IDENTITIES CONCERNING CANONICAL CONFORMAL MAPPINGS

By Yûsaku KOMATU

1. In general, a domain of finite connectivity which is finitely many-sheeted and possesses no degenerate boundary component is called a Riemann half-surface, if the projection of its whole boundary lies on a fixed circle alone.

Every Riemann half-surface becomes a closed Riemann surface, when it is inverted with respect to any one of its boundary components and the original boundary is then sewn with its inverted image by idential coordination. We may suppose further without loss of generality that the fixed circle bearing the projection of the boundary of a Riemann half-surface is coincident with the real axis of a complex plane. The closed Riemann surface obtained by the duplication process is then generated by an irreducible algebraic equation with merely real coefficients.

Let now $\mathfrak{h}$ be a Riemann half-surface laid over the w-plane, and $\mathfrak{F}$ be the closed Riemann surface obtained from $\mathfrak{h}$ by the duplication process. Let further

$$E(w, \mathfrak{w}) = 0$$

be the irreducible algebraic equation generating $\mathfrak{F}$ and

$$\mathfrak{w} = A(w)$$

be the algebraic function defined by the equation. Every analytic function meromorphic on the whole surface $\mathfrak{F}$ is necessarily a rational function with respect to w and $\mathfrak{w}$.

Let $\mathfrak{h}^*$ be any Riemann half-surface laid over the w^*-plane, yielding the corresponding closed Riemann surface $\mathfrak{F}^*$. If $\mathfrak{h}^*$ is conformally equivalent to $\mathfrak{h}$, then any analytic function

$$w^* = f(w)$$

mapping $\mathfrak{h}$ onto $\mathfrak{h}^*$ is surely prolongable, in virtue of inversion principle, beyond its boundary into $\mathfrak{F}$, and the function thus prolonged maps, of course, the whole surface $\mathfrak{F}$ onto $\mathfrak{F}^*$. Hence, according to the fact mentioned above, it must be a rational function with respect to w and $\mathfrak{w}$, i.e. there holds a relation of the form

$$f(w) = R(w, A(w)),$$

where $R(w, \mathfrak{w})$ designates a rational function with respect to its both arguments.

2. Let now D be any domain, laid on the z-plane, with no degenerate boundary conponent. We consider two analytic functions

$$w = w(z) \quad \text{and} \quad w^* = w^*(z)$$

which map D onto the Riemann half-surfaces $\mathfrak{h}$ and $\mathfrak{h}^*$, respectively. Consequently, $\mathfrak{h}$ and $\mathfrak{h}^*$ are conformally equivalent. The function $w^* = f(w)$ obtained by eliminating z form both relations $w = w(z)$ and $w^* = w^*(z)$ maps rally $\mathfrak{h}$ onto $\mathfrak{h}^*$, and hance it is of the nature mentioned above. We thus conclude that a relation of the form

$$w^*(z) = R(w(z), A(w(z)))$$

must hold, R designating, as stated above, a rational function of its both arguments. The last relation can be regarded as a functional dependence between the mapping functions $w(z)$ and $w^*(z)$.

Analytic functions mapping a given basic domain D onto Riemann half-surfaces can be constructed in various ways, especially in connection with the functions mapping D onto canonical domains of several types. Such a method has indeed been availed already by H. Lenz[1)] for proving the Schottky's mapping theorem. Following his idea and applying it to more gener general classes, we shall illustrate in the present Note several examples of such functions, among which the functional relations of the above-mentioned form must be valid.

3. Before entering into the main discourse, we begin with giving an immediate example. Suppose that D is an n-ply connected jordan domain bounded by n disjoint contours C_j

$(j=1,\cdots,n)$. It is well-known that D can be mapped conformally (of course, with respect to local parameters) onto an n-sheeted circle. By making use of an auxiliary linear transformation, it can therefore be mapped also onto an n-sheeted upper or lower half-plane, which is by itself regarded as a Riemann half-surface.

By the way, it may further be noticed that a mapping function of this type contains yet a definite number of parameters which can be chosen at our disposal. In fact, we can impose upon the mapping function, for instance, a condition that the points z_j arbitrarily chosen on C_j, respectively, correspond to the points with a common projection. Moreover, the mapping is then determinate merely within any linear transformations with real coefficients.

