REMARKS ON MR. ULLEMAR'S SECOND HARMONIC MEASURE

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Recently Leo Ullemar has introduced his so-called second harmonic measure Ω in order to give a perfect condition for the existence of Dirichlet-finite automorphic functions.[1] In this note we shall give an existence proof and establish a simple relation between Ω and other domain functions.

Let G be a connected domain bounded by a finite number of Jordan curves Γ and E be a closed subset of Γ and $E' = \Gamma - E$. We shall now redefine Ω in its original form with a slight modification corresponding to a sort of normalization condition.

Seek a supremum value of $u(\zeta)$, ζ being fixed in G, where u ranges over a class $\mathfrak{F}$ of harmonic functions satisfying the conditions listed below:

(a) $u(z) = 0$ on E'

(b) $D_G(u) \equiv \iint_G \left\{ \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \right\} dx\,dy \leq \pi$.

And we put $\Omega(\zeta, E, G) = \sup u(\zeta)$ and we call Ω the second harmonic measure.

In his paper Ullemar did not give an extremal function in explicit form, but we shall bring it into a more complete form. We shall deal with this extremal problem in its dual form explained below: Let $\mathfrak{F}_E$ be a family consisting of single-valued harmonic functions being

(a') $=$ (a) .

(b') $D_G(u) < \infty$,

(c') $u(\zeta) = 1$ for a fixed $\zeta \in G$.

Seek an extremal function and value giving $\inf D_G(u)$ when $u(z)$ ranges over the class $\mathfrak{F}_E$.

Since imposed condition (a') and (b') are linear and concern with the Dirichlet-finiteness and (c') concerns with the convex normalization, we can believe that its solution must have a deep relation to a sort of reproducing kernel function. Existence of this associated kernel function of a family satisfying (a') and (b') is easily deduced by an elementary method, but we shall here choice a more direct way for the sake of later usages.

In the first place we suppose that E and E' consist of a finite number of analytic curves. The upper wave symbol $\widetilde{\square}$ means the image of $\square$ by the inversion process with respect to E. G and $\widetilde{G}$ are identified at the corresponding points on E and $\widetilde{E}$ and $\overline{G}$ means a domain thus obtained, that is, $\overline{G} = G + E + \widetilde{G}$. Let $g_S(z, \zeta)$ be the Green function of the domain S with pole $\zeta \in S$.

Putting $\hat{N}(z,\zeta) = g_{\overline{G}}(z,\zeta) + g_{\overline{G}}(z,\tilde{\zeta})$, we can easily infer that $\hat{N}(z,\zeta) \equiv 0$ on E' and $\frac{\partial}{\partial n}\hat{N}(z,\zeta) \equiv 0$ on E. Thus, if $u(z)$ satisfies (a') and (b'), then, with $G_r = G - K_r$, K_r : $|z-\zeta| \leq r$,

$$D_{G_r}(u(z), \hat{N}(z,\zeta))$$
$$= -\int_{E+E'} u(z) \frac{\partial}{\partial n} \hat{N}(z,\zeta)\,ds - \int_{K_r} u(z) \frac{\partial}{\partial n} \hat{N}(z,\zeta)\,ds,$$

which leads us to a relation

$$D_G(u(z), \hat{N}(z,\zeta)) = 2\pi\, u(\zeta),$$

if r tends to zero. On the other hand we know that

$$D_G(u(z), g_G(z,\zeta)) = 0.$$

Let $\hat{K}_G(z, \ ) = \frac{1}{2\pi}(\hat{N}(z,\zeta) - g_G(z,\zeta))$, then two identities above lead to a relation —— named the reproducing property of $\hat{K}_G(z,\zeta)$, that is,

$$D_G(u(z), \hat{K}_G(z,\zeta)) = u(\zeta).$$

Especially we have

$$D_G(\hat{K}_G(z,\zeta), \hat{K}_G(z,\zeta)) = \hat{K}_G(z,\zeta)$$

and $\hat{K}_G(z,\zeta) \equiv 0$ on E'. Thus $\hat{K}_G(z,\zeta)$ is a reproducing kernel of a family satisfying (a') and (b').

