)u)-1-i\alpha\,G(t)\cdot u\}\,dt\right].$$

The existence of the bracket in the last expression is easily proved as follows : letting the distribution function of U_i be $F(x)$, we have

$$|g(G(t)u)-1-i\alpha\,G(t)u|$$
$$=\left|\int_{-\infty}^{\infty}\{\exp(iuG(t)x-1-ixG(t)u\}\,dF(x)\right|$$
$$=\left|\int_{-\infty}^{\infty}\{\cos(uG(t)x)-1+i(\sin(uG(t)x)-uG(t)x)\}\,dF(x)\right|$$
$$\leqq\frac{1}{2}u^2|G(t)|^2\int_{-\infty}^{\infty}x^2dF(x)+2u^2|G(t)|^2\int_{-\infty}^{\infty}x^2dF(x),$$

from which the existence of the integral in (2.12) is shown.

Hence the characteristic function of $\int_{-\infty}^{\infty}G(t)dY(t)$ is given by (2.12). Putting $G(t)=\Phi(s-t)$, (9) immediately follows.

Next, we consider the characteristic function of $(X(t),X'(t))$. If $\Phi(t)$ is differentiable and such that in addition to (2.6),

(2.13) $$\int_{-\infty}^{\infty}|\Phi'(t)|\,dt<\infty,$$
$$\int_{-\infty}^{\infty}|\Phi'(t)|^2dt<\infty$$

and $\{\Phi(t+h)-\Phi(t)\}/h$ converges in mean $L_2(-\infty,\infty)$ to $\Phi'(t)$ as $h\to 0$. Then

$$\underset{h\to 0}{l.i.m.}\ \frac{X(t+h)-X(t)}{h}=X'(t),$$
$$X'(t)=\int_{-\infty}^{\infty}\Phi'(t-s)\,dY(s),$$

and

$$uX(t)+vX'(t)$$
$$=\int_{-\infty}^{\infty}\{u\Phi(t-s)+v\Phi'(t-s)\}\,dY(s).$$

Taking $u\Phi(t-s)+v\Phi'(t-s)$ as $G(t)$ in (2.12), we have

$$E\left[\exp(i\lambda u\Phi(t-s)+i\lambda v\Phi'(t-s))\right]$$
$$=\exp\left[c\int_{-\infty}^{\infty}\{g(\lambda\Phi(s)u+\lambda\Phi'(s)v)-1-i\alpha(\lambda\Phi(s)u+\lambda\Phi'(s)v)\}\,ds\right].$$

Putting $\lambda=1$, we find the characteristic function of $(X(t),X'(t))$.

Theorem 2. If $\Phi(t)$ satisfies (2.6) and (2.13), and $\{\Phi(t+h)-\Phi(t)\}/h$ converges to $\Phi'(t)$ in mean $L_2(-\infty,\infty)$, then the characteristic function of $\{X(t), X'(t)\}$ is given by

$$(2.14)\quad f(u,v)$$
$$= \exp\Big[c\int_{-\infty}^{\infty}\{g(\Phi(s)u+\Phi'(s)v)-1$$
$$-i\alpha(\Phi(s)u+\Phi'(s)v)\}ds\Big].$$

From this we can show the variances of $X(t)$, $X'(t)$ to be $\beta c\int_{-\infty}^{\infty}\Phi^2(s)ds = \beta c b$, $\beta c\int_{-\infty}^{\infty}\{\Phi'(s)\}^2 ds$ and the covariance of $X(t)$ and $X'(t)$ to be

$$\beta c\int_{-\infty}^{\infty}\Phi(s)\Phi'(s)ds$$

which is zero, if $\Phi(s)$ converges to zero as $|s|\to\infty$.

3. We consider the integral

$$(3.1)\quad \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{-is\lambda}-e^{-is\mu}}{-is}dY(s),$$

which exists in in L_2 sense since the integrand belongs to $L_2(-\infty,\infty)$, and we define $Y^*(t)$ by putting (3.1) as $Y^*(\lambda)-Y^*(\mu)$. For the integral (3.1) and $Y^*(t)$, see Doob [1]. By Doob, it was shown that the stochastic process $Y^*(t)$ has orthogonal increments and

$$(3.2)\quad E\big[|\Delta Y^*(t)|^2\big]=\beta c\Delta t,$$

where $\Delta Y^*(t)=Y^*(t+\Delta t)-Y^*(t)$.

Let $\Phi(t)\in L_2(-\infty,\infty)$ and its Fourier transform be

$$\varphi(u)=\underset{A\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\Phi(t)e^{-iut}dt.$$

Then we can define the process

$$(3.3)\quad \int_{-\infty}^{u}\varphi(t)dY^*(t)=Z(u).^{(2)}$$

We find obviously that

$$E\big[|Z(u)-Z(v)|^2\big]=c\beta\int_v^u|\varphi(t)|^2dt$$
$$=H(u)-H(v),$$

where

$$(3.4)\quad H(u)=c\beta\int_{-\infty}^{u}|\varphi(v)|^2dv.$$

The process $Z(t)$ is also an orthogonal process.

