OPERATOR ALGEBRA OF FINITE CLASS II

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In the present note, first we shall supply the proof of the last part of the previous note [4] (the proofs of Theorem 2 and its Corollary resp.), here we shall describe more general form than Theorem 2 of [4], and next we shall prove some relations between semi-traces and traces in D^*-algebra. The definitons and the notations in the note [4] will be used in this note.

1. Let $\mathfrak{A}$ be a D^*-algebra and τ be a finite semi-trace of $\mathfrak{A}$. And let $\{x^a, x^b, j, \mathfrak{H}\}$ be the representation of $\mathfrak{A}$ generated by τ, and let R^a be C^*-algebra generated by $\{x^a ; x \in \mathfrak{A}\}$. All other algebras $\mathfrak{L}$, $\mathfrak{L}^a$, $\mathcal{R}^a$, and W^a are defined by the same way in §1 of [4]. Let Ω be the character space of $\mathcal{R}^a$ and $N = \{\omega \in \Omega ; \omega(x^a) = 0 \text{ for all } x \in \mathfrak{A}\}$ then Ω_0 ($= \Omega - N$) can be embeded into the trace space of R^a with weak* topology on R^a by the canonical mapping φ $(: \varphi(\omega)(A) = \omega(A)$ for all $A \in R^a)$. Putting $\Omega' =$ weak* closure of $\varphi(\Omega_0)$, Ω' is locally compact with respect to the weak* topology on R^a. If K' is a compact set in Ω', then it is covered by finite number of nbds (with compact closures in Ω'): $U(\varphi(\omega_i), A_j, \varepsilon) = \{\omega' \in \Omega' ; |\omega'(A_j) - \varphi(\omega_i)(A_j)| < \varepsilon\}$. $(\omega_i \in \Omega_0, A_j \in R^a, \varepsilon > 0 ;$ $i = 1,2,\dots,n; j = 1,2,\dots,m)$. Hence $K = \varphi^{-1}(K' \cap \varphi(\Omega_0)) \subset \bigcup_{ij} U(\omega_i, A_j, \varepsilon)$ where $U(\omega_i, A_j, \varepsilon) = \{\omega \in \Omega_0 ; |\omega(A_j) - \omega_i(A_j)| < \varepsilon\}$ and each nbd has compact closure in Ω. Since $\bar{K}$ is closed in Ω, it is also compact. Let L' be a set of all continuous functions on Ω' with compact supports. Then for any $f \in L'$ $f'(\omega)$ $(= f(\varphi(\omega))$ for $\omega \in \Omega_0$ and $= 0$ for $\omega \in N)$ is continuous on Ω and vanishes outside of a compact set in Ω. Putting $F(f) = \int_\Omega f'(\omega)\, d\mu(\omega)$, $F(.)$ is positive linear on L', and hence there exists a positive Radon measure ν on Ω' such that $F(f) = \int_{\Omega'} f(\omega')\, d\nu(\omega')$ for all $f \in L'$. For any $x \in \mathfrak{A}$ $x^a \in \mathcal{R}^a$ and $x^a(\omega')$ $(= \omega'(x^a)) \in L'$ (cf. (5^o) of proof of Th. 1 of [4]). If we denote any element in Ω' merely by ω, we have Th. 2 of the previous note [4] as a special case, that is, $\mathfrak{A} = L$ (group algebra of G consisting of all continuous functions on G with compact support), $\tau =$ positive Radon measure of finite class on G and Ω' $=$ G^*.

While if $\mathfrak{A}$ has a centering $\natural$ such that $\tau(x^\natural) = \tau(x)$ for all $x \in \mathfrak{A}$ and all traces τ of $\mathfrak{A}$, then all $\omega \in \Omega'$ are characters of $\mathfrak{A}$.

