OPERATOR ALGEBRAS OF TYPE I

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Recently J.Dixmier [1], I.Kaplansky [2] and I.E.Segal [4] have studied of operator algebras on a Hilbert space in the large. In this paper we shall consider operator algebras of type I. Let $\mathfrak{A}$ be a such algebra, then $\mathfrak{A}$ can be directly decomposed into subalgebras $\mathfrak{A}_\alpha$ indexed by cardinal numbers α such that each $\mathfrak{A}_\alpha$ is of type I_α [1]. On the other hand, if $\mathfrak{A}$ is of type I then $\mathfrak{A}$ can be directly decomposed into subalgebras $\mathfrak{A}_\beta$ indexed by cardinal numbers β such that each $\mathfrak{A}_\beta$ is of uniform multiplicity β [4]. The one of purposes of this paper is to give the relation of above two decompositions (Theorem 2,3). Another purpose is to study the unitary equivalency of two operator algebras of type I (Theorem 4).

1. **Definitions and some lemmas.** By a W^* -algebra we mean a weakly closed self-adjoint algebra of bounded linear operators on a Hilbert space. In this paper, we shall consider W^* - algebras which contain the identity operator I. According to J.Dixmier [1] and I.Kaplansky [2] , we shall give following definitions : A non-zero projection P in a W^* - algebra $\mathfrak{A}$ is abelian if $P\mathfrak{A}P$ is commutative. A W^* - algebra is of type I if every direct summand has an abelian projection. Let P be any abelian projection in a W^* -algebra of type I, then by Zorn's lemma there exists a maximal abelian projection which contains P [2].

Lemma 1. Let P be an abelian projection in a W^* -algebra $\mathfrak{A}$, then P is maximal if and only if $PE \neq 0$ for any non-zero central projection E in $\mathfrak{A}$.

Proof. Let P be any maximal abelian projection and E a non-zero central projection such that $EP = 0$. From the definition of type I, there exists an abelian projection Q such that $E \geq Q$. Our assumption implies that $PQ = 0$. If we put $P_1 = P + Q$, then P_1 is an abelian projection and $P_1 > P$. This is a contradiction.

Conversely, let P be an abelian projection such that $PE \neq 0$ for any non-zero central projection E. Let P_1 be an abelian projection such that $P \leq P_1$, then there exists a central projection F such that $P = FP_1$ [2]. Obviously $(1 - F)P = 0$. This implies that $I - F = 0$, that is, $I = F$. Thus we have $P = IP = P_1$, that is, P is maximal. This proves the lemma.

We say that two projection P and Q of a W^* -algebra are equivalent, written $P \sim Q$, if there exists $V \in \mathfrak{A}$ with $VV^* = P$ and $V^*V = Q$. We write $P \succsim Q$ if there exists a projection $P' \geq P$ with $P' \sim Q$.

Lemma 2. Let P be an abelian projection of a W^*-algebra $\mathfrak{A}$ and let $P \sim Q$. Then Q is abelian projection and Q is maximal if and only if P is maximal.

Proof. Since $P \sim Q$ there exists an element $V \in \mathfrak{A}$ such that $VV^* = P$ and $V^*V = Q$. Then, for any A, $B \in \mathfrak{A}$, we have

$$QAQQBQ = V^*VV^*VAV^*VV^*VV^*VBV^*VV^*V = V^*PVAV^*PPVBV^*PV = V^*PVBV^*PPVAV^*PV = QBQQAQ,$$

that is, Q is abelian. Let E be any central projection satisfying $QE = 0$, then

$$PE = VV^*VV^*E = VQEV^* = 0.$$

Therefore if P is maximal abelian then $E = 0$, that is, Q is maximal abelian by Lemma 1.

Lemma 3. All maximal abelian projections of a W^* -algebra are equivalent each other.