Theorem 3. The stationary process

$$X(t)=\int_{-\infty}^{\infty}\Phi(t-s)dY(s)$$

can be represented as

$$(3.5)\quad X(t)=\int_{-\infty}^{\infty}e^{i\xi t}dZ(\xi),$$

where $Z(\xi)$ is one in (3.3) and the integral is taken in L_2 sense.

This is easily seen from the following lemma, which is known (3).

Lemma 1. If $W(t)$ is an orthogonal process and $W^*(x)$ is a stochastic process defined similarly as in (3.1), and $g(x)\in L_2(-\infty,\infty)$, $g^*(t)$ is its Fourier transform, then

$$\int_{-\infty}^{\infty}g(x)dW^*(x)=\int_{-\infty}^{\infty}g^*(t)dW(t).$$

Since the Fourier transform of $\Phi(t-s)$ is $e^{itu}\varphi(u)$, by the lemma, we have

$$\int_{-\infty}^{\infty}\Phi(t-s)dY(s)=\int_{-\infty}^{\infty}e^{itu}\varphi(u)dY^*(u)$$
$$=\int_{-\infty}^{\infty}e^{it\xi}d\Big(\int_{-\infty}^{\xi}\varphi(u)dY^*(u)\Big),$$

which proves Theorem 3.

By the well known inversion formula, $Z(\xi)$ can be represented by means of $X(t)$, but we shall give a formula by means of $Y(t)$.

Theorem 4. Under the conditions in Theorem 3, we have

$$(3.6)\quad Z(u)-Z(v)=\int_{-\infty}^{\infty}\varphi^*(t;u,v)dY(t)$$

$$(3.7)\quad =\int_v^u\varphi(t)dY^*(t),$$

where

$$(3.8)\quad \varphi^*(t;u,v)=\frac{1}{\sqrt{2\pi}}\int_v^u\varphi(\xi)e^{-i\xi t}d\xi$$

(3.7) is nothing but (3.3). (3.6) is also immediate by Lemma 1, since $\varphi^*(t;u,v)$ is the Fourier transform of

$$\varphi_1(t)=\varphi(t),\quad v<t<u,$$
$$=0,\quad \text{otherwise}.$$

The auto-correlation of $X(t)$ is

$$\rho(u)=E\big[X(t+u)X(t)\big]=\int_{-\infty}^{\infty}e^{iu\lambda}dH(u)$$

$H(u)$ being the spectral function (3.4).

Since $\Phi(t)$ is real, we get

$$(3.9)\quad \rho(u)=\int_0^{\infty}\cos u\lambda\, dH_1(\lambda),$$

where

$$H_1(\lambda)=c\beta\int_{-\lambda}^{\lambda}|\varphi(t)|^2dt.$$

4. In this section, we shall discuss the continuity of $X(t)$. It is obvious that $X(t)$ is continuous in mean L_2, since $f(u)$ is the Fourier transform of an absolutely integrable function. We shall prove under some conditions, that $X(t)$ is continuous almost surely.

Theorem 5. Suppose that
(i) $\Phi(t)$ is monotone for $|t| > t_0 > 0$, and $\Phi(t) \to 0$ as $|t| \to \infty$,
(ii) $|\Phi(t)| \leqq A_\ell |t|^{-\ell}$ for every $\ell > 0$, for large $|t|$, A_ℓ being a constant independent of t but depends on ℓ (and is not necessarily bounded with respect to ℓ),
(iii) $\Phi(t)$ satisfies the Lipschitz condition

$$|\Phi(t) - \Phi(t')| \leqq K|t-t'|^p, \quad (p>0)$$

K being a constant independent of t. and
(iv) the random variables U_i is bounded, $0 \leqq U_i \leqq B$. Then $X(t)$ is continuous with probability 1.

Proof. Let E be a finite set in $[t' - \frac{1}{n}, t' + \frac{1}{n}]$, n being a positive integer fixed for a moment. t' is any real number.

$$X(t) - X(t')$$
$$= \int_{-\infty}^{\infty} \Phi(t-s)\, dY(s) - \int_{-\infty}^{\infty} \Phi(t'-s)\, dY(s)$$
$$= \int_{-\infty}^{\infty} \{\Phi(t-s) - \Phi(t'-s)\}\, dy(s)$$
$$(4.1) \quad = \int_{-\infty}^{-A} + \int_{-A}^{A} + \int_{A}^{\infty} = I_1 + I_2 + I_3,$$

say, where $A = 3n^{p'/2}$, p' being any positive number less than p in (iii). We have obviously

$$P\{\sup_{t\in E} |X(t) - X(t')| > n^{-p'/2}\}$$
$$(4.2) \quad \leqq P\{\sup_{t\in E} |I_1| > \tfrac{1}{3} n^{-p'/2}\}$$
$$+ P\{\sup_{t\in E} |I_2| > \tfrac{1}{3} n^{-p'/2}\}$$
$$+ P\{\sup_{t\in E} |I_3| > \tfrac{1}{3} n^{-p'/2}\}.$$