The Schwarz's inequality leads to an inequality

$$(u(\zeta))^2 \leq D_G(u(z))\, D_G(\hat{K}_G(z,\zeta)),$$

which offers a source of the solutions of our problem and Ullemar's original problem, that is, in the case of our problem we have

$$D_G\left(\frac{u(z)}{u(\zeta)}\right) \geq \frac{1}{\hat{K}_G(\zeta,\zeta)} = D_G\left(\frac{\hat{K}_G(z,\zeta)}{\hat{K}_G(\zeta,\zeta)}\right)$$

which infers the fact that

$$\hat{K}_G(z,\zeta)/\hat{K}_G(\zeta,\zeta)$$

is the desired extremal function and $1/K_G(\zeta,\zeta)$ is the desired minimum value. Uniqueness is easily deduced by the reproducing property. $K_G(z,\zeta) > 0$ in G is evident by its construction.

In the case of Ullemar's problem we have, by $D_G(u) \leq \pi$,

$$|u(\zeta)| \leq \sqrt{\pi \hat{K}_G(\zeta,\zeta)},$$

which infers an identity

$$\Omega(\zeta, E, G) = \sqrt{\pi \hat{K}_G(\zeta,\zeta)}$$

and that

$$\sqrt{\pi}\,\frac{\hat{K}_G(z,\zeta)}{\sqrt{\hat{K}_G(\zeta,\zeta)}}$$

is an extremal function of Ullemar's problem.

For the existence proof of Ω in a more general domain G we need two sorts of exhaustion, one is the case that E is considered as the ideal boundary and the other is the case that E' is considered as the ideal boundary. In each case the so-called "Gebietserweiterungsprinzip" plays an important and essential role.

In the first place we assume that E' consists of a finite or infinite number of compact or non-compact analytic curves. In this case the situation is easy.

Let $\{G_n\}$ be an exhaustion of G in the following sense: E'_n is a subset of E' and each connected component of E', on which there is only at most one component of E', is subdivided into at most three components by E'_n. E_n separates E from G, and consists of a finite number of anaytic curves. When n tends to ∞, then G_n monotonically and increasingly tends to G.

Then we have for $m > n$

$$D_{G_n}(\hat{K}_m(z,\zeta), \hat{K}_n(z,\zeta)) = \hat{K}_m(\zeta,\zeta)$$
$$= D_{G_m}(\hat{K}_m(z,\zeta), \hat{K}_m(z,\zeta)),$$

therefore

$$0 < D_{G_n}(\hat{K}_n - \hat{K}_m)$$
$$= D_{G_n}(\hat{K}_n) - 2D_{G_n}(\hat{K}_m, \hat{K}_n) + D_{G_n}(\hat{K}_m)$$
$$\leq D_{G_n}(\hat{K}_n) - D_{G_m}(\hat{K}_m)$$
$$= \hat{K}_n(\zeta,\zeta) - \hat{K}_m(\zeta,\zeta),$$

which implies the first "Gebietserweiterungsprinzip", that is, for $m > n$

$$\hat{K}_m(\zeta,\zeta) < \hat{K}_n(\zeta,\zeta)$$

or

$$\Omega(\zeta, E_m, G_m) < \Omega(\zeta, E_n, G_n).$$

This inequality is also easily obtained by $D_{G_n}(\hat{K}_m) < D_{G_m}(\hat{K}_m)$ from Ullemar's problem.

From this "Gebietserweiterungsprinzip", we infer that, if n tends to ∞, then

i) $\hat{K}_n(\zeta,\zeta) \downarrow \hat{K}_G(\zeta,\zeta)$ being either

ii) ① $\hat{K}_G(\zeta,\zeta) \equiv 0$ or ② $\hat{K}_G(\zeta,\zeta) > 0$.

① There is no non-constant Dirichlet-finite harmonic function satisfying $u(z) \equiv 0$ on E'.

②. There is at least one non-constant Dirichlet-finite harmonic function, say $\hat{K}_G(z,\zeta)$, satisfying $U(z) \equiv 0$ on E'. And $\hat{K}_G(z,\zeta)$ is bounded and non-negative on the closure of G.

iii) Converses of ii) remain true, that is, if there exists a non-constant Dirichlet-finite harmonic function on the closure of G, then there exists a non-constant bounded non-negative and Dirichlet-finite harmonic function on the closure of G vanishing identically on E'.