Proof. Let P and Q be any maximal abelian projections of a W^* -algebra. According to I.Kaplansky [2], there exists a central projection E such that

$$EP \succsim EQ \quad \text{and } (I - E)P \precsim (I - E)Q,$$

that is, there exists a projection Q_1 such that $EP \geq Q_1 \sim EQ$. Obviously EP is abelian too, then there exists a central projection F such that $Q_1 = FEP$ [2]. We may assume that $F \leq E$ without loss of generality. If $E - F \neq 0$, then $Q(E - F) = 0$. Put

$$Q' = Q_1 + (I - E)Q,$$

then $Q' \sim Q$ and Q' is maximal by the preceding lemma. On the other hand $Q'(E - F) = 0$. This contradicts to maximality of Q'. Thus we have $E = F$. It follows that $EQ \sim Q_1 = EP$. By an analogous way, we can show that $(I - E)P \sim (I - E)Q$. This implies that $P \sim Q$. This proves the lemma.

Let $\mathfrak{A}$ be a W* -algebra of type I. If there exists a family $\{P_\mu\}$ of mutually orthogonal maximal abelian projections of $\mathfrak{A}$ whose power is α and satisfying $\vee P_\mu = I$, then the power of another such family is α [1]. We say that the algebra $\mathfrak{A}$ is of type I_α. Notice that a power of any family of mutually orthogonal maximal abelian projections in a W* -algebra of type I_α is not greater than α.

2. A characterization of W* -algebras of type I_α. In this section, we shall consider W* -algebras of type I_α and the relations of direct decompositions of W* - algebras of type I stated in [3] and [4]. According I.E.Segal [4], we shall give following definitions : An operator algebra $\mathfrak{A}$ on a Hilbert space $\mathfrak{h}$ is called an α-fold copy of an operator algebra $\mathcal{B}$ on a Hilbert space $\mathcal{K}$, α being a cardinal number greater than 0, if

(1) there is a set S of power α such that $\mathfrak{h}$ consists of all functions f on S to $\mathcal{K}$ for which the series $\sum_{x \in S}\|f(x)\|^2$ is convergent, with (f,g) defined as $\sum_{x\in S}(f(x), g(x))$, and

(2) $\mathfrak{A}$ consists of all operators A of the form $(Af)(x) = Bf(x)$ for some B in $\mathcal{B}$.

In the following, by $\mathfrak{A}'$ we mean the commutor of an operator algebra $\mathfrak{A}$. A W*-algebra on an Hilbert space is said to be of minimal multiplicity α if α is the least upper bound of the cardinal numbers β such that there exist β mutually orthogonal projections P_μ in $\mathfrak{A}'$ such that the operation of contracting $\mathfrak{A}$ to $P_\mu\mathfrak{h}$ is an algebraic isomorphism. It is said to be of uniform multiplicity α if for every non-zero central projection E of the contraction of $\mathfrak{A}$ to $E\mathfrak{h}$ has minimal multiplicity α. Specially, a W* -algebra is called hyper-recudible if it is of uniform multiplicity 1. A W* -algebra $\mathfrak{A}$ is hyper-reducible if and only if $\mathfrak{A}'$ is commutative. In the following of this section, we shall consider operator algebras only on a fixed Hilbert space $\mathfrak{h}$. We shall prove the following theorem :

THEOREM 1. Let $\mathfrak{A}$ be a W* -algebra of type I_α, then $\mathfrak{A}'$ is of uniform multiplicity α.

Proof. Let $\mathfrak{A}$ be a W* -algebra of type I_α, then there exist α mutually orthogonal maximal abelian projections P_μ. By a lemma due to I.E.Sagal [4], the contraction of $\mathfrak{A}'$ to each $P_\mu\mathfrak{h}$ is isomorphic to $\mathfrak{A}'$. Thus $\mathfrak{A}'$ is of minimal multiplicity β with $\beta \leq \alpha$.