We shall estimate each of the right hand side of (4.2). For this, we consider the mean value of $\sup_{t\in E} |I_1|^2$. Noticing that almost all sample functions $y(s)$ are non-decreasing,

$$E\{\sup_{t\in E} |I_1|^2\}$$
$$\leqq E\{\int_{-\infty}^{-A} \sup_{t\in E} |\Phi(t-s) - \Phi(t'-s)|\, dy(s)\}^2$$
$$= c\beta \int_{-\infty}^{-A} \sup_{t\in E} |\Phi(t-s) - \Phi(t'-s)|^2 ds$$
$$(4.3) \quad \leqq 2c\beta \int_{-\infty}^{-A} \sup_{t\in E} |\Phi(t-s)|^2 ds$$
$$+ c\beta \int_{-\infty}^{-A} |\Phi(t'-s)|^2 ds$$

We take, in advance,

$$|t'| < n^{p'/2}.$$

Then $|t| < |t'| + \frac{1}{n} < 2n^{p'/2}$ and

$$t - s > A - |t| > n^{p'/2}.$$

We may assume $\Phi(u)$ monotone for $u > t - A$, taking previously n large enough. Then the right side of (4.3) is not greater than

$$4c\beta \int_{A+t'}^{\infty} |\Phi(u)|^2 du \leqq 4c\beta \int_{n^{p'/2}}^{\infty} |\Phi(u)|^2 du$$
$$\leqq 4A_\ell c\beta \int_{n^{p'/2}}^{\infty} u^{-2\ell} du = C_\ell n^{-(2\ell+1)p'/2}$$
$$= C_\ell n^{-\lambda}.$$

We can take $\lambda = (2\ell+1)p'/2$ arbitrarily large, taking ℓ large.

By Tchebycheff's inequality, we have, making use of above estimates,

$$P\{\sup_{t\in E} |I_1| > \tfrac{1}{3} n^{-p'/2}\}$$
$$\leqq 9 n^{p'} E\{\sup_{t\in E} |I_1|^2\}$$
$$(4.4) \quad \leqq 9 C_\ell n^{-\lambda+p'}$$

Similary we have

$$(4.5) \quad P\{\sup_{t\in E} |I_3| > \tfrac{1}{3} n^{-p'/2}\} \leqq 9 C_\ell n^{-\lambda+p'}.$$

Next

$$P\{\sup_{t\in E} |I_2| > \tfrac{1}{3} n^{-p'/2}\}$$
$$\leqq P\{\int_{-A}^{A} \sup_{t\in E} |\Phi(t-s) - \Phi(t'-s)|\, dy(s) > \tfrac{1}{3} n^{-p'/2}\},$$

which, by (iii), does not exceed

$$P\{\int_{-A}^{A} dy(s) > \frac{1}{3K} n^{p/2}\}$$
$$= \sum_{k=0}^{\infty} e^{-2Ac} \frac{(2Ac)^k}{k!} P(U_1 + \cdots + U_k > \tfrac{1}{3K} n^{p/2}).$$

Since $U_1 + \cdots + U_k \leqq kB$ by the assumption (iv), and

$$e^{-\mu}\sum_{k=n}^{\infty}\frac{\mu^k}{k!}\leqq\frac{\mu^n}{n!}$$

which is easily verified, the last expression is not greater than

$$\sum_{k>n^{p/2}/(3kB)} e^{-2Ac}\frac{(2Ac)^k}{k!}$$

$$(4.6)\qquad \leqq \frac{(2Ac)^{[n^{p/2}/3kB]}}{[n^{p/2}/3kB]!}. \quad {}^{(4)}$$

Putting $\frac{1}{3kB}=\mu$, Stirling's formula shows that the right side of (4.6)

$$\sim\frac{1}{\sqrt{2\pi}\,(\mu n^{p/2})^{1/2}}\left(\frac{6c\,n^{p/2}}{\mu n^{p/2}}\right)^{\mu n^{p/2}}$$

$$(4.7)\qquad \leqq\left\{\frac{C'_\ell}{n^{(p-p')/2}}\right\}^{\mu n^{p/2}},$$

where C'_ℓ is a constant depending only on ℓ. (4.4), (4.5) and (4.7) with (4.2) show that

$$P\{\sup_{t\in E}|X(t)-X(t')|>n^{-p/2}\}$$

$$\leqq 18\,C_\ell n^{-\lambda+p}+\left\{\frac{C'_\ell}{n^{(p-p')/2}}\right\}^{\mu n^{p/2}},$$

from which it results

$$P\{\sup_{|t-t'|<\frac{1}{n}}|X(t)-X(t')|>n^{-p/2}\}$$

$$(4.8)\qquad \leqq 18\,C_\ell n^{-\lambda+p}+C''_\ell n^{-\lambda_n},$$

where $\lambda_n\to\infty$ and C''_ℓ is a constant independent of n, t', by choosing a denser and denser set E.(5)

Let ℓ be so chosen that $\lambda-p=\gamma>3$, and let $\lambda_n>\gamma$ for $n\geqq n_0$. Then by (4.8), we get

$$P\{\sup_{\substack{|t-\frac{j}{n}|<\frac{1}{n},\\ |j|\leqq n^{1+p/2}}}|X(t)-X(\tfrac{j}{n})|>n^{-p/2}\}$$

$$\leqq C n^{-\gamma+2}\quad\text{for}\quad n\geqq n_0,$$

C being a constant independent of n. The right hand side is a general term of a convergent series, Borel-Catelli lemma shows that

$$\sup_{\substack{|t-\frac{j}{n}|<\frac{1}{n},\\ |j|\leqq n^{1+p/2}}}|X(t)-X(\tfrac{j}{n})|\leqq n^{-p/2}$$

with probability 1. This means that $X(t)$ is uniformly continuous in every finite interval, with probability 1.

