ON THE CAUCHY'S PRODUCT SERIES THEOREM

ON EULER'S SUMMABILITY

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Cauchy's, Mertens's and Abel's theorem on the Cauchy's product series are well known. Prof. K. Knopp[1] have extended Abel's and Mertens's theorem by Euler's method of summation, but not Cauchy's. In this paper I extend Cauchy's theorem.

Theorem. If $\sum_{n=0}^{\infty} a_n = A\ (|E,p|)^{2)}$,

$$\sum_{n=0}^{\infty} b_n = B\ (|E,P|),$$

then $\sum_{n=0}^{\infty} C_n = C\ (|E,P|)$ and $AB = C$, where $\sum_{n=0}^{\infty} C_n$ is the Cauchy's product series of $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$.

To prove this theorem, we shall use two lemmas as follows:

Lemma 1. If $\sum_{n=0}^{\infty} a_n = A\ (|E,P|)$, then

$$\sum_{n=1}^{\infty} a_n = A - a_0 \quad (|E,P|),$$

and conversely.

Proof. Let us put $\sum_{n=1}^{\infty} a_n u^n = \sum_{n=0}^{\infty} b_n^{(p)} v^{n+1}$ where $u = \dfrac{v}{q+1-qv}$, $q = 2^p - 1$, then we have $\sum_{n=0}^{\infty} a_n^{(p)} v^{n+1} = a^0 \dfrac{v}{q+1-qv}$

$$+ \frac{v}{q+1-qv} \sum_{n=0}^{\infty} b_n^{(p)} v^{n+1}. \qquad (1)$$

Multiplying both sides of (1) by $q+1-qv$, we express by power series of v as follows:

$$\sum_{n=0}^{\infty} \{(q+1)\ a_n^{(p)} - q\ a_{n-1}^{(p)}\} v^{n+1} = a_0 v + \sum_{n=0}^{\infty} b_n^{(p)} v^{n+2}$$

where we put $a_{-1}^{(p)} = 0$. Comparing coefficients of both sides, we have

$$(q+1)\ a_0^{(p)} = a_0$$

$$b_n^{(p)} = (q+1)\ a_{n+1}^{(p)} - q a_n^{(p)} \qquad (n \geqq 0) \qquad (2)$$

Similarly, multiplying both sides of (1) by $q+1$, and expressing by power series of v, we have

$$(q+1)\ a_n^{(p)} = a_0 \left(\frac{q}{q+1}\right)^n + \sum_{\nu=0}^{n-1} b_\nu^{(p)} \left(\frac{q}{q+1}\right)^{n-1-\nu} \qquad (3)$$

Now, by (2), we have

$$|b_n^{(p)}| \leqq (q+1)|a_{n+1}^{(p)}| + q|a_n^{(p)}|,$$

whence $\sum_{n=0}^{m} |b_n^{(p)}| \leqq (q+1)\sum_{n=0}^{m} |a_{n+1}^{(p)}| + q \sum_{n=0}^{m} |a_n^{(p)}|$.

And $\sum_{n=0}^{\infty} |a_n^{(p)}|$ is convergent, since

$$\sum_{n=0}^{\infty} a_n = A\ (|E,P|).$$

Therefore $\sum_{n=0}^{\infty} |b_n^{(p)}|$ is convergent.

On the other hand, by (2), we have

$$\sum_{\nu=0}^{n} b_\nu^{(p)} = \sum_{\nu=0}^{n} a_\nu^{(p)} + (q+1)\ a_{n+1}^{(p)} - a_0. \qquad (4)$$

Since, from $\sum_{n=0}^{\infty} a_n = A(|E,P|)$, we have $a_{n+1}^{(p)} \to 0$ as $n \to \infty$, we get

$\sum_{n=1}^{\infty} a_n = A - a_0\ (E,P)$. Consequently

$\sum_{n=1}^{\infty} a_n = A - a_0\ (|E,p|)$.

Conversely, since $\sum_{n=1}^{\infty} a_n = A - a_0\ (|E,P|)$, we have $b_n^{(p)} \to 0$ as $n \to \infty$. Applying Kojima-Schur's theorem to (3) we have $a_n^{(p)} \to 0$ as $n \to \infty$. Hence, by (4) and

$\sum_{n=0}^{\infty} b_n^{(p)} = A - a_0\ (|E,p|)$, we have

$\sum_{n=0}^{\infty} a_n^{(p)} \to A$, i.e., $\sum_{n=0}^{\infty} a_n = A\ (E,p)$.

Now, we show that $\sum_{n=0}^{\infty} a_n$ is absolutely summable (E,p).