Next, in the proof of Cor. of Th. 2, the domain Ω of the measure ν is misprint of G^*. In that proof we have shown that for $\omega \in G^*$ $\omega(st) = \omega(s)\omega(t)$ for $t \in G$ and $s \in$ center of G. Now we give more direct and precise proof of the fact. For $\omega \in G^*$, there correspond two representations $\{x^a, x^b, j, \mathfrak{H}\}$ of L and $\{s^a, s^b, j, \mathfrak{H}\}$ of G with same $\mathfrak{H}$ such that $\omega(x) = (x^a\xi, \xi)$ and $\omega(s) = (s^a\xi, \xi)$ for all $x \in L$ and $s \in G$, where ξ is the normalizing vector. Since $s \to s^a$ was defined by $s^a x^\theta = (x_s)^\theta$, $(x_s)^a y^\theta = y^b (x_s)^\theta = y^b s^a x^\theta = s^a y^b x^\theta = s^a x^a y^\theta$ for all $x, y \in L$. Hence $(x_s)^a = s^a x^a$. If s is in center of G, $(s^a e_\alpha^a y^a \xi, \xi) \xrightarrow[\alpha]{} (s^a y^a \xi, \xi)$ and the left side $=$ $((e_\alpha)_s^a y^a \xi, \xi) = \omega((e_\alpha)_s y) = \omega((e_\alpha)_s)\omega(y)$ (since $(e_\alpha)_s \in$ center of L) $=$ $((e_\alpha)_s^a \xi, \xi)(y^a\xi, \xi) \to (s^a\xi, \xi)(y^a\xi, \xi)$. Hence $(s^a\xi, \xi)(y^a\xi, \xi) = (s^a y^a \xi, \xi)$ for all $y \in L$. Put $y = (e_\alpha)_t$ and take the limit with respect to the directed set $\{\alpha\}$ of the both sides. Then $(s^a\xi, \xi)(t^a\xi, \xi) = (s^a t^a \xi, \xi)$ or $\omega(s)\omega(t) = \omega(st)$.

We have called that v in $\mathfrak{H}$ is bounded if and only if $|x^b v| \leq M |x^\theta|$ for all $x \in \mathfrak{A}$ and a const. $M > 0$ (cf. §1 of [4] in which x^a must be replaced by x^b at P.123, right side, lines 24 and 28). Now we describe supplementary remarks with respect to the bounded elements in $\mathfrak{H}$. Let $\mathfrak{L}$ be the set of all bounded elements in $\mathfrak{H}$. If for $v \in \mathfrak{L}$ we put $v^* = jv$, then $v^* \in \mathfrak{L}$ and $v^{*a} = v^{a*}$. For, $(x^b jv, y^\theta) = (jv, (yx^*)^\theta) = ((xy^*)^\theta, v) =$ $= (x^\theta, y^b v) = (x^\theta, v^a y^\theta) = (v^{a*} x^\theta, y^\theta)$ i.e. $x^b jv = v^{a*} x^\theta$ for all $x \in \mathfrak{A}$ and we have $v^* \in \mathfrak{L}$ and $v^{*a} = v^{a*}$. For any $v \in \mathfrak{L}$, $(x^a v, y^\theta) = (v, y^b x^{*\theta}) = (y^{*b} v, x^{*\theta})$ $= (v^a y^{*\theta}, x^{*\theta}) = (jy^\theta, v^{*a} jx^\theta) = (jv^{*a} jx^\theta, y^\theta)$ Hence putting $v^b = jv^{*a}j$, v^b is a bounded operator on $\mathfrak{H}$ and $x^a v = v^b x^\theta$ for all $x \in \mathfrak{A}$.

2. Let $\mathfrak{A}$ be a D^*-algebra with the approximate identity $\{e_\alpha\}$, let τ be a semi-trace of $\mathfrak{A}$ and let $\{x^a, x^b, j, \mathfrak{H}\}$ be the corresponding representation of $\mathfrak{A}$. Moreover let

$\mathfrak{A}^a$ and W^a be uniform and weak closure of $\mathcal{L}^a (= \{v^a; v \in \mathcal{L}\})$ respectively.

PROPOSITION 1. The following conditions are equivalent each other: (1°) τ is trace. (2°) There exists a constant $M > 0$ such that $\tau(e_\alpha^* e_\alpha) \le M$ for all α. (3°) $I \in \mathcal{L}^a$. (4°) $\mathcal{L}^a = W^a$.