4. As illustrated above, any function mapping D onto an n-sheeted half-plane produces really a Riemann half-surface as its image. However, the mapping function itself is not univalent, unless $n=1$. We shall now show that a function mapping D onto a Riemann half-surface can also be constructed in several ways by making use of unvalent functions alone[2]); a mapping function of D onto a Riemann half-surface is, of course, not univalent in general.

Our basic idea will now be explained. Let, in the ζ-plane, a family of curves be given in the form

$$\mathfrak{I}\Omega(\zeta)=c$$

where $\Omega(\zeta)$ is analytic in ζ and c designates the family-parameter distinguishing each member. We consider an analytic function $\zeta=F(z)$ mapping D univalently onto a domain bounded by the arcs belonging to the family. In view of its boundary behavior, we have the relations

$$\mathfrak{I}\Omega(F(z))=c_j \qquad \text{for } z\in C_j,$$

each c_j denoting a value of c; more generally, the image of each component C_j may consist of several pieces of curves with different values of c: $c_j^{(1)}$, $c_j^{(2)}$ etc. when there curves intersect. Now, by taking the differential along the boundary $C=\sum_{j=1}^{n}C_j$, we get

$$\mathfrak{I}d\Omega(F(z))=0 \qquad \text{for } z\in C.$$

Namely, the differential $d\Omega(F(z))$ taken along C remains real along the whole boundary C.

Another analytic function $\zeta=\hat{F}(z)$ of the same mapping character as $\zeta=F(z)$ will be obtained, for instance, by replacing Ω by another similar one, $\hat{\Omega}$ say, or by changing the conditions of normalization imposed at the distinguished points in D. We then get an analogous relation

$$\mathfrak{I}d\hat{\Omega}(\hat{F}(z))=0 \qquad \text{for } z\in C,$$

d designating here again the differential-operator along C.

Thus, the function defined by

$$w(z)=\frac{d\hat{\Omega}(\hat{F}(z))}{d\Omega(F(z))}\equiv\frac{\hat{\Omega}'(\hat{F}(z))\hat{F}'(z)}{\Omega'(F(z))F'(z)}$$

is analytic in D and satisfies the boundary condition

$$\mathfrak{I}w(z)=0 \qquad \text{for } z\in C.$$

Hence, the function $w(z)$ maps the basic domain D onto a Riemann half-surface, provided it does no coincide with a constant.

Consequently, our problem of constructing a function which maps D onto a Riemann half-surface may be reduced to that of constructing the differentials along C which remain real along the whole boundary C.

5. Some of modifications will be supplemented. Let now, in the ζ-plane, two families of curves be given in the form

$$\mathfrak{I}\Omega(\zeta)=c \quad \text{and} \quad \mathfrak{R}\Omega(\zeta)=c,$$

$\Omega(\zeta)$ being analytic in ζ and c designating the family-parameter. We consider an analytic function $\zeta=F(z)$ mapping D univalently onto a domain bounded merely by the arcs belonging to either of the families, more generally, it may be permitted that an image of a boundary component extends over the arcs of both families. In view of the boundary behavior, the differential $d\Omega(\zeta)$ taken along C remains real or purely imaginary along the whole boundary C. Consequently, its squere remains real everywhere along C. We hence get the relation

$$\mathfrak{I}(d\Omega(F(z)))^2=0 \qquad \text{for } z\in C.$$

Combining the differential $d\Omega(F(z))$ with a similar one from a pair $\hat{F}(z)$ and $\hat{\Omega}(\zeta)$ of the same character, we define a function

$$w(z)=\left(\frac{d\hat{\Omega}(\hat{F}(z))}{d\Omega(F(z))}\right)^2$$

Then, it is evidently an analytic function mapping D onto a Riemann half-surface.

We can also combine an expression $(d\Omega(F(z)))^2$ of the last-mentioned type with those of the previous type. For instance, a function defined by

$$w(z)=\frac{(d\Omega(F(z)))^2}{d\Omega_1(F_1(z))d\Omega_2(F_2(z))}\equiv\frac{d\Omega(F(z))}{d\Omega_1(F_1(z))}\frac{d\Omega(F(z))}{d\Omega_2(F_2(z))},$$

where the differentials in the denominator are meromorphic in D and possess the vanishing imaginary parts along C, is evidently a function mapping D onto a Riemann half-surface.