Let $\{Q_\mu\}$ be a family of projections in $\mathfrak{A}$ which are mutually orthogonal and the contraction on each $Q_\mu\mathfrak{h}$ is isomorphic to $\mathfrak{A}'$. Obviously $EQ_\mu \neq 0$ for any non-zero central projection E of $\mathfrak{A}$ and for each Q_μ. Let P be any maximal abelian projection, then for every μ there exists a central projection E_1 such that

$$E_1P \gtrsim E_1Q \quad \text{and} \quad (I - E_1)P \lesssim (I - E_1)Q.$$

By Lemma 2, there exist abelian projections P_1 and P_2 such that $P \geq EQ$, $P_2 \leq (I - E)Q$ and $P_1 \sim E_1P$ and $P_2 \sim (I - E_1)P$. Since there exists an element $V \in \mathfrak{A}$ such that $P_1 = V^*EPV$, we have $E_1P_1 = P_1$, that is, $P \leq E_1$. Since P_1 is abelian, there exists a central projection E_2 satisfying $E_1Q_\mu = E_2P_1$ and furthermore, we can assume that $E_1 \geq E_2$ without loss of generality. If $E_1 - E_2 \neq 0$, then

$$E_1Q_\mu(E_1 - E_2) = E_2P_1(E_1 - E_2) = 0,$$

that is, $Q_\mu(E_1 - E_2) = 0$. This is a contradiction since $\mathfrak{A}'$ is isomorphic to the contraction of $\mathfrak{A}'$ to $Q_\mu\mathfrak{h}$. Thus we have $E_1 = E_2$. It follows that

$$E_1Q_\mu = E_2P_1 = E_1P_1 = P.$$

Obviously we have $P_\mu \sim P$ by putting $P_\mu = P_1 + P_2$. By Lemma 2 P_μ is a maximal abelian projection and $Q_\mu \geq P_\mu$. It is clear that $P_\mu P_\nu = 0$ if $\mu \neq \nu$. Then the power of $\{P_\nu\}$ is $\leq \alpha$. It follows that $\mathfrak{A}'$ is of minimal multiplicity β with $\beta \leq \alpha$. This proves that $\mathfrak{A}'$ is of minimal multiplicity α.

Let E be any central projection of $\mathfrak{A}'$ and let $\mathfrak{A}'_E$ be the contraction of $\mathfrak{A}'$ to $E\mathfrak{h}$. Then $(\mathfrak{A}'_E)'$ is identically with the contraction of $\mathfrak{A}$ to $E\mathfrak{h}$, therefore $(\mathfrak{A}'_E)'$ is of type I [3]. Above proof shows that $(\mathfrak{A}'_E)'$ is of minimal multiplicity of α. In other words, $\mathfrak{A}'$ is of uniform multiplicity α. This proves the theorem.

THEOREM 2. A W* -algebra of type I_α if and only if the commutor is unitary equivalence to an α-fold copy of a hyper-reducible algebra.

Proof. Let $\mathfrak{A}$ be a W* -algebra of type I_α, then there exists a family $\{P_\mu\}$ of power α such that P_μ are mutually orthogonal maximal abelian projections in $\mathfrak{A}$ and $\vee P_\mu = I$. Let $\mathfrak{A}_\mu$ and $\mathfrak{A}'_\mu$ be the contractions of $\mathfrak{A}$ and $\mathfrak{A}'$ to $P_\mu\mathfrak{h}$ respectively. Then we have $(\mathfrak{A}'_\mu)' = \mathfrak{A}_\mu$. Since P_μ are abelian is commutative. This implies the hyper-reducibility of $\mathfrak{A}'_\mu$. By Lemma 3, all P_μ are equivalent each other and for fixed any P_μ, P_ν there exists an (necessary partially isometric) operator $V \in \mathfrak{A}$ such that $VV^* = P_\mu$ and $V^*V = P_\nu$. Let A_μ and A_ν be contractions of any $A \in \mathfrak{A}'$ to $P_\mu\mathfrak{h}$ and $P_\nu\mathfrak{h}$ respectively, then

$$A_\nu = P_\nu A = V^*VV^*VA = V^*P_\mu AV = V^*A_\mu V.$$

Therefore by corresponding A_μ to A_ν, is unitary equivalent to $\mathfrak{A}'_\nu$. Since $\mathfrak{h} = \sum \oplus P_\mu\mathfrak{h}$, it is clear that $\mathfrak{A}'$ is unitary equivalent to an α-fold copy of any fixed $\mathfrak{A}'_{\mu}$. This proves the necessity.