Proof. (1°) $\to$ (2°) is clear. First we prove (2°) $\to$ (3°). Since $\tau(((e_\alpha x - x)y)^*(e_\alpha x - x)y) \le \| e_\alpha x - x\|^2 \tau(y^* y) \underset{\alpha}{\to} 0$, $| e_\alpha^a (xy)^\theta - (xy)^\theta |^2 \underset{\alpha}{\to} 0$. By Lemma 1.1 of [3], there exists a directed set $x_\beta \in \mathfrak{A}$ such that $x_\beta^a \underset{\beta}{\to} I$ (strongly). Hence $(x_\beta y)^\theta = x_\beta^a y^\theta \underset{\beta}{\to} y^\theta$ in $\mathfrak{H}$ and $\{(xy)^\theta; x, y \in \mathfrak{A}\}$ is dense in $\mathfrak{H}$. For any $\zeta \in \mathfrak{H}$ and $\varepsilon > 0$ we fined $x, y \in \mathfrak{A}$ and α_0 such that $|\zeta - (xy)^\theta| < \varepsilon/3$ and $|e^a_\alpha (xy)^\theta - (xy)^\theta| < \varepsilon/3$ for $\alpha > \alpha_0$. Therefore $|e_\alpha^a \zeta - \zeta| \le |e^a_\alpha \zeta - e^a_\alpha (xy)^\theta| + |e^a_\alpha (xy)^\theta - (xy)^\theta| + |(xy)^\theta - \zeta| < 2\varepsilon/3 + \varepsilon/3 = \varepsilon$ for $\alpha \ge \alpha_0$ and hence $e_\alpha^a \to I$ strongly. Putting $f_\alpha(\zeta) = (\zeta, e_\alpha^\theta)$ for $\zeta \in \mathfrak{H}$, $|f_\alpha(\zeta)| \le |\zeta| \cdot |e_\alpha^\theta| \le |\zeta| \tau(e_\alpha^* e_\alpha)^{1/2} \le M^{1/2} |\zeta|$. Hence $\|f_\alpha\| \le M^{1/2}$ for all α. Moreover $f_\alpha((y x^*)^\theta) = (x^{*b} y^\theta, e_\alpha^\theta) = (y^\theta, x^b e_\alpha^\theta) = (y^\theta, e^a_\alpha x^\theta) \longrightarrow (y^\theta, x^\theta)$. Since $\{(y x^*)^\theta; x, y \in \mathfrak{A}\}$ is dense in $\mathfrak{H}$, $\{f_\alpha\}$ weakly converges to a bounded linear functional f on $\mathfrak{H}$ and there exists $\xi \in \mathfrak{H}$ such that $f(\zeta) = (\zeta, \xi)$. Hence $(\zeta, e_\alpha^\theta) \to (\zeta, \xi)$ for all $\zeta \in \mathfrak{H}$ or $e_\alpha^\theta \to \xi$ (weakly). In the equality $(\zeta, e_\alpha^a x^\theta) = (\zeta, x^b e_\alpha^\theta)$, the left side $\underset{\alpha}{\to} (\zeta, x^\theta)$ and the right side $\underset{\alpha}{\to} (\zeta, x^b \xi)$. Hence $x^\theta = x^b \xi$ for all $x \in \mathfrak{A}$. Putting $A x^\theta = x^b \xi$, $A x^\theta = x^\theta$ for all $x \in \mathfrak{A}$ and $A = I$ or $I^\theta = \xi$ [2] and $\xi^a = A$. Thus $I \in \mathcal{L}^a$. Since $\mathcal{L}^a$ is an ideal in W^a, (3°) $\to$ (4°) is clear. Finally we prove that (4°) $\to$ (1°). (4°) $\to I \in \mathcal{L}^a \to I^\theta = \xi \in \mathcal{L} \to x^b \xi = x^b I^\theta = I x^\theta = x^\theta$. While, $x^a \xi = j x^{*b} j \xi = j x^{*b} \xi = j x^{*\theta} = j j x^\theta = x^\theta = x^b \xi$ and $\tau(x^* y) = (y^\theta, x^\theta) = (y^a \xi, x^a \xi) = ((x^* y)^a \xi, \xi)$. That is, τ is a trace of $\mathfrak{A}$ (cf. Th. 1 of [3]).

REMARK 1. In the previous proof we have that $e_\alpha^a \to I$ strongly. Hence any semi-trace τ of a D*-algebra $\mathfrak{A}$ satisfies

(†) $\tau((e_\alpha x - x)^*(e_\alpha x - x)) \underset{\alpha}{\to} 0$ for all $x \in \mathfrak{A}$

which is stronger than the condition:

(‡) there exists a subsequence $\{e_{\alpha_n}\} \subset \{e_\alpha\}$ depending on each $x \in \mathfrak{A}$ such that $\tau((e_{\alpha_n} x)^* e_{\alpha_n} x) \longrightarrow \tau(x^* x)$ $(n \to \infty)$

which is a condition of semi-trace (cf. §1 of [4]). But the condition (‡) must be assumed in the definition of semi-trace. For, (‡) is necessary in the proof of the fact that the two-sided representation corresponding to a semi-trace is proper (cf. Th.2 of [3]) and the properness is used in the proof of $e_\alpha^a \longrightarrow I$.

REMARK 2. Prop. 1 implies that if G is a unimodular locally compact group, then $B(G) = W(G)$ if and only if G is discrete.

Now we show a theorem of Godement (Th. 7 of [1]) in the following case.

PROPOSITION 2. τ is a finite pure semi-trace of $\mathfrak{A}$ if and only if it is pure trace.

