NOTE ON IRREDUCIBLE DECOMPOSITION OF A POSITIVE

LINEAR FUNCTIONAL

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In this paper we shall introduce a stationary natural mapping in W^*-algebra generated by a two-sided representation of a D^*-algebra $\mathfrak{A}$ with a motion G (e.g. cf. [8]) — a D^*-algebra $\mathfrak{A}$ is mean by a normed*-algebra with an approximate identity and a motion G is mean by a group of *automorphisms on $\mathfrak{A}$ (the motion has been introduced by Segal for C^*-algebra). Next, applying the stationary natural mapping and the decomposition theorem of Segal (cf. Th.4 and its proof of [7]) we shall prove an ergodic decomposition of a G-stationary semi-trace of separable $\mathfrak{A}$ under a restriction which generalizes an irreducible decomposition of finite semi-trace (cf. Th.1 of [9], I), ergodic decomposition of G-stationary trace (cf. Th.6 of [8]) and ergodic decomposition of invariant regular measure on a compact metric space with a group of homeomorphisms (cf. Th. in App. II of [3] and Th.7 of [7]).

1.[0)] Let $\mathfrak{A}$ be a D^*-algebra with an approximate identity $\{e_\alpha\}_{\alpha\in D}$ and with a motion $G(=\{s\})$ i.e. D is a directed set and $e_\alpha^* = e_\alpha$, $\|e_\alpha\| \leq 1$ for all $\alpha \in D$, $\|e_\alpha x - x\| \to 0$ for all $x \in \mathfrak{A}$, and any $s,t \in G$ are automorphisms on $\mathfrak{A}$ such that $\|x^s\| = \|x\|$, $x^{s*} = x^{*s}$ and $(x^s)^t = x^{st}$ for all $x \in \mathfrak{A}$. Let τ be a G-stationary semi-trace of $\mathfrak{A}$, i.e. τ is a linear functional on the self-adjoint subalgebra generated by $\{xy;\, x,y\in\mathfrak{A}\}$ (i.e. $\mathfrak{A}^2$) such that $\tau(x^*x) \geq 0$, $\tau(yx) = \tau(xy) = \overline{\tau(y^*x^*)}$, $\tau((e_\alpha x)^* e_\alpha x) \xrightarrow[\alpha]{} \tau(x^*x)$, $\tau((xy)^*(xy)) \leq \|x\|^2 \tau(y^*y)$ and $\tau(x^s y^s) = \tau(xy)$ for all $x, y \in \mathfrak{A}$ and $s \in G$.

Putting $\mathfrak{N} = \{x \in \mathfrak{A};\, \tau(x^*x) = 0\}$, $\mathfrak{N}$ is a two-sided ideal in $\mathfrak{A}$. Let $\mathfrak{A}^\theta$ be qoutient algebra of $\mathfrak{A}$ $(= \mathfrak{A}/\mathfrak{N})$ and for any $x \in \mathfrak{A}$ let x^θ be the class containing x. Letting $(x^\theta, y^\theta) = \tau(y^*x)$ for all $x, y \in \mathfrak{A}$, $\mathfrak{A}^\theta$ is an incomplete Hilbert space. Let $\mathfrak{H}$ be competion of $\mathfrak{A}^\theta$. Putting $x^a y^\theta = (xy)^\theta$, $x^b y^\theta = (yx)^\theta$ and $j y^\theta = y^{*\theta}$ for all $x, y \in \mathfrak{A}$, $\{x^a, x^b, j, \mathfrak{H}\}$ defines a two-sided representation of $\mathfrak{A}$. Moreover putting $U_s y^\theta = (y^s)^\theta$ for all $s \in G$ and $y \in \mathfrak{A}$, $\{U_s, \mathfrak{H}\}$ is a dual unitary representation of G. For, $(U_s y^\theta, x^\theta) = (y^{s\theta}, x^\theta) = \tau(x^* y^s) = \tau(x^{s^{-1}*} y) = (y^\theta, U_{s^{-1}} x^\theta)$ and $U_{st} y^\theta = (y^{st})^\theta = U_t y^{s\theta} = U_t U_s y^\theta$. Then we have:

(1) $(x^s)^a = U_s x^a U_{s^{-1}}$ and $(x^s)^b = U_s x^b U_{s^{-1}}$ for all $x \in \mathfrak{A}$ and $s \in G$.

For, $U_s x^a U_{s^{-1}} y^\theta = U_s x^a (y^{s^{-1}})^\theta = U_s (x y^{s^{-1}})^\theta = (x^s y)^\theta = x^{sa} y^\theta$ and similarly for the latter. Putting W^a, W^b and W_G W^*-algebras generated by $\{x^a, x \in \mathfrak{A}\}$, $\{x^b;\, x \in \mathfrak{A}\}$ and $\{U_s, s \in G\}$ respectively, $W^a = W^{b'}$, $W^{a'} = W^b$, $jAj = A^*$ for all $A \in W^a \cap W^b$ and the τ is G-ergodic[1)] if and only if $W^a \cap W^b \cap W_G' = \{\lambda I\}$ (cf. Th.2 and Th.5 of [8]) where for any set F of bounded operators on $\mathfrak{H}$ F' is the commutor of F.

Let $\mathfrak{L}$ be the family of all bounded elements v in $\mathfrak{H}$ (i.e. v belongs to $\mathfrak{L}$ if and only if $\|x^b v\| \leq M\|x^\theta\|$ for all cf. [8] and [9]) whose corresponding bounded operators on $\mathfrak{H}$ be v^a and v^b such that $v^a x^\theta = x^b v^\theta$, $v^b x^\theta = x^a v$. Then $\{x^\theta;\, x \in \mathfrak{A}\} \subset \mathfrak{L}$ and $x^{\theta a} = x^a$ for all $x \in \mathfrak{A}$, and the following relations are equivalent each other : for any v_1 and v_2 in $\mathfrak{L}$ $v_1^a = v_2^a$, $v_1^b = v_2^b$ (both as operator) and $v_1 = v_2$ (as point in $\mathfrak{H}$). Now we can define in $\mathfrak{L}$ a $*$-involution and a ring product : v^* and $v_1 v_2$ $(= v_1^a v_2 = v_2^b v_1)$ for all $v, v_1, v_2 \in \mathfrak{L}$ satisfying that $v^* = jv$, $v^{*a} = v^{a*}$, $v^{*b} = v^{b*}$ (v^{a*}, v^{b*} are adjoint operators of v^a and v^b), $jv^a j = v^{b*}$, $(v_1 v_2)^a = v_1^a v_2^a$, $(v_1 v_2)^b = v_2^b v_1^b$ and $(\lambda_1 v_1 + \lambda_2 v_2)^d = \lambda_1 v_1^d + \lambda_2 v_2^d$ (for $d = a$ or b) (cf. p.35 of [8], p.61 of [9], II).

(2) $U_s v \in \mathfrak{L}$ and $(U_s v)^a = U_s v^a U_{s^{-1}}$,

$(U_s v)^b = U_s v^b U_{s^{-1}}$ for all $s \in G$ and $v \in \mathcal{L}$.