We may also consider a differential with an analogous nature of higher order. Let, in the ζ-plane, a family of curves be defined, for instance, by relation of the form

$$\mathfrak{J}\Pi(\Omega_0(\zeta), d\Omega_1(\zeta), \cdots, d^m\Omega_m(\zeta))=0,$$

where the Ω's are analytic in ζ and Π designates a rational function of its arguments satisfying a homogeneity relation of the form

$$\Pi(\Omega_0, t\Omega_1, \cdots, t^m\Omega_m)$$
$$\equiv t^\lambda\Pi(\Omega_0, \Omega_1, \cdots, \Omega_m) \quad (\lambda \text{ being real})$$

for a real parameter t. Let $\zeta=F(z)$ be an analytic function mapping D onto a domain bounded merely by the arcs belonging to the family. Then, the function defined by

$$w(z)$$
$$=\Pi\left(\Omega_0(F(z)), \frac{d\Omega_1(F(z))}{d\hat{\Omega}(\hat{F}(z))}, \cdots, \frac{d^m\Omega_m(F(z))}{(d\hat{\Omega}(\hat{F}(z)))^m}\right)$$

maps D onto a Riemann half-surface, provided it does not reduce to a constant; here $d\hat{\Omega}(\hat{F}(z))$ designates a differential expression meromorphic in D and real-valued along the whole boundary C.

6. By means of the method explained above, we shall proceed to construct some concrete examples of analytic functions mapping a given basic domain D onto Riemann half-surfaces. According to our preceding discussions, we enumerate, for that purpose, the concrete examples of the differentials meromorphic in D and real-valued along the whole boundary C and containing further a number of parameters which can be chosen arbitrarily.

First, let z_∞ be any point contained in D and a be any real number with $0\leqq a<\pi$. It is well-known that the domain D can be mapped conformally and univalently onto the whole ζ-plane cut along the segments making the angle a with the positivereal axis, i.e. the arcs belonging to the family of parallel straight lines defined by

$$\mathfrak{J}(e^{-ia}\zeta)=c,$$

in such a way that the point z_∞ corresponds to the point at infinity, and further that the begining coefficient in the Laurent expansion of the mapping function around z_∞ is equal to unity and the coefficient of the constant term varishes. Let the mapping function is denoted by

$$\zeta=\Phi_a(z; z_\infty),$$

it is uniquely determined under the imposed condition of normalization.

Based on the boundary behavior, we have the relations

$$\mathfrak{J}(e^{-ia}\Phi_a(z; z_\infty))=c_j \quad \text{for } z\in C_j,$$

c_j being constant, which imply

$$\mathfrak{J}(e^{-ia}d\Phi_a(z; z_\infty))=0 \quad \text{for } z\in C;$$

as before, d designates here also the differential-operator along C. Thus, the differential expression

$$e^{-ia}d\Phi_a(z; z_\infty)$$

is surely of the desired nature, i.e. meromorphic in D and real-valued along C which contains the parameters z_∞ and a.

Choosing another pair of the parameters $(\hat{z}_\infty; \hat{a})$, we consider a differential $e^{-i\hat{a}}d\Phi_{\hat{a}}(z; \hat{z}_\infty)$. An analytic function defined by

$$w(z)=\frac{e^{-i\hat{a}}d\Phi_{\hat{a}}(z; \hat{z}_\infty)}{e^{-ia}d\Phi_a(z; z_\infty)}\equiv e^{-i(\hat{a}-a)}\frac{\Phi'_{\hat{a}}(z; \hat{z}_\infty)}{\Phi'_a(z; z_\infty)}$$

maps then D onto a Riemann half-surface provided $(\hat{z}_\infty; \hat{a})\neq(z_\infty; a)$. To ensure this, it has only to be shown that the function $w(z)$ thus defined does not reduce to a constant. In fact, if it were $w(z)\equiv\alpha$, α being a constant we had

$$\Phi_{\hat{a}}(z; \hat{z}_\infty)\equiv\alpha e^{i(\hat{a}-a)}\Phi_a(z; z_\infty)+\text{const},$$

and hence, in virtue of the imposed conditions of normalization at $\hat{z}_\infty$

and z_∞, $\Phi_{\hat{a}}(z;\hat{z}_\infty)\equiv\Phi_a(z;z_\infty)$ or $(\hat{z}_\infty;\hat{a})=(z_\infty;a)$, contrary to the hypothesis.

