A SUPPLEMENT TO "ON TRANSFERENCE OF BOUNDARY VALUE PROBLEMS"

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In a recent paper[1)] it has been shown that for some simple domains Dirichlet and Neumann problems are readily transferable each other by means of elementary operations. Since attention has been restricted to give the connection between the solutions of both boundary value problems, the explicit formulas for the solutions have not been brought forwards in practical forms. However, it is possible to derive them separately also in elementary ways, what will be supplemented in the present paper.

1. Rectilinear slit domain.

Let the basic domain be the whole z-plane slit along a rectilinear segment

$$\mathfrak{R}z=0, \quad -1 \leqq \mathfrak{J}z \leqq +1,$$

and let first the boundary condition of a Neumann problem be assigned in the form

$$\frac{\partial v}{\partial \nu}(\pm 0+iy)=V^{\pm}(y) \qquad (-1<y<1),$$

$$\int_{-1}^{1}(V^{+}(y)+V^{-}(y))\,dy=0.$$

It is solved, as stated in the previous paper, by $v(z)=\mathfrak{R}g(z)$, where $g(z)$ is defined by

$$g(z) = c-\frac{1}{\pi}\left\{\int_{-\pi/2}^{\pi/2}\mathcal{V}^{-}(\varphi)-\int_{\pi/2}^{3\pi/2}\mathcal{V}^{+}(\varphi)\right\}\cos\varphi \lg\frac{1}{e^{i\varphi}-\zeta}\,d\varphi,$$

$$\mathcal{V}^{\pm}(\varphi)=V^{\pm}(y)\equiv V^{\pm}(\sin\varphi),$$

$$\zeta=z-\sqrt{1+z^2},$$

c being any constant and the square root representing such a branch that $z=\infty$ corresponds to $\zeta=0$.

Now substitute an integration variable η defined by

$$\eta=\sin\varphi, \quad e^{i\varphi}=i\eta\pm\sqrt{1-\eta^2},$$

where the upper and lower of the double sign is taken for $-\pi/2<\varphi<\pi/2$ and for $\pi/2<\varphi<3\pi/2$, respectively, and $\sqrt{1-\eta^2}$ is supposed to represent always a non-negative real number. In view of the relation

$$\frac{1}{e^{i\varphi}-\zeta} =-\frac{1}{2}\frac{1}{z-i\eta}\left(1+(z+\sqrt{1+z^2})(i\eta\pm\sqrt{1-\eta^2})\right),$$

a required formula for the solution of the Neumann problem is obtained in the form

$$g(z)=c-\frac{1}{\pi}\int_{-1}^{1}\Big\{(V^{+}(\eta)+V^{-}(\eta))\lg\frac{1}{z-i\eta}$$

$$+V^{+}(\eta)\lg(1+(z+\sqrt{1+z^2})(i\eta+\sqrt{1-\eta^2}))$$

$$+V^{-}(\eta)\lg(1+(z+\sqrt{1+z^2})(i\eta-\sqrt{1-\eta^2}))\Big\}\,d\eta.$$

Let next the boundary condition of a Dirichlet problem be assigned in the form

$$u(\pm 0+iy)=U^{\pm}(y) \qquad (-1<y<1).$$

It is solved by $u(z)=\mathfrak{R}f(z)$, where $f(z)$ is defined by

$$f(z) =\frac{1}{2\pi}\left\{\int_{-\pi/2}^{\pi/2}\mathcal{U}^{-}(\varphi)+\int_{\pi/2}^{3\pi/2}\mathcal{U}^{+}(\varphi)\right\}\frac{e^{i\varphi}+\zeta}{e^{i\varphi}-\zeta}\,d\varphi,$$

$$\mathcal{U}^{\pm}(\varphi)=U^{\pm}(\eta)\equiv U^{\pm}(\sin\varphi),$$

$$\zeta=z-\sqrt{1+z^2},$$

the square root representing the same branch as before. The same change of integration variable as above leads to

$$\frac{e^{i\varphi}+\zeta}{e^{i\varphi}-\zeta}=\frac{\sqrt{1+z^2}\mp\sqrt{1-\eta^2}}{z-i\eta},$$

$$\frac{d\varphi}{d\eta}=\pm\frac{1}{\sqrt{1-\eta^2}},$$

whence follows a required formula

$$f(z)=\frac{1}{2\pi}\int_{-1}^{1}\frac{1}{z-i\eta}\Big\{U^+(\eta)-U^-(\eta)+(U^+(\eta)+U^-(\eta))\frac{\sqrt{1+z^2}}{\sqrt{1-\eta^2}}\Big\}d\eta.$$