For, $x^b U_s v = U_s U_{s^{-1}} x^b U_s v = U_s x^{s^{-1}b} v$ and $\| x^b U_s v \| = \| (x^{s^{-1}})^b v \| = \| v^a U_{s^{-1}} x^\theta \| \leqq \| v^a \| \| x^\theta \|$.

Next $U_s v^a U_{s^{-1}} x^\theta = U_s v^a (x^{s^{-1}})^\theta = U_s (x^{s^{-1}})^b v = U_s U_{s^{-1}} x^b U_s v = x^b U_s v = (U_s v)^a x^\theta$. The latter follows from the similar method.

Let $W^{a(u)}$ and $W^{b(u)}$ be the sets of all unitary operators in W^a and W^b respectively, and put $U^{\#} = UjUj$ for all $U \in W^{a(u)}$. Then $(U^{\#} v)^a = (UjUjv)^a = U v^a U^{-1}$ for all $v \in \mathcal{L}$ (cf. Lem 3 of [8]). It is evident that for any $U \in W^{a(u)}$ $jUj \in W^{b(u)}$ [2] and $(UjUj)^{-1} = jU^{-1}jU^{-1} = U^{-1}jU^{-1}j$.

Put $\mathfrak{G}$ = unitary group generated by $\{U^{\#} ; U \in W^{a(u)}\}$ and $\{U_s ; s \in G\}$.

Lemma 1. For any $U' \in \mathfrak{G}$ and $v \in \mathcal{L}$, $U'v \in \mathcal{L}$ and there exists a unitary operator U on $\mathfrak{H}$ such that $(U'v)^a = U v^a U^{-1}$ for all $v \in \mathcal{L}$.

Proof. For $U' = U_s U^{\#}$ (for some $s \in G$ and $U \in W^{a(u)}$), $U'v \in \mathcal{L}$ follows from (2) and the fact that $\mathcal{L}^a$ is ideal in W^a, and $(U'v)^a = (U_s U^{\#} v)^a = (U_s UjUjv)^a = U_s (UjUjv)^a U_{s^{-1}} = U_s U v^a U^{-1} U_{s^{-1}} = (U_s U) v^a (U_s U)^{-1}$. For $U'' = U^{\#} U_s$, similarly $U'' v \in \mathcal{L}$ and $(U''v)^a = (U^{\#} U_s v)^a = (UjUjU_s v)^a = U (U_s v)^a U^{-1} = U U_s v^a U_s^{-1} U^{-1} = (UU_s) v^a (UU_s)^{-1}$. Since general element in $\mathfrak{G}$ has product form of a finite number of the above forms U' and U'', we can prove for any U' in $\mathfrak{G}$.

Let $\mathfrak{Z}$ be the closed linear manifold of all the vectors ξ in $\mathfrak{H}$ such that $U'\xi = \xi$ for all $U' \in \mathfrak{G}$, and let Z be the projection from $\mathfrak{H}$ onto $\mathfrak{Z}$. For any $\xi \in \mathfrak{H}$, put K_ξ = closed convex hull of $\{U'\xi ; U' \in \mathfrak{G}\}$. Then

Lemma 2. (Godement's lemma; cf. [2]). (i) $K_\xi \cap \mathfrak{Z}$ consists of only one point ξ_0, (ii) $\|\xi_0\| = \inf\{\|\zeta\| ; \zeta \in K_\xi\}$, (iii) $Z\xi = \xi_0$.

(3) $jU = Uj$ for all $U \in \mathfrak{G}$ and $jZ = Zj$

For, $jU_s x^\theta = j(x^s)^\theta = x^{s*\theta} = x^{*s\theta} = U_s j x^\theta$ and $jU^{\#} x^\theta = jUjUjx^\theta = jjUjUx^\theta = UjUx^\theta = UjUjjx^\theta = U^{\#} j x^\theta$ for all $s \in G$ and $U \in W^{a(u)}$. For any $\xi \in \mathfrak{H}$ taking $\xi_n = \sum \lambda_i^{(n)} U_i^{(n)} \xi \in K_\xi$ $(U_i^{(n)} \in \mathfrak{G})$ and $\xi_n \to \xi_0 (= Z\xi)$, $jZ\xi = j\xi_0 = j\lim \xi_n = \lim j\xi_n = \lim \sum \lambda_i^{(n)} U_i^{(n)} j\xi \in K_{j\xi}$. While $U'j\xi_0 = jU'\xi_0 = j\xi_0$ for all $U' \in \mathfrak{G}$ and $j\xi_0 \in K_{j\xi} \cap \mathfrak{Z}$.

(4) $x^a\xi = x^b\xi$ for all $x \in \mathfrak{A}$ and $\xi \in \mathfrak{Z}$.[3]

For, $UjUj\xi = \xi$ implies $jUj\xi = U^{-1}\xi$. Let $x \in \mathfrak{A}$ be $x^* = x$ and $\|x^a\| \leqq 1$. Putting $U_1 = x^a + i(I - x^{a^2})^{1/2}$ and $U_2 = x^a - i(I - x^{a^2})^{1/2}$, U_1 and U_2 belong to $W^{a(u)}$. Hence

$$(jx^a j - ij(I - x^{a^2})^{1/2} j)\xi = (x^a - i(I - x^{a^2})^{1/2})\xi$$

$$(jx^a j + ij(I - x^{a^2})^{1/2} j)\xi = (x^a + i(I - x^{a^2})^{1/2})\xi$$

and $jx^a j\xi = x^a\xi$, $x^b\xi = x^a\xi$. This holds for all s.a. $x \in \mathfrak{A}$. Since any $x \in \mathfrak{A}$ can be represented as $y + iz$ (y and z being self adjoint in $\mathfrak{A}$), $x^a\xi = (y^a + iz^a)\xi = (y^b + iz^b)\xi = x^b\xi$ for all $x \in \mathfrak{A}$.

(5) $K_v \subset \mathcal{L}$ for any $v \in \mathcal{L}$ and $Z\mathcal{L} \subset \mathcal{L}$.

For, let $\{\xi_n\} \subset K_v$ such that $\xi_n = \sum_{i=1}^{m(n)} \lambda_i^{(n)} U_i^{(n)} v$ $(U_i^{(n)} \in W^{a(u)}, \sum_{i=1}^{m(n)} \lambda_i^{(n)} = 1$ and $\lambda_i^{(n)} \geqq 0)$ and $\xi_n \to \xi$. Then

$$\| x^b \xi_n \| = \| \sum_{i=1}^{m(n)} \lambda_i^{(n)} x^b U_i^{(n)} v \| = \| \sum_{i=1}^{m(n)} \lambda_i^{(n)} U_i^{(n)} v^a U_i^{(n)-1} x^\theta \| \leqq \| v^a \| \cdot \| x^\theta \|$$

and $\| x^b \xi_n \| \longrightarrow \| x^b \xi \| \leqq \| v^a \| \cdot \| x^\theta \|$ for all $x \in \mathfrak{A}$. Hence $\xi \in \mathcal{L}$ and we have the former. The latter is evident by the former.