By the way, we indicate here that an explicit identity

$$\Phi_a(z;z_\infty) = e^{ia}(\Phi_0(z;z_\infty)\cos a - i\Phi_{\pi/2}(z;z_\infty)\sin a),$$

due to H. Grunsky and M. Schiffer[3]), implies

$$e^{-ia}\frac{d\Phi_a(z;z_\infty)}{d\Phi_0(z;z_\infty)} = \cos a + \sin a\cdot e^{-i\pi/2}\frac{d\Phi_{\pi/2}(z;z_\infty)}{d\Phi_0(z;z_\infty)},$$

an excellent example of the general relation $w^*(z) = R(w(z), A(w(z)))$, in which the rational function $R(w, w)$ is here simply given by $\cos a + w\sin a$ a linear polynomial in w independent of w.

7. Next, let z_0 and z_∞ be any distinct points contained in D and let β be any real number. As well-known, D can be mapped conformally and univalently onto the whole ζ-plane cut along the arcs belonging to the family of logarithmic spirals centred at the origin with the fixed inclination β, i.e. to the family defined by

$$\arg\zeta - \beta\lg|\zeta| = c$$

which may be written also in the form

$$\mathfrak{J}(e^{-ib/2}\lg\zeta) = c\cos\frac{b}{2},$$

b being defined by

$$\beta = \tan\frac{b}{2} \quad \text{or} \quad e^{ib} = \frac{(1+i\beta)^2}{1+\beta^2},$$

in such a way that the points z_0 and z_∞ correspond to the origin and the point at infinity, and further that the beginning coefficient in the Laurent expansion of the mapping function around z_∞ is equal to unity. Let the mapping function be denoted by

$$\zeta = \Psi_\beta(z;z_0,z_\infty);$$

it is uniquely determined under the imposed condition of normalization.

In particular, the function with $\beta=0$ or $\beta=\infty$ is regarded as the one mapping D onto the whole ζ-plane cut along radial or circular slits centred at the origin, respectively.

Similarly as above, we have a differential

$$e^{-ib/2}d\lg\Psi_\beta(z;z_0,z_\infty)$$

of the desired nature which contains the parameters z_0, z_∞ and β.

It is noted that an analytic function defined by

$$w(z) = \frac{e^{i\hat{b}/2}d\lg\Psi_{\hat{\beta}}(z;\hat{z}_0,\hat{z}_\infty)}{e^{ib/2}d\lg\Psi_\beta(z;z_0,z_\infty)} \equiv e^{-i(\hat{b}-b)/2}\frac{\Psi_\beta(z;z_0,z_\infty)\Psi'_{\hat{\beta}}(z;\hat{z}_0,\hat{z}_\infty)}{\Psi_{\hat{\beta}}(z;\hat{z}_0,\hat{z}_\infty)\Psi'_\beta(z;z_0,z_\infty)}$$

never reduces to a constant unless $(\hat{z}_0,\hat{z}_\infty;\hat{\beta})=(z_0,z_\infty;\beta)$; here $\hat{b}$ is, of course, given by $\hat{\beta}=\tan(\hat{b}/2)$. In fact, if $w(z)\equiv\alpha$, a constant, it becomes

$$\lg\Psi_{\hat{\beta}}(z;\hat{z}_0,\hat{z}_\infty) \equiv \alpha e^{i(\hat{b}-b)/2}\lg\Psi_\beta(z;z_0,z_\infty)+\text{const},$$

and hence, in virtue of the imposed conditions of normalization, $\Psi_{\hat{\beta}}(z;\hat{z}_0,\hat{z}_\infty) \equiv \Psi_\beta(z;z_0,z_\infty)$, implying $(\hat{z}_0,\hat{z}_\infty;\hat{\beta}) = (z_0,z_\infty,\beta)$. Consequently, we can conclude that the function $w(z)$ thus defined maps indeed D onto a Riemann half-surface.

On the other hand, we may notice that a function defined by

$$w(z) = e^{-i(b/2-\hat{a})}\frac{d\lg\Psi_\beta(z;z_0,z_\infty)}{d\Phi_{\hat{a}}(z;\hat{z}_\infty)}$$

maps D onto a Riemann half-surface, regardless of the choice of the parameters contained. In fact, as readily shown, any function of this form can never reduce to a constant.