Here it is again to be noted that $\sqrt{1-\eta^2}$ denotes always a non-negative real number while $\sqrt{1+z^2}$ represents a branch tending to $\pm\sqrt{1-y^2}$ as $z\to\pm 0+iy$, respectively.

Thus, the formulas for the solutions of both boundary value problems having been separately established in explicit forms, it is ready to verify the interrelation stated in theorem 1 in the previous paper. In fact, supposing that $U^\pm(y)=\pm V^\pm(y)$ and taking the condition for solvability of the Neumann problem into account, actual calculation leads to a relation

$$\begin{aligned}&g'(z)-f(z)\\&=\frac{1}{2\pi}\int_{-1}^{1}\frac{1}{z-i\eta}\Big\{V^+(\eta)+V^-(\eta)+(V^+(\eta)-V^-(\eta))\frac{\sqrt{1-\eta^2}}{\sqrt{1+z^2}}\Big\}d\eta\\&\quad-\frac{1}{2\pi}\int_{-1}^{1}\frac{1}{z-i\eta}\Big\{V^+(\eta)+V^-(\eta)+(V^+(\eta)-V^-(\eta))\frac{\sqrt{1+z^2}}{\sqrt{1-\eta^2}}\Big\}d\eta\\&=-\frac{1}{\sqrt{1+z^2}}\frac{1}{2\pi}\int_{-1}^{1}(V^+(\eta)-V^-(\eta))\frac{z+i\eta}{\sqrt{1-\eta^2}}d\eta,\end{aligned}$$

which is the desired one.

The circumstance is quite similar with regard to theorem 2 in the previous paper.

2. Circular slit domain.

Let the basic domain be the whole z-plane slit along a circular arc

$$|z|=1,\quad \alpha\leqq\arg z\leqq 2\pi-\alpha\quad(0<\alpha<\pi),$$

and let first the boundary condition of a Neumann problem be assigned in the form

$$\frac{\partial v}{\partial \nu}((1\pm 0)e^{i\theta})=V^\pm(\theta)\quad(\alpha<\theta<2\pi-\alpha),$$

$$\int_{\alpha}^{2\pi-\alpha}(V^+(\theta)+V^-(\theta))d\theta=0.$$

It is solved by $v(z)=\Re g(z)$, where $g(z)$ is defined by

$$g(z)=c+\frac{1}{\pi}\Big\{\int_{-\alpha/2}^{\alpha/2}\mathcal{V}^+(\varphi)-\int_{\alpha/2}^{2\pi-\alpha/2}\mathcal{V}^-(\varphi)\Big\}\times\frac{k(1-2ke^{i\varphi}+e^{2i\varphi})}{(1-ke^{i\varphi})(k-e^{i\varphi})}\lg\frac{1}{e^{i\varphi}-\zeta}d\varphi,$$

$$\mathcal{V}^\pm(\varphi)=V^\pm(\theta)\equiv V^\pm\Big(2\,\mathrm{arccot}\frac{k\sin\varphi}{1-k\cos\varphi}\Big),$$

$$2k\zeta=1+z-\sqrt{(z-e^{i\alpha})(z-e^{-i\alpha})},\quad k=\cos\frac{\alpha}{2},$$

c being any constant and the square root representing such a branch that $z=\infty$ corresponds to $\zeta=k$.