These results can be obtained by an analogous way of proof as the ones used in Virtanen[1], Mori[1] and Ozawa[1]. There is no need to repeat this.

The second "Gebietserweiterungsprinzip" is the same one due to Ullemar.

Let G_1 and G be the domain such that $G \subset G_1$ and $E=E_1$, then Ullemar proved

$$\Omega(z, E, G) \leq \Omega(z, E, G_1).$$

By our construction of $\hat{K}(z,z)$ this is equivalent to an inequality

$$\hat{K}_G(z,z) \leq \hat{K}_{G_1}(z,z).$$

Let $\phi(z,\zeta)$ be equal to $\hat{K}_G(z,\zeta)/\hat{K}_G(\zeta,\zeta)$ on G and 0 on $G_1 - G$, and $\phi^+(z,\zeta)$ be a harmonized function of ϕ, then we have $\phi^+(\zeta,\zeta) \geq \phi(\zeta,\zeta) = 1$

and

$$D_G\left(\frac{\hat{K}_G(z,\zeta)}{\hat{K}_G(\zeta,\zeta)}\right) \geq D_{G_1}(\phi(z,\zeta)) \geq D_{G_1}(\phi^+(z,\zeta))$$

by Dirichlet principle, therefore

$$D_{G_1}(\phi^+(z,\zeta)) \geq D_{G_1}\left(\frac{\phi^+(z,\zeta)}{\phi^+(\zeta,\zeta)}\right) \geq D_{G_1}\left(\frac{\hat{K}_{G_1}(z,\zeta)}{\hat{K}_{G_1}(\zeta,\zeta)}\right),$$

which infer the truth of the second "Gebietserweiterungsprinzip".

Thus we have the following results: For an increasing and exhausting sequence $\{G_n\}$ of domain G,

$$\text{i)}\quad \hat{K}_{G_n}(z,z) \nearrow \hat{K}_G(z,z)$$

being either

$$\text{ii)}\quad ①\ \hat{K}_G(z,z) \equiv 0 \quad \text{or}\ ②\ \neq 0,\infty \quad \text{or}\ ③\ \equiv \infty.$$

Although ① occurs in the case that E is an absolutely rare pointset and ② occurs in the case that E and E' are both absolutely large pointset and each case has the same conclusion as in the corresponding case of the first exhaustion respectively, a case ③ must occur in the case that E' is an absolutely rare pointset and E is an absolutely large pointset.

Since the above way of establishment in the second "Gebietserweiterungsprinzip" is essentially the same as in Ullemar's paper, we should now add another quite different way of proof. Hadamard's variation formulas developed in recent years rapidly by American researchers, say S. Bergman, M. Schiffer, D.C.Spencer, etc., are much adequate to this problem. Refer to S.Bergman's book [1] or an appendix by M.Schiffer "Some recent developments in the Theory of conformal Mapping" in R.Courant's book "Dirichlet principle, Conformal mapping, and Minimal surfaces".

Following Bergman's book, we suppose that the inner normal displacement δn of the boundary curves is positive, that is, $\delta n \geqq 0$ is equivalent to $G \supset G^*$, where G^* is obtained from basic domain G by the boundary displacement δn . The analyticity assumption for boundaries which should be postulated in Hadamard's variation theory is assumed to be fulfiled in the sequel.

The following formula for the variation of the Green function is well known:

$$\delta g_G(z,\zeta) = -\frac{1}{2\pi}\int_{E+E'} \frac{\partial g_G(z,t)}{\partial n_t}\,\frac{\partial g_G(t,\zeta)}{\partial n_t}\,\delta n_t\, ds_t,$$

where $\frac{\partial}{\partial n_t}$ denotes the inner normal derivative at t. In our case $\delta n_t \equiv 0$ on E, hence

$$\delta g_G(z,\zeta) = -\frac{1}{2\pi}\int_{E'} \frac{\partial g_G(z,t)}{\partial n_t}\cdot\frac{\partial g_G(t,\zeta)}{\partial n_t}\,\delta n_t\, ds_t.$$