By the way, we indicate here also that an explicit identity

$$\lg\Psi_\beta(z;z_0,z_\infty) = \frac{1+e^{ib}}{2}\lg\Psi_0(z;z_0,z_\infty)+\frac{1-e^{ib}}{2}\lg\Psi_\infty(z;z_0,z_\infty)$$

with $\beta=\tan(b/2)$ due to H.Grunsky[4]), implies

$$e^{-ib/2}\frac{d\lg\Psi_\beta(z;z_0,z_\infty)}{d\lg\Psi_0(z;z_0,z_\infty)} = \cos\frac{b}{2}+\sin\frac{b}{2}\cdot e^{-i\pi/2}\frac{d\lg\Psi_\infty(z;z_0,z_\infty)}{d\lg\Psi_0(z;z_0,z_\infty)},$$

an excellent example of the general relation $w^*(z) = \mathcal{R}(w(z), A(w(z)))$, in which the rational function $R(w, \omega)$ is here simply given by $\cos(\ell/2) + w\sin(\ell/2)$, again a linear polynomial in w independent of ω.

8. Let z_∞ denote here again any point contained in D and a be any real number with $0 \leq a < \pi$. It is known[5] that there exists a function mapping D onto the whole ζ-plane cut along the segments making the angle a or $a+\pi/2$ with the positive-real axis, i.e. the arcs belonging to either of the families of curves defined by

$$\mathcal{J}(e^{-ia}\zeta) = c \quad \text{and} \quad \mathcal{R}(e^{-ia}\zeta) = c,$$

in such a way that the point z_∞ corresponds to the point at infinity and the boundary components C_j with $1 \leq j \leq p$ correspond to the segments of the former family while the remaining components C_j with $p < j \leq n$ correspond to those of the latter, and further that the begining coefficient in the Laurent expansion of the mapping function around z_∞ is equal to uinty and the coefficient of the constant term vanishes. Let the mapping function be denoted by

$$\zeta = \Phi_a^p(z; z_\infty);$$

it is uniquely determined under the imposed conditions.

Based on the imposed boundary behavior, the expression

$$e^{-2ia}(d\Phi_a^p(z; z_\infty))^2$$

is a desired one which contains, for each value of p with $1 \leq p < n$, the parameters z_∞ and a. It will readily be shown that an analytic function defined by

$$w(z) = e^{-2i(\hat{a}-a)}\left(\frac{d\Phi_{\hat{a}}^{\hat{p}}(z; \hat{z}_\infty)}{d\Phi_a^p(z; z_\infty)}\right)^2$$

$$\equiv e^{-2i(\hat{a}-a)} \frac{\Phi_{\hat{a}}^{\hat{p}\prime}(z; \hat{z}_\infty)^2}{\Phi_a^{p\prime}(z; z_\infty)^2}$$

maps D onto a Riemann half-surface provided $(\hat{z}_\infty; \hat{a}; \hat{p}) \neq (z_\infty; a; p)$.

By the way, it is noted that the function $\Phi_a(z; z_\infty)$ or $\Phi_{a+\pi/2}(z; z_\infty)$, the parameter value $a+\pi/2$ in the latter being to be considered with respect to modulo π, may be regarded as a particular one of $\Phi_a^p(z; z_\infty)$ with $p = n$ or $p = 0$, respectively. The values of the indices p and $\hat{p}$ in the defining expression are accordingly permitted to be equal to 0 or n. It may also be noted that an analytic function defined by

$$w(z) = e^{-i(2a-a^*-a^{**})} \frac{(d\Phi_a^p(z; z_\infty))^2}{d\Phi_{a^*}^0(z; z_\infty^*)\, d\Phi_{a^{**}}^n(z; z_\infty^{**})}$$

maps D onto a Riemann half-surface regardless of the values of its parameters.

On the other hand, the previous results can further be extended. We have hitherto supposed that, by a mapping of D onto a canonical domain on the ζ-plane, an image of each boundary component corresponds to an arc belonging to one and the same family. However, this restriction is removable, as already noticed in the general discussion. In fact,[6] it has only to be imposed that a mapping in consideration, $\zeta = \zeta_a(z)$, transfers each boundary component into a set consisting of the pieces lying on either of the families $\mathcal{J}(e^{-ia}\zeta) = c$ and $\mathcal{R}(e^{-ia}\zeta) = c$. Therefore, the complementary component of $\zeta_a(D)$ with respect to $\zeta_a(C_j)$ may contain a number of polygons of which the sides belong to the families. In such a circumstance, the expression $e^{-2ia}(d\zeta_a(z))^2$ is also of the desired nature.