Putting $(v^a)^\natural = (Zv)^a$ for all $v \in \mathcal{L}$, by the proof of (5) $\| (v^a)^\natural x^\theta \| = \| (Zv)^a x^\theta \| = \| x^b Z v \| \leqq \| v^a \| \cdot \| x^\theta \|$ for all $x \in \mathfrak{A}$ and we have

(6) $\| v^{a\natural} \| \leqq \| v^a \|$ for all $v \in \mathcal{L}$

Let $\mathcal{R}$ and $\mathcal{R}^\natural$ be the uniform closures of $\mathcal{L}^a$ and $\mathcal{L}^{a\natural}$ respectively, then

Proposition 1. The mapping $\natural$ is uniquely extended to a linear mapping on $\mathcal{R}$ onto $\mathcal{R}^\natural$ such that :

(i) $A \in \mathcal{R}^\natural$ implies $A^\natural = A$.
(ii) $A^{*\natural} = A^{\natural *}$ and $(A^* A)^\natural \geqq 0$.
(iii) $(UAU^{-1})^\natural = A^\natural$ for all $U \in W^{a(u)}$ and all $U = U_s$ $(s \in G)$.
(iv) $(AB)^\natural = (BA)^\natural$ and $(A^\natural B)^\natural = (AB^\natural)^\natural = A^\natural B^\natural$ for all $A, B \in \mathcal{R}$.

(v) $(A\xi,\xi) = (A^\natural\xi,\xi)$ for all $A \in \mathcal{R}$ and $\xi \in \mathfrak{Z}$.

Proof. (i) follows immediately from (6). (ii): $v^{a*\natural} = (Zjv)^a = (jZv)^a$ (by (3)) $= (Zv)^{a*} = v^{a\natural *}$. While $((v^*v)^{a\natural}x^\theta, x^\theta) = (x^b Z v^*v, x^\theta) = \lim(\sum \lambda_i^{(n)} x^b U_i^{(n)} v^*v, x^\theta)$.
Since $(x^b U' v^*v, x^\theta) = (U v^{*a} v^a U^{-1}x^\theta, x^\theta) = \|v^a U^{-1}x^\theta\|^2 \geq 0$ (where U is as in lemma 1), $((v^*v)^{a\natural}x^\theta, x^\theta) \geq 0$.
Taking $v_n \in \mathcal{L}$ such that $\|v_n^a - A\| = \|v_n^{a*} - A^*\| \to 0 \ (n\to\infty)$ we have (ii).
(iii): Since for any $u \in W^{a(u)}$ $K_{u^*v} \subset K_v$, $ZU^\# v \in K_v$ and $ZU^\# v \in K_v \cap \mathfrak{Z}$. Hence by lemma 2 $ZU^\# v = Zv$ and $(uv^a u^{-1})^\natural = (u^\# v)^{a\natural} = (ZU^\# v)^a = (Zv)^a = v^{a\natural}$ for all $u \in W^{a(u)}$ and $v \in \mathcal{L}$. While for $s \in G$, similarly $ZU_s v = Zv$ and $(U_s v^a U_{s^{-1}})^\natural = (U_s v)^{a\natural} = (ZU_s v)^a = (Zv)^a = v^{a\natural}$. Taking $v \in \mathcal{L}$ as the previous we have (ii). (iv): For any $v, w \in \mathcal{L}$ and $x, y \in \mathfrak{A}$, $(Zv^a w, x^\theta) = (v^a w, Zx^\theta) = (w, v^{*a}Zx^\theta) = (w, v^{*b}Zx^\theta)^{3)} = (v^b w, Zx^\theta) = (Zw^a v, x^\theta)$, hence $(v^a w^a)^\natural = (Zvw)^a = (Zwv)^a = (w^a v^a)^\natural$
$((v^{a\natural}w^a)^\natural x^\theta, y^\theta) = (x^b Z v^{a\natural} w, y^\theta) = (Z v^{a\natural} w, (yx^*)^\theta) = (w, v^{a\natural *} Z(yx^*)^\theta) = (Zw, v^{a\natural *} Z(yx^*)^\theta)$ (because $v^{a\natural *}Z(yx^*)^\theta \in \mathfrak{Z}$), $= (v^{a\natural} Zw, (yx^*)^\theta) = (x^b v^{a\natural} Zw, y^\theta) = (v^{a\natural}(Zw)^a x^\theta, y^\theta) = (v^{a\natural} w^{a\natural} x^\theta, y^\theta)$.
For any $A, B \in \mathcal{R}$, taking $\{v_n\}, \{w_n\} \subset \mathcal{L}$: $\|v_n^a - A\| \to 0$ and $\|w_n^a - B\| \to 0$ we can prove $(AB)^\natural = (BA)^\natural$, $(A^\natural B)^\natural = A^\natural B^\natural$ and clearly $= (AB^\natural)^\natural$. (v): For $v, w \in \mathcal{L}$, $(v^a Zw, Zw) = (w^{a\natural} v, Zw) = (Zv, w^{a\natural *} Zw) = (w^{a\natural} Zv, Zw) = (v^{a\natural} Zw, Zw)$. Since $Z\mathcal{L}$ is dense in $\mathfrak{Z}$ and $\|v_n^a - A\| \to 0$ implies $\|v_n^{a\natural} - A^\natural\| \to 0$, (v) holds.

Lemma 3. If $\mathfrak{Z}$ has the following properties:

(7) $\{x^a\mathfrak{Z};\ x \in \mathfrak{A}\}$ is dense in $\mathfrak{H}$.

Then the mapping $v^a \to v^{a\natural}$ is strongly continuous on a sphere of $\mathcal{L}^a$.

Proof. Since $Z\mathcal{L}$ is dense in $\mathfrak{Z}$, (7) is equivalent to that $\{x^a Zv;\ x \in \mathfrak{A}, v \in \mathcal{L}\}$ is dense in $\mathfrak{H}$. If $v_\gamma^a \to v^a$ strongly and $\|v_\gamma^a\| \leq M$, then $\|(v_\gamma^a - v^a)^\natural w^{a\natural} x^\theta\| = \|x^b (v_\gamma^a - v^a)^\natural Zw\|$, (since $((v_\gamma^a - v^a)^\natural Zw, x^\theta) = ((Zw)^a Z(v_\gamma - v), Zx^\theta) = (v_\gamma - v, (Zw)^{a*} Zx^\theta) = (Z(v_\gamma^a - v^a)Zw, x^\theta)$ for all $x \in \mathfrak{A}$, $(v_\gamma^a - v^a)^\natural Zw = Z(v_\gamma^a - v^a)Zw$) $= \|x^b Z(v_\gamma^a - v^a)Zw\| \to_\gamma 0$ and $v_\gamma^{a\natural} \to_\gamma v^{a\natural}$ strongly.

(8) The approximate identity $\{e_\alpha\}$ in $\mathfrak{A}$ satisfies that e_α belongs to the center of $\mathfrak{A}$ and $e_\alpha^s = e_\alpha$ for all $s \in G$ and $\alpha \in D$.