Similarly, we have

$$\delta g_{\tilde{G}}(z,\zeta) = -\frac{1}{2\pi}\int_{E'+\tilde{E}'} \frac{\partial g_{\tilde{G}}(z,t)}{\partial n_t}\cdot\frac{\partial g_{\tilde{G}}(t,\zeta)}{\partial n_t}\,\delta n_t\, ds_t.$$

In this variation formula, we must remark that $ds_{\tilde{t}} = -ds_t$ for $t \in E'$ and $\delta n_{\tilde{t}} = \delta n_t$ and $\tilde{t}$ rotates with the inverse direction of t and moreover

$$\frac{\partial g_{\tilde{G}}(z,\tilde{t})}{\partial n_{\tilde{t}}} = -\frac{\partial g_{\tilde{G}}(\tilde{z},t)}{\partial n_t},$$

from which we can infer

$$\delta g_{\tilde{G}}(z,\zeta) = -\frac{1}{2\pi}\int_{E'}\left(\frac{\partial g_{\tilde{G}}(z,t)}{\partial n_t}\,\frac{\partial g_{\tilde{G}}(t,\zeta)}{\partial n_t} + \frac{\partial g_{\tilde{G}}(\tilde{z},t)}{\partial n_t}\,\frac{\partial g_{\tilde{G}}(t,\tilde{\zeta})}{\partial n_t}\right)\delta n_t\, ds_t.$$

Similarly, we have

$$\delta g_{\tilde{G}}(z,\tilde{\zeta}) = -\frac{1}{2\pi}\int_{E'}\left(\frac{\partial g_{\tilde{G}}(z,t)}{\partial n_t}\cdot\frac{\partial g_{\tilde{G}}(t,\tilde{\zeta})}{\partial n_t} + \frac{\partial g_{\tilde{G}}(\tilde{z},t)}{\partial n_t}\,\frac{\partial g_{\tilde{G}}(t,\zeta)}{\partial n_t}\right)\delta n_t\, ds_t.$$

Putting $z = \zeta$, then easy calculation leads to a relation

$$\begin{aligned}
&\delta(2\pi\, \hat{K}_G(\zeta,\zeta)) \\
&= -\frac{1}{2\pi}\int_{E'}\left(\frac{\partial g_{\tilde{G}}(\zeta,t)}{\partial n_t} + \frac{\partial g_{\tilde{G}}(\tilde{\zeta},t)}{\partial n_t} - \frac{\partial g_G(\zeta,t)}{\partial n_t}\right)\cdot \\
&\qquad \left(\frac{\partial g_{\tilde{G}}(\zeta,t)}{\partial n_t} + \frac{\partial g_{\tilde{G}}(\tilde{\zeta},t)}{\partial n_t} + \frac{\partial g_G(\zeta,t)}{\partial n_t}\right)\delta n_t\, ds_t \\
&= -\int_{E'}\frac{\partial}{\partial n_t}\hat{K}_G(t,\zeta)\left(\frac{\partial g_{\tilde{G}}(\zeta,t)}{\partial n_t} + \frac{\partial g_{\tilde{G}}(\tilde{\zeta},t)}{\partial n_t} + \frac{\partial g_G(\zeta,t)}{\partial n_t}\right)\delta n_t\, ds_t.
\end{aligned}$$

Since $\hat{K}_G(t,\zeta) > 0$ in G and $\hat{K}_G(t,\zeta) \equiv 0$ on E', thus $\partial/\partial n_t\, \hat{K}_G(t,\zeta) \geqq 0$ and similarly $\partial/\partial n_t\, g_{\Box}(t,s) \geqq 0$ on E', therefore $\delta \hat{K}_G(\zeta,\zeta) \geqq 0$ if $\delta n_t < 0$, that is, $G^* \supset G$, which infers the desired result:

$$\hat{K}_{G^*}(\zeta,\zeta) \geqq \hat{K}_G(\zeta,\zeta)$$

and equality sign occurs if and only if $G = G^*$

Although the desired result can be obtained by Hadamard's variational formula as above, the formula is not used in its full force. So we shall supplement this lack in order to obtain a more precise result.