A generalization similar as above from $\Phi_a(z; z_\infty)$ to $\Phi_a^p(z; z_\infty)$ can be applied to $\Psi_\beta(z; z_0, z_\infty)$ yielding $\Psi_\beta^p(z; z_0, z_\infty)$. Namely, let z_0 and z_∞ denote again any distinct points in D and β be any real number. There exists then[7] a function mapping D onto the whole ζ-plane cut along the arcs belonging to either of the families of logarithmic spirals centred at the origin with the inclination β and $-\beta^{-1}$, i.e. the families defined by

$$\arg\zeta - \beta\, \lg|\zeta| = c \quad \text{and} \quad \arg\zeta + \beta^{-1}\lg|\zeta| = c$$

or by

$$\mathcal{J}(e^{-i\ell/2}\lg\zeta) = c\cos\frac{\ell}{2} \quad \text{and} \quad \mathcal{R}(e^{-i\ell/2}\lg\zeta) = c\sin\frac{\ell}{2},$$

ℓ being defined, as before, by $\beta = \tan(\ell/2)$ or $e^{i\ell} = (1+i\beta)^2/(1+\beta^2)$, in such a way that the points z_0 and z_∞ correspond to the origin and the point at infinity, respectively, and the boundary components C_j with $1 \leq j \leq p$ correspond to the arcs belonging to the former family while the remaining components C_j with $p < j \leq n$ correspond to those of the latter, and further that the beginning coefficient in the Laurent expansion of the mapping around z_∞ is equal to unity. The mapping function denoted by

$$\zeta = \Psi_\beta^p(z; z_0, z_\infty)$$

is uniquely determined under these imposed conditions.

Quite similarly as above, the differential expression

$$e^{-ib}(d\lg \Psi_\beta^p(z; z_0, z_\infty))^2$$

is surely of the desired character containning, for each value of p with $1 \leqq p < n$, the parameters z_0, z_∞ and β.

Several functions mapping D onto Riemann half-surfaces can then be defined by means of the method employed above; for instance,

$$w(z) = e^{-i(\hat{b}-b)}\left(\frac{d\lg \Psi_{\hat\beta}^{\hat p}(z, \hat z_0, \hat z_\infty)}{d\lg \Psi_\beta^p(z; z_0, z_\infty)}\right)^2$$

with $(\hat z_0, \hat z_\infty; \hat\beta; \hat p) \neq (z_0, z_\infty; \beta; p)$,

$$w(z) = e^{-i(b-2a)}\left(\frac{d\lg \Psi_\beta^p(z; z_0, z_\infty)}{d\Phi_a^{\hat p}(z; \hat z_\infty)}\right)^2,$$

or

$$w(z) = e^{-i(b-a-\hat b/2)}\frac{(d\lg \Psi_\beta^p(z; z_0, z_\infty))^2}{d\Phi_a(z; z_\infty^*)d\lg \Psi_{\hat\beta}^{\hat p}(z; \hat z_0, \hat z_\infty)},$$

etc.

Remark and extension analogous as stated above concerning can also be applied to the present case[8])

9. We shall now turn to another class of examples. Let $P(\zeta)$ be a polynomial in ζ of degree $n-1$. If it has no multiple zero, the equation

$$|P(\zeta)| = c$$

defines a family of Cassinian, and each member of the family consists of at most $n-1$ simple closed curves. Let c_j $(j=1,\cdots,n)$ be real numbers such that $c_1 = 0$ and

$$\int_{C_j} d\arg \chi(z) = -2\pi \qquad \text{for } j=2,\cdots,n,$$

where $\chi(z)$ is defined by

$$\chi(z) = \exp \sum_{j=2}^{n} c_j(\omega_j(z) + i\tilde\omega_j(z)),$$

$\omega_j(z)$ denoting the harmonic measure of C_j with respect to D and $\tilde\omega_j(z)$ a harmonic function conjugate to $\omega_j(z)$. The quantities c_j $(j=2,\cdots,n)$ are determined uniquely under these conditions. Then, C. de la Vallée Poussin[9]) has proved a mapping theorem stating that, if $\chi'(z) \neq 0$ along C, there exists a polynomial $P(\zeta)$ of degree $n-1$ such that D can be mapped conformally and univalently onto a domain of which the boundary components corresponding to C_j $(j=1,\cdots,n)$ are the regular closed curves lying on

$$|P(\zeta)| = e^{c_j},$$

respectively; the mapping function is determined uniquely except an arbitrary motion.