Conversely, let $\mathfrak{A}'$ be unitary equivalent to an α-fold copy of a hyper-reducible algebra $\mathfrak{A}'_{\mu_0}$ on a Hilbert space $\mathfrak{H}_{\mu_0}$. Then we may assume without loss of generality that $\mathfrak{H} = \Sigma_\mu \oplus \mathfrak{H}_\mu$ where the power of indices is α and each $\mathfrak{H}_\mu$ reduces $\mathfrak{A}'$ and the contraction of $\mathfrak{A}'$ to each $\mathfrak{H}_\mu$ is unitary equivalent to $\mathfrak{A}'_{\mu_0}$. Thus we may assume that the contraction of $\mathfrak{A}'$ to $\mathfrak{H}_\mu$ is hyper-reducible.

Now let P_μ be the projection on $\mathfrak{H}_\mu$, then P_μ commuts with every element of $\mathfrak{A}'$, that is, $P_\mu \in (\mathfrak{A}')' = \mathfrak{A}$. Obviously the commutor of $\mathfrak{A}'_\mu$ is $P_\mu \mathfrak{A} P_\mu$. Since $\mathfrak{A}'_\mu$ is hyper-reducible, $P_\mu \mathfrak{A} P_\mu$ is commutative. In other words, P_μ is an abelian projection. For any non-zero central projection E, $PE \neq 0$ by the fact that the contraction of $\mathfrak{A}'$ to $P_\mu \mathfrak{H} = \mathfrak{H}_\mu$ is isomorphic to $\mathfrak{A}'$. Therefore P_μ is a maximal abelian projection by Lemma 1. All P_μ are mutually orthogonal and the power of them is α and $\vee P_\mu = I$. This proves that $\mathfrak{A}$ is of type I_α, that is, the sufficiency was proved.

According to J.Dixmier[1], for any W* - algebra $\mathfrak{A}$ of type I we can decompose it into subalgebras of type I_α : we have

$$\mathfrak{A} = \sum \oplus \mathfrak{A}_\alpha = \sum \oplus E_\alpha \mathfrak{A}$$

where E_α are central projection and $E_\alpha \mathfrak{A}$ are of type I_α respectively. Since the family $\{E_\alpha\}$ is a family mutually orthogonal central projections, we can decompose $\mathfrak{A}'$ by $\{E_\alpha\}$: we have

$$\mathfrak{A}' = \sum \oplus E_\alpha \mathfrak{A}' = \sum \oplus \mathfrak{A}'_\alpha$$

It is clear that the commutor of $\mathfrak{A}'_\alpha$ on $E_\alpha \mathfrak{H}$ is $\mathfrak{A}_\alpha$, therefore by Theorem 2 each $\mathfrak{A}'_\alpha$ is an α-fold copy of hyper-reducible algebra. Thus the above decomposition of $\mathfrak{A}'$ is identical with the one due to I.E.Segal[4]. The converse statement is obviously true. Thus we have a following theorem :

THEOREM 3. Let $\mathfrak{A}$ be of a W* -algebra of type I. Let

$$\mathfrak{A} = \sum \oplus E_\alpha \mathfrak{A}_\alpha = \sum \oplus E_\alpha \mathfrak{A}$$

be a decomposition of $\mathfrak{A}$ such that $\mathfrak{A}_\alpha$ are of type I_α respectively, then

$$\mathfrak{A}' = \sum \oplus E_\alpha \mathfrak{A}' = \sum \mathfrak{A}'_\alpha$$

is a decomposition of $\mathfrak{A}'$ such that each $\mathfrak{A}'_\alpha$ is an α-fold copy of a hyper-reducible algebra and conversely.

3. An application. In this section we shall prove the following theorem which is well known in the case of factors in the sence of F.J.Murray and J. von Noumann[3].

THEOREM 4. Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be W* -algebras of type I on Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ respectively. Moreover we assume that $\mathfrak{A}_1$ and $\mathfrak{A}'_1$ are *-isomorphic to $\mathfrak{A}_2$ and $\mathfrak{A}'_2$ respectively, then $\mathfrak{A}_1$ is unitary equivalent to $\mathfrak{A}_2$.