If $\{e_\alpha\}$ satisfy (8), then (7) is always satisfied. For, clearly $e_\alpha^\theta \in \mathfrak{Z}$ and $e_\alpha^a x^\theta \to x^\theta$ strongly in $\mathfrak{H}$, and $\{x^a e_\alpha^\theta;\ x \in \mathfrak{A}, \alpha \in D\}$ is dense in $\mathfrak{H}$.

THEOREM 1. Under the assumption (7) or (8), the mapping $\natural$ (on $\mathcal{L}^a$) is uniquely extended to a linear mapping on W^a onto $W^\natural$ ($= W^a \cap W^{a'} \cap W'_G$) satisfying the conditions (i) - (v) in the Prop. 1, where we take W^a and $W^\natural$ in the place of $\mathcal{R}$ and $\mathcal{R}^\natural$ respectively which coincides with $\natural$ on $\mathcal{R}$ introduced in Prop. 1, and moreover

(vi) $I^\natural = I$, and $(A^*A)^\natural = 0$ for $A \in W^a$ implies $A = 0$.

Proof. Since $\mathcal{L}^a$ is dense in W^a under the bounded strong topology (cf. [4]), by lemma 3 and its proof $\natural$ (on $\mathcal{L}^a$) can be uniquely extended onto W^a. Since the uniform convergence in $\mathcal{L}^a$ implies boundedly strong convergence (in the operator topology), the introduced mapping $\natural$ (on W^a) coincides with $\natural$ (on $\mathcal{R}$). If $v_\beta \in \mathcal{L}$ and $\|v_\beta^a\| \leq M$, then $v_\beta^a \to A$ (strongly) if and only if $v_\beta^{a*} \to A^*$. For, $w_2^a v_\beta^a Zw_1 = w_2^a v_\beta^b Zw_1 = v_\beta^b w_2^a Zw_1$ and $\{v_\beta^b w_2^a Zw_1\}_\beta$ is Cauchy directed set for all $w_1, w_2 \in \mathcal{L}$; since $\{x^a Zv;\ x \in \mathfrak{A}, v \in \mathcal{L}\}$ is dense in $\mathfrak{H}$ and $\|v_\beta^b\| = \|jv_\beta^{a*}j\| = \|v_\beta^a\| \leq M$, there exists a strongly limit B of v_β^b. Since for any $\xi, \zeta \in \mathfrak{H}$ $(jBj\xi, \zeta) = \lim(jv_\beta^b j\xi, \zeta) = \lim(v_\beta^{a*}\xi, \zeta) = \lim(\xi, v_\beta^a\zeta) = (\xi, A\zeta) = (A^*\xi, \zeta)$, $jBj = A^*$ and hence $v_\beta^{a*} = jv_\beta^b j \to jBj = A^*$. The converse is clear. If $(Zv)^a\xi = 0$ for all $v \in \mathcal{L}$, then $(v^{a*\natural}\xi, x^\theta) = ((Zv)^{a*}\xi, x^\theta) = (\xi, x^a Zv) = 0$ for all $x \in \mathfrak{A}$, $v \in \mathcal{L}$, and $\xi = 0$. Hence there exists $\{u_\gamma\} \subset Z\mathcal{L}$ such that $\|u_\gamma^a\| \leq 1$ and $u_\gamma^a \to I$ (strongly) by Satz 5 in [5] and Th. 1 in [4]. For any $u \in Z\mathcal{L}$, $A \in W^\natural$ and $U' \in \mathfrak{G}$, $U'A = AU'$, and hence $U'Au = AU'u = Au$ or $Au \in Z\mathcal{L}$. By the construction of $\natural$ on W^a, $A^\natural$ is boundedly strong limit of a $\{v_\beta^{a\natural}\}$ ($v_\beta \in \mathcal{L}$) and hence $A^\natural u_\gamma^a = (Au_\gamma^a)^\natural = (ZAu_\gamma)^a = Au_\gamma^a$. Since $u_\gamma^a \to I$ strongly, $A^\natural = A$. The fact $A^\natural \in W^\natural$ for any $A \in W^a$ follows from that $\mathcal{L}^a$ is dense in W^a under the bounded strong topoloby. Since for any $A \in W^a$ we can take $\{v_\beta\} \subset \mathcal{L}$ such that $\|v_\beta^a\| \leq M$, $v_\beta^a \to A$

and $v_\beta^{a*} \to A^*$ strongly, for any $\xi \in \mathfrak{H}$
$\|(A^*A - v_\beta^{a*} v_\beta^a)\xi\| \leq \|(A^* - v_\beta^{a*})A\xi\|$
$+ M\|(A - v_{\beta'}^a)\xi\| + M\|(v_{\beta'}^a - v_\beta^a)\xi\|$ [4]
hence $v_\beta^{a*} v_\beta^a \to A^*A$ strongly and $\|v_\beta^{a*} v_\beta^a\| = \|v_\beta^a\|^2 \leq M^2$. Since (i) - (v) fold in $\mathcal{L}^a$ (cf. Proof of Prop. 1), we have also (i) - (v) for $A \in W^a$.

(vi): Since $I \in W^\natural$, $I^\natural = I$ is evident. Let $A \in W^a$ satisfies $(A^*A)^\natural = 0$, then $((A^*A)^\natural Z v, Zv) = (A^*AZv, Zv)$ (by (v)) $= \|AZv\|^2 = 0$ and $x^b A Z v = A x^b Z v = A x^a Z v = 0$ for all $x \in \mathfrak{A}$ and $v \in \mathcal{L}$. Hence $A = 0$.

Now we have following

Corollary 1. Let τ be arbitrary G-stationary trace of a D^*-algebra $\mathfrak{A}$ with a motion G and let W^a, W^b and W_G be the W^*-algebras generated by the representations $\{x^a, \mathfrak{H}\}$, $\{x^b, \mathfrak{H}\}$ and $\{u_s, \mathfrak{H}\}$. Then there exists a G-stationary natural mapping on W^a onto $W^a \cap W^b \cap W_G'$ satisfying the properties (i) - (vi) on W^a.

Proof. There exists a strictly normalizing vector $\xi \in \mathfrak{H}$ such that $j\xi = \xi$, $x^\theta = x^a\xi = x^b\xi$, $\tau(x) = (x^a\xi, \xi)$ for all $x \in \mathfrak{A}$ and $\{x^a\xi ; x \in \mathfrak{A}\}$ is dense in $\mathfrak{H}$ (e.g. cf. Th.1 in [8]). We now prove $u_\beta^\theta \to \xi$ strongly in $\mathfrak{H}$ for any approximate identity $\{u_\beta\}$ in $\mathfrak{A}$. $(u_\beta^\theta, x^\theta) = (u_\beta^a\xi, x^a\xi) = \tau(u_\beta x^*) \to \tau(x^*) = (\xi, x^a\xi)$ and $\|u_\beta^\theta\| = \|u_\beta^a\xi\| \leq \|\xi\|$ for all β. Hence $u_\beta^\theta \to \xi$ weakly, and u_β^θ being uniformly bounded, converges strongly. Clearly e_α^s is also approximate identity in $\mathfrak{A}$ for all $s \in G$. Hence $(e_\alpha^s)^\theta = u_s e_\alpha^\theta \to u_s\xi$, $(e_\alpha^s)^\theta \to \xi$ and hence $u_s\xi = \xi$ for all $s \in G$. Therefore ξ belongs to the manifold $\mathfrak{Z}$, and the condition (7) is always satisfied and by Th.1 we have Cor.1.

