ON SOME EXAMPLES OF SEMIGROUPS

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By a semigroup is meant a set S of elements a, b, . . . closed under an associative binary operation:

$$(ab)c = a(bc).$$

We shall say that S is left regular, right regular, left simple or right simple if it satisfies the following condition respectively.

I. Left regularity: $ax = ay$ for some a implies $x = y$.
II. Right regularity: $xa = ya$ for some a implies $x = y$.
III. Left simplicity: For any a,b, there exists some x such that $xa = b$.
IV. Right simplicity: For any a,b, there exists some x such that $ax = b$.

The structure of semigroups S satisfying any two postulates taken from the above four is well-known in the following four cases.

(1) S is left regular and right regular,
(2) S is left regular and right simple,
(3) S is right regular and left simple,
(4) S is left simple and right simple.

In the first case (1), the structure theorem is seen in [1].

In the second case (2), S is represented as the following manner:

$$S \cong G \times R,$$

where G is a group, and R is a semigroup in which we have

$$ab = b \quad \text{for any} \quad a, b \in R.$$

This semigroup R is called right singular.

Dually in the third case (3), S is represented as the following manner:

$$S \cong G \times L,$$

where G is again a group, and L is a semigroup in which we have

$$ab = a \quad \text{for any} \quad a, b \in L.$$

This semigroup L is called left singular.

Remarks. Consider the following two postulates.

III'. For any a, b, the equation

$$xa = b$$

has a unique solution.

IV'. For any a, b, the equation

$$ax = b$$

has a unique solution.

It is easily velified that the postulate

III' implies II and III.

Dually we have the postulate

IV' implies I and IV.

Conversely the above-mentioned semigroup $G \times L$ satisfies the postulate III', i.e.,

II and III implies III'.

Dually we have

I and IV implies IV'.

Consequently,

III' is equivalent to II and III,

IV' is equivalent to I and IV.

In the fourth case (4), S becomes a group, in more detail it is read as follows:

A semigroup becomes a group if and only if it is both left simple and right simple. (Cf. [2])

Two cases are left out from the above classification, and these cases are the following

(5) S is left regular and left simple,

and dually (6) S is right regular and right simple.

Mr. Tamura had presented in [3] the question whether right simple semigroup is always a group or not.

The object of the present paper is to prove the existence of a right simple semigroup which is not a group, even when it satisfies the right regularity postulate. Thus Tamura's problem is solved negatively. In §1, we shall prove this fact by illustrating an example of semigroup which is right regular and right simple. In §2, we shall be concerned with an example of left regular and left simple semigroup, and in §3, the another interesting example is dealt with.

§1. An example of a right regular and right simple semigroup which is not a group.

Let Ω be an arbitrary denumerable set. Consider transformations x on Ω onto Ω such that every kernal has the cardinal $\aleph_0$:

For any $\omega \in \Omega$, $\overline{\overline{x^{-1}(\omega)}} = \aleph_0$.

All such x forms a set S of transformations. If x, y $\in$ S, then

$$(xy)(\omega) = x(y(\omega))$$

is defined, and

$$\overline{\overline{(xy)^{-1}(\omega)}} = \overline{\overline{y^{-1}(x^{-1}(\omega))}}$$
$$= \aleph_0 \cdot \overline{\overline{x^{-1}(\omega)}} = \aleph_0 \cdot \aleph_0 = \aleph_0 .$$

Thus S becomes a semigroup.

Now we shall prove that S is right regular and right simple.

Let us suppose xa = ya. Then for any $\omega \in \Omega$, we can find an element $\xi \in \Omega$ such that $a(\xi) = \omega$, since a is an onto-map. Thus we have

$$x(\omega) = x(a(\xi)) = (xa)(\xi)$$
$$= (ya)(\xi) = y(a(\xi)) = y(\omega)$$

for any $\omega \in \Omega$. It proves x = y, that is,

xa = ya implies x = y.

Next we shall show the solubility of any equation

$$ax = b$$

for arbitrary given two elements a, b from S.

For each $\omega \in \Omega$, we define

$$A_\omega = \{ \alpha \ ; \ a(\alpha) = \omega \} ,$$
$$B_\omega = \{ \beta \ ; \ b(\beta) = \omega \} .$$

So by definition we have

$$\overline{\overline{A}}_\omega = \aleph_0 , \quad \text{and} \quad \overline{\overline{B}}_\omega = \aleph_0$$

for each $\omega \in \Omega$. Decompose B_ω into denumerable disjoint union indexed by A_ω:

$$B_\omega = \bigcup_{\lambda \in A_\omega} B_{\omega,\lambda} \quad \text{(disjoint union).}$$

The possibility of this decomposition is certified by the denumerability of B_ω and A_ω.

Now define a transformation x on Ω into Ω by the following manner:

If $\xi \in B_{\omega,\lambda}$, then $x(\xi) = \lambda$ ($\omega \in \Omega$, $\lambda \in A_\omega$). For any $\lambda \in \Omega$, there exists $\omega \in \Omega$ such that $A_\omega \ni \lambda$. Then for any $\xi \in B_{\omega,\lambda}$, we have $x(\xi) = \lambda$. Thus x is an onto-map, i.e., $x(\Omega) = \Omega$. And

$$(ax)(\xi) = a(x(\xi)) = a(\lambda) = \omega ,$$
$$b(\xi) = \omega ,$$

since $\xi \in B_{\omega,\lambda} \subset B_\omega$. This shows

$$ax = b.$$

$x^{-1}(\lambda) = B_{\omega,\lambda}$ has the cardinal $\aleph_0$ for each $\lambda \in \Omega$. Therefore x belongs to S. Thus the proof of right simplicity of S is completed.