Now substitute an integration variable θ defined by

$$e^{i\theta}=\frac{e^{i\varphi}(1-ke^{i\varphi})}{k-e^{i\varphi}},$$

$$2ke^{i\varphi}=1+e^{i\theta}\mp 2ie^{i\theta/2}\sqrt{\sin\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2}},$$

where the upper and lower of the double sign is taken for $-\alpha/2<\varphi<\alpha/2$ and for $\alpha/2<\varphi<2\pi-\alpha/2$, respectively, and $\sqrt{\sin((\theta+\alpha)/2)\sin((\theta-\alpha)/2)}$ is supposed to represent always a non-negative real number. In view of the relation

$$\frac{1}{e^{i\varphi}-\zeta}=\frac{k}{\sin^2\alpha}$$

$$\times\left\{\sin^2\alpha+\left(e^{i\theta}-\cos\alpha\pm 2ie^{i\theta/2}\sqrt{\sin\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2}}\right)\right.$$

$$\left.\times\left(z-\cos\alpha+\sqrt{(z-e^{i\alpha})(z-e^{-i\alpha})}\right)\right\}$$

$$\div(e^{i\theta}-z),$$

a required formula for the solution of the Neumann problem is obtained in the form

$$g(z)=c+\frac{1}{\pi}\int_\alpha^{2\pi-\alpha}\Big\{(V^+(\theta)+V^-(\theta))\lg\frac{1}{e^{i\theta}-z}$$

$$+V^+(\theta)\lg\left(\sin^2\alpha+\left(e^{i\theta}-\cos\alpha+2ie^{i\theta/2}\sqrt{\sin\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2}}\right)\right.$$

$$\left.\times\left(z-\cos\alpha+\sqrt{(z-e^{i\alpha})(z-e^{-i\alpha})}\right)\right)$$

$$+V^-(\theta)\lg\left(\sin^2\alpha+\left(e^{i\theta}-\cos\alpha-2ie^{i\theta/2}\sqrt{\sin\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2}}\right)\right.$$

$$\left.\times\left(z-\cos\alpha+\sqrt{(z-e^{i\alpha})(z-e^{-i\alpha})}\right)\right)\Big\}d\theta.$$

Let next the boundary condition of a Dirichlet problem be assigned in the form

$$u((1\pm 0)e^{i\theta})=U^\pm(\theta)\qquad(\alpha<\theta<2\pi-\alpha).$$

It is solved by $u(z)=\mathcal{R}f(z)$, where $f(z)$ is defined by

$$f(z)$$

$$=\frac{1}{2\pi}\left\{\int_{-\alpha/2}^{\alpha/2}\mathfrak{U}^+(\varphi)+\int_{\alpha/2}^{2\pi-\alpha/2}\mathfrak{U}^-(\varphi)\right\}\frac{e^{i\varphi}+z}{e^{i\varphi}-z}d\varphi,$$

$$\mathfrak{U}^\pm(\varphi)=U^\pm(\theta)\equiv U^\pm\left(2\operatorname{arccot}\frac{k\sin\varphi}{1-k\cos\varphi}\right),$$

$$2kz=1+z-\sqrt{(z-e^{i\alpha})(z-e^{-i\alpha})},\quad k=\cos\frac{\alpha}{2},$$

the square root representing the same branch as before. The same change of integration variable leads to

$$\frac{e^{i\varphi}+z}{e^{i\varphi}-z}=\frac{1}{e^{i\theta}-z}\left(e^{i\theta}+z\right.$$

$$\left.-2e^{i\theta}\frac{1-z-\sqrt{(z-e^{i\alpha})(z-e^{-i\alpha})}}{1-e^{i\theta}\pm 2ie^{i\theta/2}\sqrt{\sin\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2}}}\right),$$

$$\frac{d\varphi}{d\theta}=\pm\frac{1-e^{i\theta}}{4ie^{i\theta/2}\sqrt{\sin\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2}}}+\frac{1}{2};$$

$$\frac{e^{i\varphi}+z}{e^{i\varphi}-z}\frac{d\varphi}{d\theta}$$

$$=\frac{e^{i\theta}\sqrt{(z-e^{i\alpha})(z-e^{-i\alpha})}\pm 2ie^{i\theta/2}z\sqrt{\sin\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2}}}{\pm 2ie^{i\theta/2}(e^{i\theta}-z)\sqrt{\sin\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2}}}$$