By (3), we have

$$(q+1)\ |a_n^{(p)}| \leqq |a_0| \left(\frac{q}{q+1}\right)^n + \sum_{\nu=0}^{n-1} |b_\nu^{(p)}| \left(\frac{q}{q+1}\right)^{n-1-\nu}$$

whence

$$(q+1)\sum_{\nu=0}^{m} |a_\nu^{(p)}| \leqq |a_0| \sum_{\nu=0}^{m} \left(\frac{q}{q+1}\right)^\nu + \sum_{\mu=1}^{m} \sum_{\nu=0}^{\mu-1} |b_\nu^{(p)}| \left(\frac{q}{q+1}\right)^{\mu-1-\nu}$$

Changing the order of summation in the second term of the right side, we have

$$\sum_{\mu=1}^{m} \sum_{\nu=0}^{\mu-1} |b_\nu^{(p)}| \left(\frac{q}{q+1}\right)^{n-1-\nu} = \sum_{\nu=0}^{m-1} \left(\frac{q}{q+1}\right)^{n-1-\nu} \times (|b_0^{(p)}| + |b_1^{(p)}| \dots + |b_\nu^{(p)}|)^{3)},$$

whence

$$(q+1)\sum_{\nu=0}^{m} |a_\nu^{(p)}| \leqq |a_0| \sum_{\nu=0}^{m} \left(\frac{q}{q+1}\right)^\nu + \sum_{\nu=0}^{m-1} \left(\frac{q}{q+1}\right)^{n-1-\nu} \times (|b_0^{(p)}| + |b_1^{(p)}| + \dots + |b_\nu^{(p)}|).$$

Since $\sum_{n=1}^{\infty} a_n' = A - a_0$ $(|E,p|)$, i.e. $\sum_{n=0}^{\infty} |b_n^{(p)}|$ is convergent, we see that the second term of the right hand is convergent by Kojima-Schur's theorem. And $\sum_{n=0}^{\infty}\left(\frac{q}{q+1}\right)$ is also convergent.

Therefore $\sum_{n=0}^{\infty} |a_n^{(p)}|$ is convergent, i.e. $\sum_{n=0}^{\infty} a_n = A$ $(|E,p|)$.

Thus lemma 1 have been proved completely.

Lemma 2[4]. Let $\sum_{n=0}^{\infty} C_n$ be the Cauchy's product series of two series $\sum_{n=0}^{\infty} a_n$, $\sum_{n=0}^{\infty} b_n$. And put $\sigma_n = C_{n-1}$ $(n = 1, 2, \ldots)$, $\sigma_0 = 0$. Then we have

$$\sigma_n^{(p)} = a_0^{(p)} b_{n-1}^{(p)} + a_1^{(p)} b_{n-2}^{(p)} + \ldots\ldots + a_{n-1}^{(p)} b_0^{(p)}$$

$$(n = 1, 2, \ldots\ldots).$$

Proof of theorem. Since

$$\sum_{n=0}^{\infty} a_n = A \quad (|E,p|),$$

$\sum_{n=0}^{\infty} b_n = B$ $(|E,p|)$ by the hypothesis, $\sum_{n=0}^{\infty} a_n^{(p)}$ and $\sum_{n=0}^{\infty} b_n^{(p)}$ are absolutely convergent. Hence, since by lemma 2 $\sum_{n=1}^{\infty} \sigma_n^{(p)}$ is the Cauchy's product series of $\sum_{n=0}^{\infty} a_n^{(p)}$ and $\sum_{n=0}^{\infty} b_n^{(p)}$, $\sum_{n=1}^{\infty} \sigma_n^{(p)}$ is absolutely convergent and $\sum_{n=1}^{\infty} \sigma_n^{(p)} = AB$ by applying Cauchy's theorem as to the product series. Therefore, from $\sigma_0^{(p)} = 0$, $\sum_{n=0}^{\infty} \sigma_n^{(p)}$ is absolutely convergent, i.e. $\sum_{n=0}^{\infty} \sigma_n = 0 + c_0 + c_1 \ldots = AB$ $(|E,p|)$.

Hence, by lemma 1, $\sum_{n=0}^{\infty} C_n$ is also absolutely summable (E,p), i.e. $\sum_{n=0}^{\infty} C_n = C(|E,p|)$ and $C = AB$.

Thus the theorem have been proved completely.

1) K. Knopp, Über das Eulersche Summierungsverfahren, Math. Zeits., 18(1923).

2) When $\sum_{n=0}^{\infty} a_n^{(p)}$ converges absolutely with sum A, where $\sum_{n=0}^{\infty} a_n^{(p)}$ are Euler's transformation of $\sum_{n=0}^{\infty} a_n$, after Prof. K. Knopp, we say that $\sum_{n=0}^{\infty} a_n$ is absolutely summable (E,p), and we write $\sum_{n=0}^{\infty} a_n = A(|E,p|)$.

3) Apply the transformation : $u = \mu - 1 - \nu$, $v = \nu$ to the left hand, then we have the result simply.

4) S. Sasaki, On the Cauchy product-series, Tôhoku. Math. J., 43(1937).

(*) Received May 27. 1953.