The above discussion proves that the transformation semigroup S is right regular and right simple, but it can not be a group, for S has no unit element.

Remark. If a right simple semigroup S has the unit, then S is obviously a group. More generally if a right simple semigroup S has an idempotent element u, then we can prove that S satisfies the left regularity postulate I. Therefore S is decomposed in the following manner:

$$S \cong G \times R,$$

where G is isomorphic with Su which is the subgroup of S, and R is a right singular semigroup which is isomorphic with the subsemigroup made up of all idempotent elements of S. ([4])

§2. An example of a left regular and left simple semigroup which is not a group.

In this §2, we shall construct a semigroup which is left regular and left simple but is not a group, by considering some special transformations on an enumerable set.

It is sufficient to consider the dual of the semigroup obtained in §1, if the existence of such semigroup is only a matter of question. Nevertheless the example which follows has some interest of its own.

Let Ω be the same one defined in §1. This time let S be the totality of all one-to-one transformations x of Ω into Ω, such that the complement of the image of Ω by x has cardinal $\aleph_0$:

$$\overline{\overline{\Omega - x(\Omega)}} = \aleph_0.$$

Then S is easily shown to form a semigroup. For being x, y $\in$ S,

$$(xy)(\Omega) = x(y(\Omega)) \subset x(\Omega)$$

implies

$$\aleph_0 = \overline{\overline{\Omega}} \geqq \overline{\overline{\Omega - (xy)(\Omega)}}$$
$$\geqq \overline{\overline{\Omega - x(\Omega)}} = \aleph_0.$$

Now we shall prove that S is left regular.

Let us assume $ax = ay$.

Since a is one-to-one into-map of Ω,

$x(\omega) \neq y(\omega)$ implies $(ax)(\omega) \neq (ay)(\omega)$, i.e.,
$ax = ay$ implies $x(\omega) = y(\omega)$ for any $\omega \in \Omega$.

This shows x = y. Thus the left regularity is proved.

Next we shall prove that S is left simple. Let a, b be any two elements of S. Put

$$A = a(\Omega), \quad B = b(\Omega),$$

and the complement A or B is denoted by A' or B' respectively. By definition, $\overline{\overline{A}} = \aleph_0$, $\overline{\overline{B}} = \aleph_0$. Decompose B' into two disjoint sets such that each part has the cardinal :

$$B' = B_1 \cup B_2, \ B_1 \cap B_2 = \phi,$$
$$\overline{\overline{B}}_1 = \overline{\overline{B}}_2 = \aleph_0.$$

Let $s(\xi)$ is a one-to-one correspondence from A to B_1. Define a transformation x as follows:

$$x(\omega) = \begin{cases} b(\xi) \text{ where } \omega = a(\xi), & \text{if } \omega \in A, \\ s(\omega), & \text{if } \omega \in A'. \end{cases}$$

Then x is a one-to-one transformation of Ω into Ω such that

$$\overline{\overline{\Omega - x(\Omega)}} = \overline{\overline{\Omega - (B_1 \cup B)}} = \overline{\overline{B}}_2 = \aleph_0.$$

This shows that x belongs to S.

The identity $xa = b$ is immediately obtained.

It is trivial that S is not a group, for S has no idempotent element.

§3. Another example of semigroup which is both left regular and left simple.

Let Ω be the totality of all non-negative real numbers:

$$\Omega = [0, \infty).$$

And let S be the totality of all positive-valued strictly increasing continuous functions f defined on Ω such that

$$\lim_{\omega \to \infty} f(\omega) = \infty .$$

Obviously

$$f(\Omega) = [f(0), \infty) \subsetneqq \Omega .$$

Define a multiplication fg in S such that

$$(fg)(\omega) = f(g(\omega)), \quad \text{for any } \omega \in \Omega .$$

Then S becomes a semigroup. Now we shall prove that S is left regular and left simple, but is not a group.

Proof of left regularity of S. If $fg = fh$ in S, then for any $\omega \in \Omega$, we have

$$(fg)(\omega) \quad (fh)(\omega),$$

and then

$$f(g(\omega)) = f(h(\omega)).$$

Since f is a one-to-one-into-map, we have

$$g(\omega) = h(\omega) \quad \text{for all } \omega \in \Omega .$$

This shows that $g = h$.

Proof of left simplicity of S. It is sufficient to show the solubility in S of the equation

$$xf = g$$

for any two functions f, g from S.

Consider the following function

$$x(\omega) = \begin{cases} \frac{g(0)}{2}\left\{1 + \frac{\omega}{f(0)}\right\} & \text{if } 0 \leqq \omega < f(0), \\ g(f^{-1}(\omega)) & \text{if } f(0) \leqq \omega < \infty . \end{cases}$$

Then it is easy to prove that x belongs to S. And

$$(xf)(\omega) = x(f(\omega)) = g(f^{-1}(f(\omega))) = g(\omega) \text{ for any } \omega \in \Omega .$$

This shows that $xf = g$.

Proof that S is not a group is obvious, for S has no idempotent element.

References.

1. Bourbaki: Algèbre, Chap.I. Structure Algébriques pp.34-35.
2. Huntington: Simplified definition of a group, Bull. of the Amer. Math. Soc. vol.8. (1901-2) pp.296-300.
3. Tamura: On independency of Axioms (in Japanese) Sûgaku Shijô Danwa No. 2, (1951) pp.22-26.
4. Yamada, Tamura and Kimura: Regurality and Simplicity of Semigroups (not yet published).

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