5. In this section, we consider the "non-vanishing" property of $X(t)$. We first prove the following theorem.

<u>Theorem 6. Let</u> $g(t)\in L_1(-\infty,\infty)$ <u>and</u> $\in L_2(-\infty,\infty)$. <u>Then</u> $g(t)X(t)$ <u>belongs to</u> $L_2(-\infty,\infty)$ <u>and</u> $L_1(-\infty,\infty)$ <u>with probability 1, and its Fourier transform is given by</u>

$$(5.1)\qquad \int_{-\infty}^{\infty}G(\xi-x)\varphi(\xi)\,dY^*(\xi)$$

$$(5.2)\qquad =\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Phi(t-s)g(t)e^{-itx}dt\right)dy(s)$$

$$-c\alpha a\,G(-x),$$

<u>where</u> $G(x)$ <u>is the (inverse) Fourier transform of</u> $g(t)$:

$$G(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}g(t)e^{itx}dt$$

<u>Proof.</u> We have

$$\int_{-\infty}^{\infty}E\{|X(t)g(t)|^2\}dt$$

$$=\int_{-\infty}^{\infty}|g(t)|^2E\{|X(t)|^2\}dt$$

$$=c\beta b\int_{-\infty}^{\infty}|g(t)|^2dt<\infty,$$

and

$$\int_{-\infty}^{\infty}E\{|X(t)g(t)|\}dt=\int_{-\infty}^{\infty}|g(t)|E\{|X(t)|\}dt$$

$$\leqq\sqrt{c\beta b}\int_{-\infty}^{\infty}|g(t)|\,dt<\infty.$$

Hence by Fubini's theorem,

$$g(t)X(t)\in L_1\cdot L_2$$

with probability 1. We, then, have

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}g(t)X(t)e^{-itx}dt$$

$$=\underset{A\to\infty}{l.i.m.}\int_{-A}^{A}dZ(\xi)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}g(t)e^{it(\xi-x)}dt$$

$$=\int_{-\infty}^{\infty}G(\xi-x)\,dZ(\xi)$$

$$=\int_{-\infty}^{\infty}G(\xi-x)\varphi(\xi)\,dY^*(\xi)$$

by (3.7). Further we have

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(t)X(t)e^{-itx}dt$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(t)e^{-itx}dt\int_{-\infty}^{\infty}\Phi(t-s)\,dY(s)$$

(5.3)
$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(t)e^{-itx}dt\int_{-\infty}^{\infty}\Phi(t-s)\,dy(s) - \frac{c\alpha}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(t)e^{-itx}dt\int_{-\infty}^{\infty}\Phi(t-s)\,ds.$$

$\Phi(t-s)g(t)$ belongs to $L_2(-\infty,\infty)$ as a function of t for almost all s, since

$$\int_{-\infty}^{\infty} ds\int_{-\infty}^{\infty}|\Phi(t-s)g(t)|^2dt$$
$$=\int_{-\infty}^{\infty}|g(t)|^2dt\int_{-\infty}^{\infty}|\Phi(t-s)|^2ds$$
$$=\int_{-\infty}^{\infty}|g(t)|^2dt\int_{-\infty}^{\infty}|\Phi(s)|^2ds<\infty.$$

Hence the first term of (5.3) can be proved to be

$$\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Phi(t-s)g(t)e^{-itx}dt\right)dy(s).$$ [6]

The second term of (5.3) is $-c\alpha a\,G(-x)$. And the theorem is proved.

The function of x

(5.4)
$$\ell(x,s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Phi(t-s)g(t)e^{-itx}dt$$

belongs to $L_2(-\infty,\infty)$ for almost all s, since as was shown $\Phi(t-s)g(t)\in L_2(-\infty,\infty)$ for almost all s[7]. We can also see that $\ell(x,s)$ is in $L_2(-\infty,\infty)$ as a function of s. For we have

$$\int_{-\infty}^{\infty}|\ell(x,s)|^2ds$$
$$\leq\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}|\Phi(t-s)g(t)|\,dt\right\}^2ds$$

which is, by Minkowski inequality,

$$\leq\left[\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}|\Phi(t-s)g(t)|^2ds\right\}^{\frac{1}{2}}dt\right]^2$$
$$=\left\{\int_{-\infty}^{\infty}|g(t)|\,dt\right\}^2\left\{\int_{-\infty}^{\infty}|\Phi(s-t)|^2ds\right\}<\infty.$$

Now we put

$$X^*(t)=\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Phi(t-s)g(t)e^{-itx}dt\right)dy(s)$$

(5.5)
$$=\int_{-\infty}^{\infty}\ell(x,s)\,dy(s),$$

this is obviously well-defined, and also is written as

(5.6)
$$X^*(x)=\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{\infty}\varphi(u)G(u-x)e^{-isu}du\right)dy(s),$$

for $\ell(x,s)$ is the Fourier transform of $\varphi(u)G(u-x)$.

