A THEOREM ON FOURIER SERIES

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(Comm. by T. Kawata)

Theorem.[1] Suppose that a function $f(x,t)$ is defined in $-\infty < x < \infty$, $0 \leqq t \leqq 1$, with period 2π and measurable with respect to x and there exists a function $S(x)$ with period 2π, integrable in $[0, 2\pi]$ and, such that

$$|f(x,t)| \leqq S(x)$$

for every x and t. Now suppose further that $f(x,t)$ is continuous with respect to t at each x belonging to a set A $(\subset [0,2\pi])$ of positive measure; then $\sigma_n(x,t)$ tends to $f(x,t)$ uniformly in $0 \leqq t \leqq 1$ at almost all x belonging to A, where $\sigma_n(x,t)$ denotes the n-th Cesaro sum of the Fourier series of $f(x,t)$ of order 1 with respect to x.

Lemma.[2] If a function $f(x,t)$ defined in $a \leqq x \leqq b$, $0 \leqq t \leqq 1$ is measurable with respect to x for every t and continuous with respect to t for every x belonging to a set A^* of positive measure. Then for any positive number ε there exists a closed set $F^{(\varepsilon)}$ such that

$$m(A^* - F^{(\varepsilon)}) < \varepsilon$$

and $f(x,t)$ is continuous on

$$\{(x,t);\ x \in F^{(\varepsilon)},\ 0 \leqq t \leqq 1\}.$$

Proof. Let $t^{(1)}, t^{(2)}, \cdots, t^{(k)}, \cdots$ be all the rational numbers in $[0,1]$. Since $f(x,t)$ is uniformly continuous on $[0,1]$ as a function of t at every $x \in A^*$, we have

$$A^* = \sum_{m=1}^{\infty} A^*_{n,m} \quad ,$$

where $A^*_{n,m}$ is the set of all x^* $(\in A^*)$ such that for every $t^{(i)}$ and $t^{(j)}$ that satisfy $|t^{(i)} - t^{(j)}| < 1/m$,

$$|f(x^*, t^{(i)}) - f(x^*, t^{(j)})| < 1/n \ .$$

$A^*_{n,m}$ are clearly measurable.

Now we take the positive numbers ε_i $(i = 1, 2, \cdots \quad)$, such that We can take a positive integer m_n for every positive integer n, such that

$$m(A^* - A^*_{n,m_n}) < \varepsilon_n .$$

On the other hand, we can take for every integer $k > 0$ a measurable set A^*_k, such that $f(x, t^{(k)})$ is continuous on A^*_k and

$$m(A^* - A^*_k) < \varepsilon_k$$

by Lusin's theorem. Then we have

$$m(A^* - \Pi_{k=1}^{\infty} A^*_k \cdot \Pi_{n=1}^{\infty} A^*_{n,m_n})$$

$$= m\left(\sum_{k=1}^{\infty}(A^* - A^*_k) + \sum_{n=1}^{\infty}(A^* - A^*_{n,m_n})\right)$$

$$< \sum_{k=1}^{\infty} \varepsilon_k + \sum_{n=1}^{\infty} \varepsilon_n = \varepsilon/2 . \qquad (1)$$

Since $\Pi_{k=1}^{\infty} A^*_k \cdot \Pi_{n=1}^{\infty} A^*_{n,m_n}$ is measurable, it contains a closed set $F^{(\varepsilon)}$, such that

$$m(\Pi_{k=1}^{\infty} A^*_k \cdot \Pi_{n=1}^{\infty} A^*_{n,m_n} - F^{(\varepsilon)}) < \varepsilon/2 . \qquad (2)$$

Then it follows from (1) and (2) that

$$m(A^* - F^{(\varepsilon)}) < \varepsilon .$$

Now let us prove that $f(x,t)$ is continuous on the set $\{(x,t);\ x \in F^{(\varepsilon)},\ 0 \leqq t \leqq 1\}$. Since the set of $t^{(j)}$ $(j = 1, 2, \cdots)$ is dense in $[0,1]$ and $f(x,t)$ is continuous with respect to t at every $x \in A^*$, we have

$$|f(x,t_1) - f(x,t_2)| \leqq 1/n ,$$

whenever $x \in A^*_{n,m}$, $0 \leqq t_1 \leqq 1$, $0 \leqq t_2 \leqq 1$ and $|t_1 - t_2| < 1/m$. Consequently $f(x,t)$ converges to $f(x,t_0)$ uniformly on $\Pi_{n=1}^{\infty} A_{n,m_n}$ when t tends to an arbitrary number t_0, such that $0 \leqq t_0 \leqq 1$. Since furthermore there exists a sequence