We now condider a function $\zeta = G(z)$ effecting such a mapping. Then, since the boundary components $G(C_j)$ $(j=1,\cdots,n)$ of the image-domain $G(D)$ lie on the curves defined by

$$\Im(i\lg P(\zeta)) = c_j,$$

respectively, the meromorphic differential

$$i\, d\lg P(G(z))$$

remains real along the whole boundary C and hence is of the desired nature.

The above-mentioned mapping theorem is further modified by G. Julia[10]) in such a manner that the polynomical $P(\zeta)$ may be replaced by a rational function $Q(\zeta)$ also of degree $n-1$ and with a fundamental circle; the uniqueness assertion is then modified in a manner that the mapping function is determined uniquely except an arbitrary linear transformation.

It is a matter of course, that such a mapping function $\zeta = H(z)$ is also available for constructing a differential of the desired nature, i.e.

$$i\, d\lg Q(H(z)).$$

10. We have enumerated several examples of analytic functions mapping a given basic domain D onto Riemann surfaces. As a consequence of the general statement, there holds always an identity of the form

$$w^*(z) = R(w(z), A(w(z)))$$

between every two functions $w(z)$ and $w^*(z)$ of this category, where $A(w)$ denotes an algebraic function of w defined by an irreducible equation generating the closed Riemann surface which is obtained by duplicating the Riemann half-surface $w(D)$ and $R(w,u)$ is a rational function of its both arguments. By definition, $A(w)$ depends only on $w(D)$ but not on $w^*(D)$, while $R(w, w)$ depends on $w(D)$ as well as $w^*(D)$.

If, in particular, D is simply-connected, i.e. $n=1$, then it can be mapped onto the upper or lower half-plane. An ordinary half-plane being, of course, a Riemann half-surface, we can then put $A(w)\equiv w$ and hence $R(w, A(w))$ reduces to a rational function of w alone. Thus, any function mapping a given simply-connected domain onto a Riemann H half-surface is especially a rational function of a function mapping it onto a half-plane.

In case the connectivity of D does not exceed three, i.e. $n \leqq 3$, it can be mapped univalently onto a slit domain bounded merely by the segments lying on the real axis. Such an image-domain being a Riemann half-surface, we may regard a mapping function as a member $w(z)$ of the class in consideration. The closed Riemann domain obtained from $w(D)$ by duplication is then two-sheeted. And the algebraic function $A(w)$ is given by a square root of a polynomial in w of degree two, four, and six, respectively, for $n=1$, 2 and 3. If a branch-point is transferred into the point at infinity by a linear transformation with real coefficients, the degree of the polynomial in question is decreased by one in each case.

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3) M. Schiffer. The span of multiply connected domain. Duke Math. Journ. 10 (1943), 206-216. Cf. also H. Grunsky, Neue Abschätzungen zur konformen Abbildung ein- und mehrfach zusammenhängender Bereiche. Schriften math. Sem. Inst. angew. Math. Univ. Berlin 1 (1923/3), 95-140.

4) H. Grunsky, loc. cit.[3])

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6) Cf. Y. Komatsu, Einige kanonische konforme Abbildungen vielfach zusammenhängender Gebiete. Proc. Japan Acad. 29 (1953), 1-5.

7) Loc. cit.[5])

8) Cf. loc. cit.[6])

9) C. de la Vallée Poussin, Sur la représentation des aires multiplement connexes. Ann. Sci. Ec. Norm. Sup. 47 (1930), 267-309. Cf. also G. Julia, Leçons sur la représentation conforme des aires multiplement connexes. Paris (1934).

10) G. Julia, Sur la représentation des aires multiplement connexes. Ann. Scuola Norm. Sup. Pisa (2) 1 (1932), 113-138. Cf. also G.Julia loc. cit. [10]).

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