__Lemma 1.__ Let $\Phi(t)\in L_1$ and further be monotone for $|t|>t_0$. Then for $|s|>2t_0$,

(5.7)
$$\int_{-\infty}^{\infty}\frac{|\Phi(t-s)|}{1+t^2}dt\leq\frac{c_1}{s^2}+c_2\Phi\left(-\frac{s}{2}\right),$$

c_1, c_2 being constants independent of s.

Noticing that $\Phi(u)$ is monotone for $u>\frac{|s|}{2}$, and $u<-\frac{|s|}{2}$, we have

$$\int_{-\infty}^{\infty}\frac{\Phi(t-s)}{1+t^2}dt$$
$$=\int_{|t|>s/2}\frac{|\Phi(t-s)|}{1+t^2}dt+\int_{-s/2}^{s/2}\frac{|\Phi(t-s)|}{1+t^2}dt$$
$$\leq\frac{1}{1+(s/2)^2}\int_{-\infty}^{\infty}|\Phi(u)|^2du$$
$$+|\Phi(-\tfrac{s}{2})|\int_{-\infty}^{\infty}\frac{dt}{1+t^2},$$

which proves the lemma.

__Lemma 2.__ __Let $F(u)\in L_1$, and let__

(5.8)
$$F(u)=O(e^{-\theta(u)}),\quad u\to\infty,$$

__where $\theta(u)$ is a positive increasing function such that__

(5.9)
$$\int_1^{\infty}\frac{\theta(u)}{u^2}du=\infty.$$

__Let $h(z)$ be analytic in $a\leq x\leq b$, $0\leq y\leq\gamma$. Suppose that the Fourier transform of $F(u)$ is equal to $h(x)$ for $a<\alpha\leq x\leq\beta<b$. Then it is equal to $h(x)$ for $a\leq x\leq b$.__

This fact is due to N. Levinson [1], (P. 75, Th. XXIII).

Now we shall prove the following theorem.

__Theorem 7.__ __Suppose that $\Phi(t)\in L_1(-\infty,\infty)$ and $\in L_2(-\infty,\infty)$,__

(5.10)
$$\int_{|u|>x}|\Phi(u)|^2du=O\left(\frac{1}{x^3}\right),\quad x\to\infty$$

__and the Fourier transform $\varphi(u)$ of $\Phi(t)$ is a function of $L_1(-\infty,\infty)$, such that__

(5.11) $\varphi(u) = O(e^{-\theta(|u|)}), \quad |u| \to \infty$

where $\theta(v)$ is a non-decreasing function of v for large v, and

(5.12) $$\int_1^\infty \frac{\theta(v)}{v^2} dv = \infty,$$

(5.13) $$\int_1^\infty e^{-\frac{1}{4}\theta(v)} dv < \infty.$$

Then $X(t)$ can not vanish over any interval with probability 1.
It is worth noticing that if (5.11) is satisfied, then $\Phi(t)$ can not vanish over any interval. From the view point that $\Phi(t)$ is a function representing an after-effect, this fact is rather curious. Although $\Phi(t)=o\ (t<o)$ practically, it seems more convenient mathematically to take non-vanishing $\Phi(t)$ for such function. function.

Proof. Let

(5.14) $$g(t) = \frac{1}{(1+it)^2}$$

The Fourier transform of $g(t)$ is

$$G(x) = \sqrt{2\pi}\, e^{-x} x, \quad x>0,$$
$$0, \quad x<0.$$

By Theorem 6, the Fourier transform of $g(t)X(t)$ is, with probability 1,

$$\int_{-\infty}^\infty l(x,s)dy(s) - c\alpha a G(-x),$$

where $l(x,s)$ is defined by (5.4). We shall estimate

$$X^*(x) = \int_{-\infty}^\infty l(x,s)\, dy(s).$$

We divide into three parts

$$X^*(x) = \int_{-\infty}^{-A} + \int_{-A}^{A} + \int_A^\infty$$
$$= J_1 + J_2 + J_3,$$

say, where $A = A(n) = Ke^{\theta(n)/2}$, n being a positive integer and K being a constant which will be determined later. We have

$$P(\sup_{n\le x\le n+1} |X^*(x)| > e^{-\frac{1}{2}\theta(n)})$$
$$\le P(\sup_{n\le x\le n+1} |J_1| > \tfrac{1}{3} e^{-\frac{1}{2}\theta(n)})$$
$$+ P(\sup_{n\le x\le n+1} |J_2| > \tfrac{1}{3} e^{-\frac{1}{2}\theta(n)})$$
$$+ P(\sup_{n\le x\le n+1} |J_3| > \tfrac{1}{3} e^{-\frac{1}{2}\theta(n)})$$
$$= H_1 + H_2 + H_3,$$