$\{t^{(\nu_n)}\}$ which tends to t_0 and $f(x,t^{(\nu_n)})$ is continuous respectively on $A^*_{\nu_n}$, $f(x,t_0)$ is continuous on

$$\prod_{k=1}^{\infty} A^*_k \left(\subset \prod_{n=1}^{\infty} A^*_{\nu_n}\right).$$

Next we take an arbitrary number $x_0 \in F^{(\varepsilon)}$. Then, as proved above, for any integer $n > 1$ there exists a positive number δ_n such that $|x-x_0| < \delta_n$, $x \in F^{(\varepsilon)}$ implies

$$|f(x,t) - f(x,t_0)| \leqq 1/n \tag{3}$$

On the other hand $|t-t_0| < 1/m_n$, $x \in F^{(\varepsilon)}$ implies

$$|f(x,t) - f(x,t_0)| < 1/n \tag{4}$$

Since $F^{(\varepsilon)} \subset A^*_{x,m_n}$. Then it follows from (3) and (4) that $|x-x_0| < \delta_n$, $x \in F^{(\varepsilon)}$, $|t-t_0| < 1/m_n$ implies

$$|f(x,t) - f(x_0,t_0)| < 2/n,$$

q. e. d.

Proof of the theorem. By the above lemma, for any integer $m > 1$, there exists a closed set F_m, such that F_m, $m(A-F_m) < 1/m$ and further such that $f(x,t)$ is continuous on the set $\{(x,t)\,;\, x \in F_m,\ 0 \leqq t \leqq 1\}$.

First we will prove

$$\frac{1}{u}\Phi_x(u,t) = \frac{1}{u}\int_{-u}^{u} |f(x+v,t) - f(x,t)|\,dv$$

tends to 0 uniformly in $0 \leqq t \leqq 1$ for almost all $x \in A$ when $u(>0)$ tends to 0. Now we have

$$\frac{1}{u}\Phi_x(u,t) = \frac{1}{u}\int_{F_{m,u,x}} |f(w,t) - f(x,t)|\,dw + \frac{1}{u}\int_{F^{(c)}_{m,u,x}} |f(w,t) - f(x,t)|\,dw \tag{5}$$

where $F_{m,u,x}$ is $[x-u,\ x+u]\cdot F_m$ and

Since $f(x,t)$ is uniformly continuous on the set $\{(x,t)\,;\, x \in F_m,\ 0 \leqq t \leqq 1\}$ because of its compactness, the first term of the right hand tends to 0 uniformly in $0 \leqq t \leqq 1$ with $u \to 0$ in case $x \in F_m$. Next by Lebesgue's theorem the additive function of a set,

$$G_m(E) = \int_{E\cdot CF_m} S(x)\,dx$$

has a derivative equal to 0 at almost all x belonging to the set F^*_m ($\subset F_m$) such that $m(F_m - F^*_m) = 0$ and further such that all the points of F^*_m are density points. Since

$$\frac{1}{u}\int_{F^{(c)}_{m,u,x}} |f(w,t) - f(x,t)|\,dw \leqq \frac{1}{u}\int_{F^{(c)}_{m,u,x}} S(w)\,dw + M(x)\frac{m(F^{(c)}_{m,u,x})}{u},$$

where $M(x) = \sup_{0 \leqq t \leqq 1} |f(x,t)|$, the second term of the right hand of (5) tends to 0 uniformly in $0 \leqq t \leqq 1$ with $u \to 0$, in case $x \in F^*_m$, by the definition of F^*_m and in virtue of $M(x) < \infty$ at any $x \in F_m$. Thus $\Phi_x(u,t)/u$ tends to 0 uniformly in $0 \leqq t \leqq 1$ for all x belonging to $\sum_{m=1}^{\infty} F^*_m$ (In the sequel we denote it by A_∞) which satisfies

$$m\left(A - \sum_{m=1}^{\infty} F^*_m\right) = 0.$$

Now let us prove that $\sigma_n(x,t)$ tends to $f(x,t)$ uniformly in $0 \leqq t \leqq 1$ with $n \to \infty$, in case We can easily obtain