Proof. Denote elements of $\mathfrak{A}_1$ and $\mathfrak{A}'_1$ by $A^{(1)}, B^{(1)}, \ldots$, and corresponding elements of $\mathfrak{A}_2$ and $\mathfrak{A}'_2$ by $A^{(2)}, B^{(2)}, \ldots$.

Since $\mathfrak{A}_1$ is of type I, there exists a family $\{E^{(1)}_\alpha\}$ of mutually orthogonal central projections such that each contraction of $\mathfrak{A}'$ to $E^{(1)}\mathfrak{H}$ is of type I_α and $\vee E^{(1)}_\alpha = I^{(1)}$. The W* -algebra $\mathfrak{A}'$ is of type I by [4] and then there exists a family $\{F^{(1)}_\beta\}$ such as $\{E^{(1)}_\alpha\}$ of $\mathfrak{A}$. If $E^{(1)}_\alpha F^{(1)}_\beta \neq 0$, then this is a non-zero central projection of $\mathfrak{A}_1$ and we shall denote $E^{(1)}_{\alpha\beta} = E^{(1)}_\alpha F^{(1)}_\beta$ in such case. Then the contractions of $\mathfrak{A}_1$ and $\mathfrak{A}'_1$ to $E^{(1)}_{\alpha\beta}\mathfrak{H}$ are of type I_α and I_β respectively. Since the notion of type is purely algebraical, $E^{(2)}_{\alpha\beta}$ have same properties. Let $\mathfrak{A}_{i\alpha\beta}$ be the contractions of $\mathfrak{A}_i$ to $E^{(i)}_{\alpha\beta}\mathfrak{H}_i$ $(i = 1,2)$, then $\mathfrak{A}_1$ is unitary equivalent to $\mathfrak{A}_2$ if $\mathfrak{A}_{1\alpha\beta}$ are unitary equivalent to $\mathfrak{A}_{2\alpha\beta}$ for all such pairs (α, β). Thus we may assume that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are of type I_α and $\mathfrak{A}'_1$ and $\mathfrak{A}'_2$ are of type I_β.

In the case of above, there exists a family $\{P_\mu\}$ of mutually orthogonal maximal abelian projections satisfying $\vee P^{(1)}_\mu = I^{(1)}$. Let $\mathfrak{A}_{1\mu}$ and $\mathfrak{A}'_{1\mu}$ be the contractions of $\mathfrak{A}_1$ and $\mathfrak{A}'_1$ to $P^{(1)}_\mu \mathfrak{H}_1$ respectively, then we have $(\mathfrak{A}_{1\mu})' = \mathfrak{A}'_{1\mu}$. Since $P^{(1)}_\mu$ is maximal abelian projection, $\mathfrak{A}_{1\mu}$ is commutative and $\mathfrak{A}'_{1\mu}$ is isomorphic $\mathfrak{A}'_1$. Therefore $\mathfrak{A}'_{1\mu}$ is of type I_β too. By theorem 1, $\mathfrak{A}_{1\mu}$ is a commutative W* - W* -algebra of uniform multiplicity β. We can define $\mathfrak{A}_{2\mu}$ by an analogous way and we can prove that $\mathfrak{A}_{2\mu}$ is a commutative W* - algebra of uniform multiplicity β. It is clear that $\mathfrak{A}_{1\mu}$ is isomorphic to $\mathfrak{A}_{2\mu}$. By a theorem due to I.E.Segal[4], it follows that $\mathfrak{A}_{1\mu}$ are unitary equivalent to $\mathfrak{A}_{2\mu}$ for all μ. Therefore $\mathfrak{A}'_{1\mu}$ are unitary equivalent to $\mathfrak{A}'_{2\mu}$ for all μ. This proves that $\mathfrak{A}'_1$ is unitary equivalent to $\mathfrak{A}'_2$ by the fact that $\vee P_\mu(i) = I(i)(i = 1,2)$. In other words, $\mathfrak{A}_1$ is unitary equivalent to $\mathfrak{A}_2$.

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