Say. (8)

But

$$E\{\sup_{n\le x\le n+1} |\int_A^\infty l(x,s)dy(s)|^2\}$$
$$\le E\{(\int_A^\infty \sup_{n\le x\le n+1} |l(x,s)|\, dy(s))^2\}$$

noticing that almost all sample functions $y(s)$ are non-decreasing, and

$$= \beta c \int_A^\infty \sup_{n\le x\le n+1} |l(x,s)|^2 ds$$

which is, by (5.4)

$$\le \beta c \int_A^\infty \frac{ds}{2\pi} \left(\int_{-\infty}^\infty |\Phi(t-s)g(t)|\, dt\right)^2$$
$$= \frac{\beta c}{2\pi} \int_A^\infty ds \left(\int_{-\infty}^\infty \frac{|\Phi(t-s)|}{1+t^2} dt\right)^2.$$

By Lemma 1, this does not exceed

$$c_1 \int_A^\infty \frac{ds}{s^4} + c_2 \int_{|s|>A} |\Phi(\tfrac{s}{2})|^2 ds$$

for some constants c_1 and c_2, which is, by (5.10)

$$\le \frac{c_3}{A^3} = c_3 e^{-\frac{3}{2}\theta(n)} \cdot \frac{1}{K^3}$$

Tchebycheff inequality shows

(5.14) $$H_1 \le \frac{c_3}{K^3} e^{-\frac{3}{2}\theta(n)} \cdot 9 e^{\theta(n)}$$
$$= c_4 e^{-\frac{1}{2}\theta(n)}.$$

Similarly we have

(5.15) $$H_3 \le c_4 e^{-\frac{1}{2}\theta(n)}.$$

Now

$$|l(x,s)| = \frac{1}{\sqrt{2\pi}} \left|\int_{-\infty}^\infty \varphi(u) G(u-x) e^{-isu} du\right|$$
$$\le \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty |\varphi(u) G(u-x)|\, du$$
$$= \frac{1}{\sqrt{2\pi}} \int_x^\infty |\varphi(u)| \cdot (u-x) e^{-(u-x)} du$$

which is, by the assumption (5.11)

$$\le c_5 \int_x^\infty e^{-\theta(u)} (u-x) e^{-(u-x)} du$$
$$\le c_5 e^{-\theta(x)} \le c_5 e^{-\theta(n)}, \quad n \le x \le n+1.$$

Hence

$$H_2 = P\{\sup_{n\le t\le n+1} |\int_{-A}^A l(x,s)dy(s)| > \tfrac{1}{3} e^{-\theta(n)/2}\}$$

$$\leqq P\left\{ \int_{-A}^{A} \ell\, dy(s) > \frac{1}{3c_5} e^{\frac{1}{2}\theta(n)} \right\}$$

$$\leqq \sum_{k \geqq N} \frac{(2Ac)^k}{k!} e^{-2Ac},$$

where $N = N(n) = \frac{1}{3c_5} e^{-\frac{1}{2}\theta(n)}$. Thus by Stirling formula

$$H_2 \leqq \frac{(2Ac)^N}{N!}$$

$$\sim \frac{1}{\sqrt{2\pi}} \left\{ \frac{3c_5 \cdot 2Kc\, e^{\theta(n)/2}}{e^{\theta(n)/2}} \right\}^N \cdot \frac{e^N (3c_5)^{1/2}}{e^{\theta(n)/4}}$$

If we take $K = (6ecc_5)^{-1}$ in advance, then for large n,

$$(5.16) \qquad H_2 \leqq c_6\, e^{-\frac{1}{4}\theta(n)}$$

(5.14), (5.15) and (5.16) show that

$$P\left(\sup_{n \leqq x \leqq n+1} |X^*(x)| > e^{-\frac{1}{2}\theta(n)} \right) \leqq c_6\, e^{-\frac{1}{4}\theta(n)}$$

Similar inequality holds for negative integer n, and the right hand side is the general term of the convergent series. Thus

$$\sup_{n \leqq x \leqq n+1} |X^*(x)| \leqq e^{-\frac{1}{2}\theta(|n|)} \qquad |n| \geqq n_0$$

holds with probability 1, and hence

$$\sup_{n \leqq x \leqq n+1} |X^*(x)| \leqq e^{-\frac{1}{2}\theta(|x|)}$$

for large $|x|$, with probability 1. Since the (inverse) Fourier transform of $X^*(x)$ is $g(t)(X(t) + c\,d\alpha)$. If $X(t)$ vanish over some interval, then $g(t)(X(t)+c\,d\alpha) = g(t) \cdot c\,d\alpha$ which is analytic in a strip containing the real axis. Therefore by Lemma 2, $X(t)$ can not vanish over any interval with probability 1. Thus our theorem is proved.