$$+\frac{1}{2}\mp\frac{1+e^{i\theta}}{2ie^{i\theta/2}\sqrt{\sin\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2}}},$$

whence follows a required formula

$$f(z)=\frac{1}{4\pi}\int_\alpha^{2\pi-\alpha}\frac{e^{i\theta}+z}{e^{i\theta}-z}\left\{U^+(\theta)-U^-(\theta)\right.$$

$$\left.-(U^+(\theta)+U^-(\theta))\frac{ie^{i\theta/2}\sqrt{(z-e^{i\alpha})(z-e^{-i\alpha})}}{(e^{i\theta}-z)\sqrt{\sin\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2}}}\right\}d\theta$$

$$+ib;$$

while the above analysis yields

$$b=\frac{1}{4\pi}\int_\alpha^{2\pi-\alpha}(U^+(\theta)-U^-(\theta))\frac{\cos\frac{\theta}{2}\,d\theta}{\sqrt{\sin\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2}}}.$$

b may designate indeed an arbitrary real constant. Here it is again to be noted that $\sqrt{\sin((\theta+\alpha)/2)\sin((\theta-\alpha)/2)}$ denotes always a non-negative real number while $\sqrt{(z-e^{i\alpha})(z-e^{-i\alpha})}$ represents a branch tending to $\pm 2ie^{i\theta/2}\times\sqrt{\sin((\theta+\alpha)/2)\sin((\theta-\alpha)/2)}$ as $z\to(1\pm 0)e^{i\theta}$, respectively.

Thus, the formulas for the solutions of both boundary value problems having been separately established, it is ready to verify the interrelation stated in theorem 3 in the previous paper. In fact, supposing that $U^\pm(\theta)=\pm V^\pm(\theta)$, and taking the condition for solvability of the Neumann problem into account, actual calculation leads to a relation

$$zg'(z)-f(z)$$

$$=\frac{1}{2\pi}\int_\alpha^{2\pi-\alpha}\frac{1}{e^{i\theta}-z}\left\{(V^+(\theta)+V^-(\theta))z\right.$$

$$
\begin{aligned}
&+(V^+(\theta)-V^-(\theta))\frac{2zie^{i\theta/2}\sqrt{\sin\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2}}}{\sqrt{(z-e^{i\alpha})(z-e^{-i\alpha})}}\Bigg\}d\theta\\
&-\frac{1}{2\pi}\int_\alpha^{2\pi-\alpha}\frac{1}{e^{i\theta}-z}\Bigg\{(V^+(\theta)+V^-(\theta))z\\
&\qquad-(V^+(\theta)-V^-(\theta))\frac{ie^{i\theta/2}\sqrt{(z-e^{i\alpha})(z-e^{-i\alpha})}}{2\sqrt{\sin\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2}}}\Bigg\}d\theta\\
&-ib\\
&=\frac{1}{\sqrt{(z-e^{i\alpha})(z-e^{-i\alpha})}}\frac{i}{4\pi}\int_\alpha^{2\pi-\alpha}(V^+(\theta)-V^-(\theta))\\
&\qquad\times\frac{e^{-i\theta/2}(1-e^{i\theta}z)}{\sqrt{\sin\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2}}}d\theta-ib,
\end{aligned}
$$

which is the required one.

The circumstance is quite similar with regard to theorem 4 in the previous paper.

3. Radial slit domain.

In case where the basic domain is the whole plane slit along a radial segment, a similar argument as above will, of course, remain valid. However, as noticed also in the previous paper, this is a sort of rectilinear slit domain, for which the standard case has been dealt with minutely in §1 and accordingly to which the present case is readily reducible by means of a motion followed by a similitude transformation. Consequently, the details may be omitted here.

REFERENCE

1) Y. Komatu, On transference of boundary value problems. Kōdai Math. Sem. Rep. (1954), 71-80.

Related topics are studied also in Y. Komatu, Über eine Übertragung zwischen Randwertaufgaben für einen Kreisring. Ibid., 101-108; Y. Komatu and H. Mizumoto, On transference of boundary value problems for a sphere. Ibid., 115-120.

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