2. In this section, we shall prove an ergodic decomposition of a G-stationary semi-trace τ of a separable D^*-algebra $\mathfrak{A}$ with a motion G. We shall use the same notations in § 1, and assume the condition (7) or (8). Since $\mathfrak{A}$ is separable, the Hilbert space $\mathfrak{H}$ is also separable (cf. Lem.5 of [8]).

Lemma 4. There exists a nonzero vector ξ in $\mathfrak{Z}$ such that $j\xi = \xi$ and $\{x^a\xi ; x \in \mathfrak{A}\}$ is dense in $\mathfrak{H}$.

The proof follows from the similar proof of a theorem of Segal (cf. the last paragraph of the proof of Th.9, p.49 of [7]): Let $\{\xi_n\}$ be a countable family of nonzero elements of $\mathfrak{Z}$ which is maximal with respect to the properties: 1) $j\xi_n = \xi_n$, 2) $\{\mathfrak{A}^a\xi_n\}_n$ are orthogonal with respect to each other. Putting $\xi = \sum \xi_n / 2^n\|\xi_n\|$ is the required one. This follows from the proof of Segal adjoining the facts, that the closure $\mathfrak{M}_n$ of $\mathfrak{A}^a\xi_n$ and projection P_n (onto $\mathfrak{M}_n$) satisfy that $\mathfrak{A}^a\mathfrak{M}_n \subset \mathfrak{M}_n$, $\mathfrak{A}^b\mathfrak{M}_n \subset \mathfrak{M}_n$, $j\mathfrak{M}_n \subset \mathfrak{M}_n$, $u_s\mathfrak{M}_n \subset \mathfrak{M}_n$ for all $s \in G$ and $P_n \in W^\natural$ $(= W^a \cap W^b \cap W_G')$, and that $\{x^a\zeta ; x \in \mathfrak{A}, \zeta \in \mathfrak{Z}\}$ spans $\mathfrak{H}$. [5]

Let $\mathcal{R}_1$ (resp. $\mathcal{R}_1^\natural$) be C^*-algebras generated by $\mathcal{R}$ (resp. $\mathcal{R}^\natural$) and I. Then the natural mapping $\natural$ on $\mathcal{R}$ is uniquely extended on $\mathcal{R}_1$ onto $\mathcal{R}_1^\natural$ which coincides with the contraction of the mapping $\natural$ on W^a. For any $s \in G$ and $A \in \mathcal{R}$ (or $\mathcal{R}_1$) putting $A^s = u_s A u_{s^{-1}}$, $A^s \in \mathcal{R}$ (or $\mathcal{R}_1$), G defines a motion on $\mathcal{R}$ (or $\mathcal{R}_1$) such that $x^{as} = x^{sa}$ for all $s \in G$ and $x \in \mathfrak{A}$. Let Ω and Ω_1 be character spaces of $\mathcal{R}^\natural$ and $\mathcal{R}_1^\natural$, and putting $\omega(A) = \omega(A^\natural)$ for all $A \in \mathcal{R}$ (or $\mathcal{R}_1$), ω are G-stationary traces of $\mathcal{R}$ (or $\mathcal{R}_1$ respectively). Then Ω (resp. Ω_1) is locally compact (resp. compact) Hausdorff space, [6] and there exists a Radon measure $d\mu$ on Ω such that

(9) $\quad (SA\xi, \xi) = \int_\Omega S(\omega)\,\omega(A)\,d\mu(\omega)$
for $S \in \mathcal{R}^\natural$ and $A \in \mathcal{R}$

The (9) follows from that $\mu(A^\natural) = \mu(A)$, $\mu(SA^\natural) = \mu(SA)$ and $\omega(SA) = \omega((SA)^\natural) = \omega(SA^\natural) = \omega(S)\omega(A)$ for all $A \in \mathcal{R}$, $S \in \mathcal{R}^\natural$ and $\omega \in \Omega$, where $\mu(A) = (A\xi, \xi)$.

Denote $\mathcal{R}$ the C^*-algebra generated by $\{x^a ; x \in \mathfrak{A}\}$.

THEOREM 2. Let τ be G-stationary semi-trace on $\mathfrak{A}$ and Ω the character space of $\mathcal{R}^\natural$. Then there exists a positive Radon measure ν on Ω such that

(10) $\quad \tau(xy) = \int_\Omega \pi_\omega(xy)\,d\nu(\omega)$
for all $x, y \in \mathfrak{A}$ and π_ω are $d\nu(\omega)$-almost all G-ergodic traces on $\mathfrak{A}$.

Proof. By a method of Segal which is done under the resolution of identity (cf. p.284-5 in [6]), for any s.a. $v \in Z\mathcal{L}$ there exists a sequence $\{q_n\}$ of linear combinations of orthogonal s.a. idempotents in $Z\mathcal{L}$ such that

$$(11) \qquad \|q_n - v\| \to 0, \text{ and } \|q_n^a - v^a\| \to 0 \quad (n \to \infty).$$

For any $v \in Z\mathcal{L}$, taking $v = v_1 + i v_2$ (: $v_1^* = v_1$ and $v_2^* = v_2$), (11) also holds for v. Denote $\mathcal{L}_p$ and $\mathcal{L}_q$ be the sets of all s.a. idempotents in $Z\mathcal{L}$ and linear extension of $\mathcal{L}_p$ respectively. Let $\mathcal{R}_p^{\natural}$ be the set of all projections in $\mathcal{R}^{\natural}$, then $\mathcal{R}_p^{\natural} = \mathcal{L}_p^a$ $(= \{p^a;\ p \in \mathcal{L}_p\})$ (cf. (4°), p.25 of [9], I). This follows from, that for $P \in \mathcal{R}_p^{\natural}$ taking $\{q_n\} \subset \mathcal{L}_q$ such that $\|q_n^a - P\| \to 0$ (which is possible by (11) and the fact that the uniform closure of $\mathcal{L}^{a\natural}$ is $\mathcal{R}^{\natural}$), $q_n^a(\omega) \to P(\omega)$ uniformly on Ω, and that $\mathcal{L}^{a\natural}$ is an ideal in $\mathcal{R}^{\natural}$. Let $C_0(\Omega)$ be the set of all continuous functions on Ω with compact supports. Then $C_0(\Omega) \subset \mathcal{L}^a$ ($\mathcal{L}^{a\natural}$ being an ideal in $\mathcal{R}^{\natural}$).

Putting $\nu_0(p^a) = \|p\|^2$ for any $p \in \mathcal{L}_p$, $\nu_0(\cdot)$ define a complete additive gage on $\mathcal{L}_p$ which can be considered as a complete additive set function on the collection $\mathcal{K}_0$ of all compact-open sets in Ω (considering $\nu_0(K_p) = \nu_0(p^a)$ where K_p is compact-open set corresponding to $p \in \mathcal{L}_p$), and it can be uniquely extended to a complete additive measure ν on the family of Borel sets generated by $\mathcal{K}_0$.