<u>Corollary. If we suppose that $\varphi(u) = O(e^{-\lambda|u|})$ λ being a positive constant, instead of (5.11) in Theorem 7, then $X(t)$ is analytic on a strip including the real axis, with probability</u> 1.

This is immediate since $\{g(z)\}^{-1}$ is function everywhere analytic.

6. We add a remark rather obvious. Let $E_T(0)$ be a set of roots of

$$X(t) = 0, \quad -T \leqq t \leqq T.$$

And let the distribution function of $X(t)$ be $D(x)$, which is independent of a parameter t as $X(t)$ is a strictly stationary process. We have the following fact

$\frac{1}{2T} E_T(0)$ <u>converges to</u> $D(+0) - D(-0)$ <u>as $T \to \infty$, with probability</u> 1.

Let

$$\chi_0(u) = 1, \quad u = 0$$
$$\qquad = 0, \quad u \neq 0,$$

and let

$$\chi_\varepsilon(u) = 1, \quad |u| < \varepsilon$$
$$\qquad = 0, \quad |u| > \varepsilon$$
$$\qquad = \frac{1}{2}, \quad |u| = \varepsilon$$

Evidently $\lim_{\varepsilon \to 0} \chi_\varepsilon(u) = \chi_0(u)$. We have

$$E_T(0) = \frac{1}{2T} \int_{-T}^{T} \chi_0(X(t))\, dt.$$

Since $X(t)$ is metrically transitive, with probability 1,

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \chi_0(X(t))\, dt = E\{\chi_0(X(t))\}$$

$$= \lim_{\varepsilon \to 0} E\{\chi_\varepsilon(X(t))\}$$

$$(6.1) = \lim_{\varepsilon \to 0} E\left\{ \lim_{A \to \infty} \frac{1}{\pi} \int_{-A}^{A} \frac{\sin \varepsilon u}{u} e^{iX(t)u}\, du \right\}$$

$$= \lim_{\varepsilon \to 0} \lim_{A \to \infty} \frac{1}{\pi} \int_{-A}^{A} \frac{\sin \varepsilon u}{u} E\left(e^{iX(t)u}\right) du$$

The interchange of the integral and the expectation is legitimate since the integral in (6.1) is uniformly bounded with respect to A, ε being fixed.

The above expression is

$$\lim_{\varepsilon \to 0} \lim_{A \to \infty} \int_{-\infty}^{\infty} \left(\frac{1}{\pi} \int_{-A}^{A} e^{ixu} \frac{\sin \varepsilon u}{u} du \right) dD(x)$$

$$= \lim_{\varepsilon \to 0} \{ D(\varepsilon) - D(-\varepsilon) \} = D(+0) - D(-0),$$

(letting $\varepsilon \to 0$ through points such that $D(x)$ and $D(-x)$ are continuous) which proves the italicized fact.

7. Here we shall prove the formula which gives the mean value of zero points of $X(t)$ over an interval. The formula is known for special cases (9). We shall prove it by using a method due to M. Kac [1],(10) who gave the mean value of real roots of an algebraic equation with random coefficients.

Suppose that $\Phi(t)$ satisfies (2.6), and $\Phi'(t)$ satisfies (2.13) and moreover $\{\Phi(t+h) - \Phi(t)\}/h$ converges to $\Phi'(t)$ in mean $L_2(-\infty, \infty)$. We suppose that $\varphi(u)$ satisfies that $\varphi(u) = O(e^{-\lambda|u|})$ in addition to the assumptions in Theorem 7. Then by the corollary in §5, $X(t)$ is analytic with probability 1 and almost all sample functions have finite zero points on every finite interval.

Let joint distribution function of $(X(t), X'(t))$ be $D(x,y) = P(X(t) \leq x, X'(t) \leq y)$, which is also independent of t. Lastly we assume that the distribution function of $X(t)$ is continuous for every t.

We shall prove the following theorem.

<u>Theorem 8. Let $\Phi(t)$ and its Fourier transform satisfy the above conditions. Put</u>

(7.1) $$W(x) = \int_{-\infty}^{\infty} |y| \, dD(x,y)$$

<u>which is assumed to exist. If $J = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon}(W(\varepsilon) - W(-\varepsilon))$ exists, then the mean value of the number of zero points of $X(t)$ over any interval (a, b) is given by</u>

(7.2) $$(b-a)J \; .$$

To prove the theorem we shall use the following lemma which is made use of by M. Kac[11].