$$2\pi\left(\sigma_n(x,t) - f(x,t)\right) = \int_{-\pi}^{\pi}\{f(x+u,t) - f(x,t)\}K_n(u)\,du$$

$$= \int_{-\frac{1}{n}}^{\frac{1}{n}}\{f(x+u,t) - f(x,t)\}K_n(u)\,du + \int_{\frac{1}{n}}^{\pi}\{f(x+u,t) - f(x,t)\}K_n(u)\,du + \int_{-\pi}^{-\frac{1}{n}}\{f(x+u,t) - f(x,t)\}K_n(u)\,du$$

where $K_n(u) = \frac{1}{n}\left(\frac{\sin nu/2}{\sin u/2}\right)^2$.

Since $|K_n(u)| \leqq n$ in $[-\pi, \pi]$,

$$\left|\int_{-\frac{1}{n}}^{\frac{1}{n}}\{f(x+u,t) - f(x,t)\}K_n(u)\,du\right| \leqq n\,\Phi_x\!\left(\tfrac{1}{n}, t\right).$$

The left hand tends to 0 uniformly with respect to t with $n \to \infty$, in case $x \in A_\infty$, since so is the right hand, as proved above.

Next, since $|K_n(u)| \leqq C/nu^2$ in $[1/n, \pi]$ (C : absolute constant), we have

$$\left| \int_{\frac{1}{n}}^{\pi} \{ f(x+u,t) - f(x,t) \} K_n(u)\, du \right|$$

$$\leqq \frac{C}{n} \int_{\frac{1}{n}}^{\pi} \frac{|f(x+u,t)-f(x,t)|}{u^2}\, du$$

$$= \frac{C}{n} \left[\frac{\Phi_x(u,t)}{u^2} \right]_{\frac{1}{n}}^{\pi}$$

$$+ \frac{2C}{n} \int_{\frac{1}{n}}^{\pi} \frac{\Phi_x(u,t)}{u^3}\, du.$$

As is proved above, when $x \in A_\infty$, there exists for any $\varepsilon > 0$ a positive number δ, such that $\Phi_x(u,t)/u < \varepsilon$ in $0 < u \leqq \delta$, $0 \leqq t \leqq 1$. Then if $\frac{1}{n} < \delta$, we have

$$\frac{1}{n} \int_{\frac{1}{n}}^{\pi} \frac{\Phi_x(u,t)}{u^3}\, du$$

$$= \frac{1}{n} \left(\int_{1/n}^{\delta} \frac{\Phi_x(u,t)}{u^3}\, du + \int_{\delta}^{\pi} \frac{\Phi_x(u,t)}{u^3}\, du \right)$$

$$\leqq \varepsilon - \frac{\varepsilon}{\delta n} + \frac{1}{\delta^3 n} \int_{\delta}^{\pi} \int_{-u}^{u} (S(x+v) + M(x))\, dv\, du$$

$$\leqq \varepsilon + \frac{\pi}{\delta^3 n} \left\{ \int_0^{2\pi} S(v)\, dv + 2\pi M(x) \right\}.$$

The rightest hand of this is smaller than 2ε when n is sufficiently large. On the other hand we have

$$\left| \left[\frac{C}{n} \frac{\Phi_x(u,t)}{u^2} \right]_{1/n}^{\pi} \right|$$

$$\leqq C n \Phi_x(1/n, t) + \frac{C}{\pi^2 n} \int_{-\pi}^{\pi} \{ S(x+u) + M(x) \}\, du,$$

which tends to 0 uniformly in $0 \leqq t \leqq 1$ with $n \to \infty$ in case $x \in A_\infty$.

Quite similarly it follows that

$$\int_{-\pi}^{-\frac{1}{n}} \{ f(x+u,t) - f(x,t) \} K_n(u)\, du$$

tends to 0 uniformly in $0 \leqq t \leqq 1$, in case $x \in A_\infty$.

From all the results obtained above and (6), it follows immediately that $\sigma_n(x,t) - f(x,t)$ tends to 0 uniformly in $0 \leqq t \leqq 1$, in case $x \in A_\infty$, q. e. d.

1) We can prove also its generalization for functions of the form $f(x, t_1, t_2, \cdots, t_m)$ in the same manner.
2) This lemma can be regarded as a generalization of Egoroff's theorem.

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