Then for any $v, w \in Z\mathcal{L}$, $v^a(\cdot)$ and $w^a(\cdot)$ are in $L_2(\Omega, \nu)$ and

$$(12) \qquad (v, w) = \int_\Omega v^a w^{*a}(\omega)\, d\nu(\omega).$$

For, by (11) we can take $\{q_n\}$ and $\{r_n\}$ in $\mathcal{L}_q$: $\|q_n - v\| \to 0$, $\|r_n - w\| \to 0$, $\|q_n^a - v^a\| \to 0$ and $\|r_n^a - w^a\| \to 0$ $(n \to \infty)$. Since $\|q_n - q_m\|^2 = \int |q_n^a(\omega) - q_m^a(\omega)|^2 d\nu(\omega) \to 0$ $(m, n \to \infty)$, there exists a $v'(\cdot)$ in $L_2(\Omega, \nu)$ such that $q_n^a(\omega) \to v'(\omega)$ in measure, and $\|q_n - v'\|_2 \to 0$ ($\|\cdot\|_2$ being $L_2(\nu)$ -norm). Since $q_n^a(\omega) \to v^a(\omega)$ uniformly on Ω, $v'(\omega) = v^a(\omega)$ a.e., and hence $\|q_n - v\|^2 = \int |q_n^a(\omega) - v^a(\omega)|^2 d\nu(\omega) \to 0$. Similarly $\|r_n - w\|^2 = \int |r_n^a(\omega) - w^a(\omega)|^2\, d\nu(\omega) \to 0$

Hence (v, w)
$= \lim_{n,m \to \infty}(q_n, r_m) = \lim \int q_n^a(\omega) r_m^{*a}(\omega)\, d\nu(\omega)$
$= \int v^a(\omega) w^{*a}(\omega) d\nu(\omega)$. For any $v \in \mathcal{L}$ and $w \in Z\mathcal{L}$, $(v, w) = (Zv, w) = \int (Zv)^a(\omega) w^{*a}(\omega)\, d\nu(\omega)$, (since $\omega(v^a w^{*a}) = \omega(v^a)\omega(w^{*a}) = (Zv)^a(\omega) w^{*a}(\omega)$), $= \int \omega(v^a w^{*a})\, d\nu(\omega)$.

For any $v, w \in \mathcal{L}$ and $u \in Z\mathcal{L}$, $(u^a v, w) = (v^a u, w) = (u, v^{*a} w) = \int_\Omega \omega(u^a v^a w^{*a})\, d\nu(\omega)$. Letting $\{u_n\} \subset Z\mathcal{L}$ such that, $u_n^a \geq 0$, $\|u_n^a\| \leq 1$ and $0 \leq \omega(u_n^a v^a v^{*a}) \to \omega(v^a v^{*a})$ (the existence of $\{u_n\}$ is possible by that $v^a v^{*a}(\omega)$ $(= \omega((v^a v^{*a})^{\natural}))$ vanishes at infinite on Ω), $\omega(u_n^a v^a v^{*a}) \leq \omega(v^a v^{*a})$ and $\int \omega(u_n^a v^a v^{*a}) d\nu(\omega) = (u_n^a v, v) \leq (v, v)$. Hence by Fatou's lemma $\omega(v^a v^{*a})$ is ν-integrable and

$$\int \omega(v^a v^{*a})\, d\nu(\omega) \leq (v, v).$$

Let $\{A_n\} \subset (Z\mathcal{L})^a$ such that $\|A_n\| \leq 1$, $A_n \geq 0$ and $A_n \to I$ strongly on $\mathfrak{H}$. Since $\omega(A_n v^a v^{*a}) \leq \omega(v^a v^{*a})$ and $\int \omega(A_n v^a v^{*a})\, d\nu(\omega) = (A_n v, v) \to (v, v) \leq \int \omega(v^a v^{*a})\, d\nu(\omega)$.

Hence

$$\int \omega(v^a v^{*a})\, d\nu(\omega) = (v, v)$$

for all $v \in \mathcal{L}$.

For any $v, w \in \mathcal{L}$, vw^* is complex finite linear combinations of the form $v_k v_k^*$ (i.e. $vw^* = \sum_1^q \lambda_k v_k v_k^* \in \mathcal{L}$). Then, taking $\{A_n\}$ in $(Z\mathcal{L})^a$ as above it can be shown that $(v, w) = \sum \lambda_k (v_k, v_k)$. This implies that $\omega(v^a w^{*a})$ $(= \sum \omega(v_k^a v_k^{*a}))$ is ν-integrable and $\int \omega(v^a w^{*a})\, d\nu(\omega) = \sum \lambda_k \int \omega(v_k^a v_k^{*a})\, d\nu(\omega) = \sum \lambda_k (v_k, v_k) = (v, w)$.

Since $\nu_0(\cdot)$ determines a unique positive linear functional $\nu_1(\cdot)$ on $C_0(\Omega)$ which is the contraction of $\nu(\cdot)$ onto $C_0(\Omega)$ and $\nu(p^a) = \nu_1(p^a) = \nu_0(p^a)$ for all $p \in \mathcal{L}_p$, $d\nu$ is a regular measure on Ω. Since for any $p \in \mathcal{L}_p$ $W^{\natural} p^a$ is contained in $\mathcal{L}^{a\natural}$ and weakly closed, (K, ν) is perfect measure space (cf. Lem. 1.4. of [7]). Hence any non-dense set in K_p or more general any non-dense set in Ω is ν-null set by the regularity of $d\nu$.

Let Γ be the character space of $W^{\natural}$, then $W^{\natural}$ is *isomorphic with $C(\Gamma)$ by $S \to S(\cdot)$, and there exists a continuous mapping ϕ from Γ on Ω_1 such that $S(\phi(\gamma)) = S(\gamma)$ for all $S \in \mathcal{R}^{\natural}$ and $\gamma \in \Gamma$. We prove that $\phi(\Gamma) = \Omega$: Since $\phi(\Gamma)$ is compact in Ω_1, if $\Omega_1 - \phi(\Gamma)$ is non-empty, then there exists a $0 \neq S \in \mathcal{R}_1^{\natural}$ such that $S(\phi(\gamma)) = 0$ for all $\gamma \in \Gamma$. Since $S \in W^{\natural}$ and $S(\gamma) = 0$ for all $\gamma \in \Gamma$, S is zero operator on $\mathfrak{H}$. This is a contradiction. Let $d\mu'$ be regular measure on Γ such that

$$(AS\xi, \xi) = \int \gamma(A) S(\gamma)\, d\mu'(\gamma)$$

for all $A \in W^a$ and $S \in W^{\natural}$

where $\gamma(A)$ are traces on W^a defined by $\gamma(A)=\gamma(A^\delta)$ for all $\gamma \in \Gamma$ and $A \in W^a$.