<u>Lemma 3. Let $\psi_\varepsilon(x)$ be 1 if $|x| < \varepsilon$ and 0 otherwise. If neither a nor b is a zero of $f(x)$, then the number of zeros of $f(x)$ in the interval (a, b) is</u>

$$\lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_a^b \psi_\varepsilon(f(x)) \, |f'(x)| \, dx$$

We shall prove the theorem.
Because of

$$\psi_\varepsilon(u) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin \varepsilon t}{t} e^{iut} dt,$$

(difference at $u = |\varepsilon|$ is not important in our following arguments for the distribution of $X(t)$ is continuous), and

$$|\gamma| = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1 - \cos \gamma t}{t^2} dt ,$$

We have

$$E\{\psi_\varepsilon(X(t)) |X'(t)|\}$$

$$= E\left\{ \frac{1}{\pi} \lim_{A \to \infty} \int_{-A}^{A} \frac{\sin \varepsilon u}{u} e^{iX(t)u} du \cdot \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1 - \cos X'(t) v}{v^2} dv \right\}$$

$$= \frac{1}{\pi} \lim_{A \to \infty} E\left[\int_{-A}^{A} \frac{\sin \varepsilon u}{u} e^{iX(t)u} du \cdot \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1 - \cos X'(t) v}{v^2} dv \right]$$

$$= \frac{1}{\pi} \lim_{A \to \infty} \frac{1}{\pi} \int_{-A}^{A} du \int_{-\infty}^{\infty} \frac{\sin \varepsilon u}{u v^2} \cdot E\{ e^{iX(t)u} (1 - \cos X'(t) v)\} dv ,$$

where the interchange of the integrals and the expectation is legitimate since $\int_{-A}^{A} \sin \varepsilon u / u \cdot e^{iX(t)u} du$ is bounded.

If $f(u)$ and $f(u,v)$ denote as before the characteristic functions of $X(t)$ and $\{X(t), X'(t)\}$ respectively, then the above expression is

$$= \frac{1}{\pi} \lim_{A \to \infty} \int_{-A}^{A} du \int_{-\infty}^{\infty} \frac{\sin \varepsilon u}{u v^2} \cdot \left[f(u) - \frac{1}{2}(f(u,v) + f(u,-v)) \right] dv .$$

Now

$$\lim_{A \to \infty} \frac{1}{\pi} \int_{-A}^{A} \frac{\sin \varepsilon u}{u} f(u) du$$

$$= \lim_{A \to \infty} \frac{1}{\pi} \int_{-A}^{A} \frac{\sin \varepsilon u}{u} du \int_{-\infty}^{\infty} e^{iux} dD(x)$$

(7.3) $$= \int_{-\varepsilon}^{\varepsilon} dD(x)$$

$$= \int_{-\infty}^{\infty} d\{D(\varepsilon, y) - D(-\varepsilon, y)\}$$

Similarly

$$\lim_{A \to \infty} \frac{1}{\pi} \int_{-A}^{A} \frac{\sin \varepsilon u}{u} f(u,v) du$$

$$= \lim_{A \to \infty} \frac{1}{\pi} \int_{-A}^{A} \frac{\sin \varepsilon u}{u} du \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{iux + ivy} dD(x,y)$$

$$= \int_{-\infty}^{\infty} e^{ivy} \, d \int_{-\varepsilon}^{\varepsilon} d_x D(x,y)$$

$$= \int_{-\infty}^{\infty} e^{-ivy} d\{D(\varepsilon, y) - D(-\varepsilon, y)\} .$$

Hence we have

$$E\{\psi_\varepsilon(X(t)) |X'(t)|\}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1 - \cos v y}{\pi v^2} dv \, d\{D(\varepsilon, y) - D(-\varepsilon, y)\}$$

$$= \int_{-\infty}^{\infty} |y| \, dD(\varepsilon, y) - \int_{-\infty}^{\infty} |y| \, dD(-\varepsilon, y)$$

Hence the theorem is proved.

<u>If $(X(t), X'(t))$ has probability density $p(x,y)$, then (7.2) becomes</u>

(7.4) $$(b-a)J = (b-a) \int_{-\infty}^{\infty} |y| \, p(0,y) \, dy$$

<u>where $\int_{-\infty}^{\infty} |y| p(x.y)\,dy \ (-\infty < x < \infty)$</u>

<u>is assumed to exist.</u>

This is due to S. O. Rice [1] when $X(t) = L(\Xi_1, \cdots, \Xi_N; t)$ $\Xi_1, \ldots, \Xi_n$ are random variables and $L(\xi_1, \ldots, \xi_N; t)$ is a function differentiable with respect to t.

References

(1) Doob , Stochastic process, Wiley, New York, 1952.

(2) M. Kac, On the average number of real roots of a random algebraic equation. Bull. Amer. Math. Soc. 49(1943).

(3) O. S. Rice, The Bell system, Technical Journal, XXIII(1944).

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(1) Doob [1] Chapt. XII.

(2) See Doob [1]

(3) Doob [1]. p.

(4) [x] denotes the integral part of x.

(5) $X(t)$ is mean continuous, and we may consider $X(t)$ a separable process, Doob [1], Chapter II.

(6) The detail proof of the inversion of the order of integrals is omitted, but we can, without difficulty, prove it by use of the definitions of ordinary and stochastic integral.

(7) Moreover, of course, $\ell(x.s)$ is bounded for almost all s since $\Phi(t-s)\,g(t) \in L_1(-\infty,\infty)$ for almost all s.

(8) We can define $P(\sup(\cdot))$ for analogous way as in the proof of theorem 5 or Doob [1] Chapter 2, but details are omitted here.

(9) O. S. Rice [1]

(10) M. Kac [1]

(11) M. Kac [1]

(*) Received Feb. 1, 1955.