At any point $t \in G$,

$$g_{\tilde{G}}(t,\zeta) > g_G(t,\zeta)$$

and both members vanish identically on E', which infer

$$\frac{\partial}{\partial n_t} g_{\tilde{G}}(t,\zeta) \geqq \frac{\partial}{\partial n_t} g_G(t,\zeta) \geqq 0$$

at $t \in E'$. Since

$$\delta(g_{\tilde{G}}(z,\zeta) - g_G(z,\zeta))$$

$$= -\frac{1}{2\pi}\int_{E'}\left(\frac{\partial g_{\tilde G}(z,t)}{\partial n_t}\cdot\frac{\partial g_{\tilde G}(t,\zeta)}{\partial n_t} - \frac{\partial g_{G}(z,t)}{\partial n_t}\frac{\partial g_{G}(t,\zeta)}{\partial n_t}\right)\delta n_t\, ds_t$$

$$-\frac{1}{2\pi}\int_{E'}\frac{\partial g_{\tilde G}(t,\tilde z)}{\partial n_t}\cdot\frac{\partial g_{\tilde G}(t,\tilde\zeta)}{\partial n_t}\,\delta n_t\, ds_t$$

and the inequality

$$\frac{\partial g_{\tilde G}(z,t)}{\partial n_t}\cdot\frac{\partial g_{\tilde G}(t,\zeta)}{\partial n_t} - \frac{\partial g_{G}(z,t)}{\partial n_t}\cdot\frac{\partial g_{G}(t,\zeta)}{\partial n_t} \geqq 0$$

holds on E', we can infer that, if $\delta n \leqq 0$,

$$\delta\left(g_{\tilde G}(z,\zeta) - g_{G}(z,\zeta)\right) \geqq 0.$$

On the other hand $\delta g_{\tilde G}(z,\tilde\zeta) \geqq 0$ holds. By these variational inequalities, we can say that

$$\delta \hat{K}_{G}(z,\zeta) \geqq 0,$$

which leads us to a more precise "Gebietserweiterungsprinzip", that is, if $G \subset G^*$ and $E = E^*$, then

$$\hat{K}_{G}(z,\zeta) \leq \hat{K}_{G^*}(z,\zeta).$$

The last inequality can be obtained in a more direct way. Let G^+ be a connected component on which

$$v(z) \equiv \hat{K}_{G}(z,\zeta) - \hat{K}_{G^*}(z,\zeta) > 0.$$

Then G^+ is bounded by the curves consisting of a part E^+ of E and the regular curve γ on which $v \equiv 0$ holds. $G - G^+$ is not evidently an empty set, for it contains ζ at which $v(\zeta) \leqq 0$. To prove is that G^+ is an empty set. On γ, $v(z) \equiv 0$ and on E^+

$$\frac{\partial}{\partial n}\hat{K}_{G}(z,\zeta) = -\frac{1}{2\pi}\frac{\partial}{\partial n}g_{G}(z,\zeta)$$
$$\geqq -\frac{1}{2\pi}\frac{\partial}{\partial n}g_{G^*}(z,\zeta)$$
$$= \frac{\partial}{\partial n}\hat{K}_{G^*}(z,\zeta),$$

that is,

$$\frac{\partial}{\partial n} v \geqq 0$$

on E^+, hence we now finished the preparation in order to bring into a contradiction. In fact,

$$0 \leq D_{G^+}(v(z))$$
$$= -\int_{E^+} v(z)\frac{\partial}{\partial n}v(z)\, ds \leqq 0.$$

Thus $v(z) \equiv 0$ on G^+ which is absurd.

What phenomena can we expect for

$$\hat{K}_{G}(z,\zeta)/\hat{K}_{G}(\zeta,\zeta) \quad\text{or}\quad \hat{K}_{G}(z,\zeta)/\sqrt{\hat{K}_{G}(\zeta,\zeta)} \quad ?$$

The former one has not the monotone.ty concerning with the variable (monotonically) domains. This is easily deduced by the maximum principle, so we omit off.

For the latter one we failed to obtain the monotoneity, but it seems to us that the monotoneity holds on the closure of G as the second "Gebietserweiterungsprinzip" in its most precise form.

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NOTE

1) Mr. J. Tamura stated an excellent but somewhat complicated existence proof on the Ullemar's extremal value at Kansûron Danwakai (Meeting of the researchers of function-theory) in Tokyo held at June 27, 1953.

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