We shall prove now that, putting $m_\gamma(A)=\gamma(A)$ for all $A \in \mathcal{R}$ and $A \in W^a$, m_γ are G-ergodic traces excepting a μ'-null set in Γ [8]. Let $\{\varphi_\gamma^a, \varphi_\gamma^b, j, \mathfrak{H}_\gamma\}$ $(\gamma \in \Gamma - N')$ be the two-sided representations of $\mathcal{R}$ and let $\{\varphi_\gamma(u_s), \mathfrak{H}_\gamma\}$ be the dual unitary representation of G with normalizing vector $\xi_\gamma \in \mathfrak{H}_\gamma$ such that $\varphi_\gamma^a(A)\xi_\gamma = \varphi_\gamma^b(A)\xi_\gamma$, $\varphi_\gamma(u_s)\xi_\gamma = \xi_\gamma$, $\varphi_\gamma^a(A^s)\xi_\gamma = \varphi_\gamma(u_s)\varphi_\gamma^a(A)\xi_\gamma$ and $m_\gamma(A)=(\varphi_\gamma^a(A)\xi_\gamma, \xi_\gamma)$ for all $A \in \mathcal{R}$. Let $W^{a(\gamma)}$, $W^{b(\gamma)}$ and $W_G(\gamma)$ be W^*-algebras generated by $\{\varphi_\gamma^a(A)\}_{A\in\mathcal{R}}$, $\{\varphi_\gamma^b(A)\}_{A\in\mathcal{R}}$ and $\{\varphi_\gamma(u_s)\}_G$. As in the proof of Lem. 4.2 in [7] (cf. p.31), if $2m_\gamma = \rho_\gamma + \sigma_\gamma$ for G-stationary traces ρ_γ and σ_γ of $\mathcal{R}$ such that $\rho_\gamma(A)$ and $\sigma_\gamma(A)$ are μ'-measurable for all $A \in \mathcal{R}$, then $\rho_\gamma(A)=(T_\gamma\varphi_\gamma^a(A)\xi_\gamma, \xi_\gamma)$ for all $A \in \mathcal{R}$ where $T_\gamma \in W^{a(\gamma)} \wedge W^{b(\gamma)} \wedge W_G(\gamma)'$ and $\|T_\gamma\| \leq 2$ (cf. Proof of Th.5 of [7]). For any $A, B \in \mathcal{R}$, $\rho_\gamma(B^*A)\ (=(T_\gamma\varphi_\gamma^a(A)\xi_\gamma, \varphi_\gamma^b(B)\xi_\gamma))$ is μ'-integrable and

$$\left|\int (T_\gamma\varphi_\gamma^a(A)\xi_\gamma, \varphi_\gamma^a(B)\xi_\gamma)\,d\mu'(\gamma)\right| \geq \left(\int\|\varphi_\gamma^a(A)\xi_\gamma\|^2 d\mu'(\gamma)\int\|\varphi_\gamma^a(B)\xi_\gamma\|^2 d\mu'(\gamma)\right)^{1/2} = \|A\xi\|\cdot\|B\xi\|.$$

Since $\{A\xi\,;\,A\in\mathcal{R}\}$ is dense in $\mathfrak{H}$, there exists a bounded operator T on $\mathfrak{H}$ such that

$$(13)\quad (TA\xi, B\xi) = \int (T_\gamma\varphi_\gamma^a(A)\xi_\gamma, \varphi_\gamma^a(B)\xi_\gamma)\,d\mu'(\gamma) \quad \text{for all } A, B \in \mathcal{R}.$$

From (13) and $T_\gamma \in W^{a(\gamma)} \wedge W^{b(\gamma)} \wedge W_G(\gamma)'$ it implies $T \in W^a \wedge W^b \wedge W_G'$, and

$$(TSA\xi, B\xi) = \int T(\gamma)S(\gamma)(\varphi_\gamma^a(A)\xi_\gamma, \varphi_\gamma^a(B)\xi_\gamma)\,d\mu'(\gamma) \quad \text{for all } S \in W^{\delta'} \text{ and } A, B \in \mathcal{R}$$

Hence $T_\gamma = T(\gamma)I_\gamma$ a.e. γ where I_γ is unit operator on $\mathfrak{H}_\gamma$. Thus we have Lem. 4.2 of Segal for G-stationary trace by the similar way. The proof of theorem of Segal (p.32-4 in [7]) is applicable for G-stationary traces in the place of state, and m_γ are G-ergodic traces on $\mathcal{R}$ (i.e. extrem points in the space of all G-stationary traces of $\mathcal{R}$) excepting a μ'-null set in Γ. For $\omega \in \Omega$ putting $\omega'(A+\lambda I)=\omega(A)+\lambda$ (for all $A \in \mathcal{R}$ and λ), $\omega' \in \Omega_1$ (cf. p.32 of [7]) and the correspondence $\omega \to \omega'$ is one-one (form Ω into Ω_1). For such a ω' we denote ω under identification. The inverse ϕ^{-1} of ϕ induces on Ω: $\phi^{-1}(\omega)=\phi^{-1}(\omega')$ for all $\omega \in \Omega$. Let Ω' be a set of all ω in Ω such that $m_{\phi^{-1}(\omega)}$ are G-ergodic traces. ($m_{\phi^{-1}(\omega)}$ is well.defined as a G-stationary traces on $\mathcal{R}$ excepting a μ'-null set set N' by the fact that $m_\gamma(A)=m_{\gamma'}(A)$ for all $A \in \mathcal{R}$ and all $\gamma, \gamma' \in \phi^{-1}(\omega)$). If $\Omega - \Omega'$ contains a non-null open set Ω_0, $\phi^{-1}(\Omega_0)$ is non-null open set in Γ and for all $\gamma \in \phi^{-1}(\Omega_0)$ m_γ are not G-ergodic on $\mathcal{R}$. This is a contradiction. Putting $m_\omega(A)=m_{\phi^{-1}(\omega)}(A)$, there exists a ν-null set N in Ω such that m_ω are G-ergodic for all $\omega \in \Omega - N$. Putting $\pi_\omega(x) = m_\omega(x^a)$ for all $x \in \mathfrak{A}$, π_ω $(\omega \in \Omega - N)$ are G-ergodic traces on $\mathfrak{A}$. Indeed, if $\pi_\omega = \lambda\tau_1 + (1-\lambda)\tau_2$ for some G-stationary traces τ_1 and τ_2 on $\mathfrak{A}$ and $0 \leq \lambda \leq 1$, then $|\tau_i(xy)|^2 \leq \tau_i(x^*x)\,\tau_i(y^*y) \leq \pi_\omega(x^*x)\,\pi_\omega(y^*y) = m_\omega(x^{a*}x^a)\,m_\omega(y^{a*}y^a) \leq \|x^a\|^2\|y^a\|^2$, and hence $|\tau_i(xe_\omega)| \leq \|x^a\|\|e_\omega^a\| \leq \|x^a\|$ and $\tau_i(xe_\omega) \to \tau_i(x)$ implies $|\tau_i(x)| \leq \|x^a\|$. Putting $\rho_i(x^a) = \tau_i(x)$ for all $x \in \mathfrak{A}$ ρ_i is well defined as a positive linear functional on $\{x^a;\,x \in \mathfrak{A}\}$ and $\rho_i(x^{as}) = \rho_i(x^{sa}) = \tau_i(x^s) = \tau_i(x) = \rho_i(x^a)$. Hence ρ_i $(i = 1, 2)$ are extended to G-stationary traces ρ_i' $(i = 1, 2)$ on $\mathcal{R}$ such that $m_\omega = \lambda\rho_1' + (1-\lambda)\rho_2'$. Therefore τ_1 and τ_2 are linearly dependent. Since $\pi_\omega(x) = m_\omega(x^a) = \omega(x^a)$, $\pi_\omega(xy)$ is ν-integrable for all x, $y \in \mathfrak{A}$ and $\tau(xy^*) = \int_\Omega \pi_\omega(xy^*)\,d\nu(\omega)$. The proof is complete.

The decomposition of finite semi-trace onto pure traces follows as a special case (: G consists of only the identity automorphism) of Th.2, because (7) is always satisfied for finite semi-traces (cf. Prop. 1 of [8]); and we have Th.1 and 2 of [9],I as special cases, since for non-separable case the present proof remains valid for the type of [9].

As an application, we can prove an ergodic decomposition (to finite ergodic measures) of invariant (not ergodic) regular measure $d\tau$ on separable locally compact space E with a group of homeomorphisms under a condition that if there exists a family of finite invariant regular measures $\{d\mu_\beta\}$ such that $d\tau$ and $\{d\mu_\beta\}$ are absolutely continuous with respect to each other in the sense that for a Borel set B in E is $d\tau$-null set if and only if $d\mu_\beta$-null set for all β.

As another application, we have $\mathfrak{J}$-ergodic decomposition of Haar measure of a locally compact group with a complete compact nbd system invariant under $\mathfrak{J}$ where $\mathfrak{J}$ is the group of all inner-automorphims.

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FOOTNOTES

0) The natural mapping $\natural$ in this section will be introduced by a similar method with Godement for algebra of a representation of a unimodular locally compact group corresponding to a positive Radon measure (cf. Jour. de Math. pure et appl. 30 (1951)).

1) A stationary trace (resp. semi-trace) τ on $\mathfrak{A}$ is called G-ergodic, if τ is not convex combinations of two other linearly independent G-stationary traces (resp. semi-traces) on $\mathfrak{A}$ where the trace τ satisfies $\sup\{\tau(x^*x);\ x \in \mathfrak{A},\ \|x\| \leqq 1\} = 1$

2) In general, $jAj \in W^b$ if $A \in W^a$, and $jBj \in W^a$ if $B \in W^b$. For $jAjx^a y^\theta = jAx^{b*}y^{*\theta} = jx^{b*}Ay^{*\theta} = jx^{b*}jjAjy^\theta = x^a jAjy^\theta$ and $jAj \in W^b$. The latter similarly follows.

3) (4) implies that $v^a\xi = v^b\xi$ for all $v \in \mathfrak{L}$ and $\xi \in \mathfrak{Z}$. For $(v^a\xi, x^\theta) = (jv^{b*}j\xi, x^\theta) = (x^{*\theta}, v^{b*}j\xi) = (v^b x^{*\theta}, j\xi) = (x^{*a}v, j\xi) = (\xi, jx^{*a}jjv) = (\xi, x^b jv) = (\xi, x^a jv) = (\xi, v^{b*}x^\theta) = (v^b\xi, x^\theta)$ for all $x \in \mathfrak{A}$.

4) J. von Neumann's theorem ([5]) stated for separable Hilbert space, but both the theorem and the cited proof remain valid for arbitrary Hilbert space.

5) For any $\varepsilon > 0$ there exists β_0 such that $\|(A^*A - v_\beta^{a*}v_\beta^a)\xi\| \leqq \|(A^* - v_\beta^{a*})A\xi\| + \|v_\beta^{a*}(A - v_{\beta'}^a)\xi\| + \|v_\beta^{a*}(v_{\beta'}^a - v_\beta^a)\xi\| \leqq \|(A^* - v_\beta^{a*})A\xi\| + \|v_\beta^{a*}(A - v_{\beta'}^a)\xi\| + \|v_\beta^{a*}(v_{\beta'}^a - v_\beta^a)\xi\| \leqq \|(A^* - v_\beta^{a*})A\| + M\|(A - v_{\beta'}^a)\xi\| + M\|(v_{\beta'}^a - v_\beta^a)\xi\| < \varepsilon$ for all $\beta, \beta' > \beta_0$.

6) Let $\mathfrak{M}$ be closed linear manifold generated by $\{x^a\xi;\ x \in \mathfrak{A}\}$ and let $\mathfrak{M}_1$ be the othogonal manifold of $\mathfrak{M}$ (i.e. $\mathfrak{M}_1 = \mathfrak{M}^\perp$). Since $jx^a\xi = jx^aj\xi = x^{*b}\xi = x^{*a}\xi$ and $U_s x^a\xi = U_s x^a U_{s^{-1}}\xi = x^{s\,a}\xi$ for all $x \in \mathfrak{A}$, $\mathfrak{M}$ and $\mathfrak{M}_1$ are invariant under j and Z. If $Z\mathfrak{M}_1$ $(= \mathfrak{Z} \cap \mathfrak{M}_1) \neq 0$, then there exists ζ in $Z\mathfrak{M}_1$ (such as $j\zeta = \zeta \neq 0$) and $\zeta \in \mathfrak{M}_1$. This is a contradiction of the maximality of $\{\xi_n\}$. Hence $Z\mathfrak{M}_1 = 0$. For any $\zeta \in \mathfrak{M}_1$, $x \in \mathfrak{A}$ and $v \in \mathfrak{L}$, $(\zeta, x^a Zv) = (x^a\zeta, Zv) = (Zx^a\zeta, v)$ (since $x^a\zeta \in \mathfrak{M}_1$, $Zx^a\zeta \in \mathfrak{M}_1$) $= 0$. Hence $\zeta = 0$ or $\mathfrak{M}_1 = 0$, i.e. $\mathfrak{M} = \mathfrak{H}$.

7) For a locally compact space E, we denote $C_\infty(E)$ (resp. $C(E)$) be B^*-algebras of all continuous functions on E vanishing at infinite (resp. all bounded continuous functions) with norm $\|f\| = \sup|f(p)|$ and $*$involution $f^*(p) = \overline{f(p)}$ (:complex conjugate). Then $\mathcal{R}^\natural$ and $\mathcal{R}_1^\natural$ are $*$isomorphic (i.e. $*$preserving isomorph) with $C_\infty(\Omega)$ and $C(\Omega_1)$ by the isomorphisms $A \to A(\cdot)$ and $A(\omega) = \omega(A)$ for all $A \in \mathcal{R}^\natural$, $\omega \in \Omega$ and $A \in \mathcal{R}_1^\natural$, $\omega \in \Omega_1$ respectively.

8) Because $\{K_p;\ p \in \mathfrak{L}_p\}$ form a complete basis of open sets in Ω.

9) We can prove by the same proof of Segal (cf. p.14 of [7]) that $\sup\{\gamma(A^*A);\ A \in \mathcal{R},\ \|A\| \leqq 1\} = 1$ excepting a μ'-null set N' in Γ.

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