Proof. We prove the part of "only if", since the converse has been proved in the previous paper (Th. 1 of [3]). Let $\{x^a, x^b, j, \mathfrak{H}\}$ be the corresponding representation of $\mathfrak{A}$. Since it is irreducible[3], $W^a \cap W^b = \{\lambda I\}$. Furthermore, since W^a is of finite class, $A^\natural$ is scalar operator for any $A \in W^a$. Let $\mathcal{Z}$ be central manifold of $\mathfrak{H}$ (i.e. $\zeta \in \mathcal{Z}$ if and only if $x^a \zeta = x^b \zeta$ for all $x \in \mathfrak{A}$). By Lemma 3 and the proof of Prop. 1 of [3], we can find a vector $0 \ne v \in \mathcal{L} \cap \mathcal{Z}$. Clearly $v^a \in W^a \cap W^b$ and v^a is scalar $= \lambda I$ $(\lambda \ne 0)$. Therefore $v = \lambda I^\theta$. Since $e_\alpha^a \underset{\alpha}{\to} I$ strongly, so is $e_\alpha^b \to I$ and $e_\alpha^b I^\theta = e_\alpha^\theta \to I^\theta$ strongly. Thus we have that τ is a trace. From the irreducibility of $\{x^a, x^b, j, \mathfrak{H}\}$ it follows that τ is a pure trace.

REMARK 3. Concerning the concept of pure trace of D*-algebra we show the following (a general form of a theorem of Godement - Th. 9 of [1]). Let τ be a trace of a D*-algebra $\mathfrak{A}$ with $\|\tau\| = 1$. If we put $\tau_y(x) = \tau(xy)$ for all x and $y \in \mathfrak{A}$ and let $\{x^a, x^b, j, \mathfrak{H}\}$ be the corresponding two-sided representation of $\mathfrak{A}$. Then $\mathcal{L}^a = W^a$ (by Prop. 1) and $\tau(x) = (x^a \xi, \xi)$ where ξ is the normalizing vector of $\mathfrak{H}$ (i.e. $\xi^a = I$). Put $(\tau_y)^\#(x) = (x^a \xi, y^{*a\natural} \xi)$. Then we have that τ is pure trace (= character) of $\mathfrak{A}$ if and only if $(\tau_y)^\#(x) = \tau(x)\tau(y)$ for all $x, y \in \mathfrak{A}$. For, if τ is pure, $T(A) = (A\xi, \xi)$ is pure trace on W^a and hence $T(AB) = T(A)T(B)$ for all $A, B \in W^a \cap W^b$ and $A^\natural = T(A)I$. Therefore $y^{a\natural} = T(y^a)I = (y^a \xi, \xi) I$. While $(\tau_y)^\#(x) = (x^a \xi, y^{*a\natural} \xi) = (y^{a\natural} x^a \xi, \xi) = (x^a \xi, \xi)(y^a \xi, \xi) = \tau(x)\tau(y)$. Conversely, if $(\tau_y)^\#(x) = \tau(x)\tau(y)$ for all $x, y \in \mathfrak{A}$, then $(\tau_y)^\#(x^* z) = \tau(x^* z)\tau(y)$, and its left side $= ((x^* z)^a \xi, y^{*a\natural} \xi) = (z^a \xi, y^{*a\natural} x^a \xi)$

and the right side $= (z^{a}\xi, \tau(y^*)x^{a}\xi)$. Since the both sides are equal for all x, y and $z \in \mathfrak{A}$, $y^{a\natural} = \tau(y)I$. Let P be projection onto the central manifold $\mathfrak{Z}$. For any $v \in \mathfrak{Z}$, there exist $x_n \in \mathfrak{A}$ such that $|x_n^{\theta} - v| \to 0\ (n \to \infty)$. Hence $Px_n^{\theta} \to Pv = v$. Since for all $x \in \mathfrak{A}$ $Px^{\theta} \in \mathcal{L}$ and $(Px^{\theta})^{a} = x^{a\natural}$, cf. the proof of Prop. 1 of [3], $(Px^{\theta})^{a} = \tau(x)I$, $Px^{\theta} = \tau(x)\xi$ and the center of $\mathcal{L}$ is scalar, i.e. $= \{\lambda\xi\}$. Thus "the center of $\mathcal{L}^{a''} = W^{a} \wedge W^{b} (= W^{a\natural})$ is $\{\lambda I\}$, and τ is pure. The proposition obtained in this remark contains the first part of Prop. 2 of [4] as a special case.

FOOTNOTES

(1) In a separable D*-algebra, the decomposition of arbitrary semi-trace into a system of pure semi-traces in the form of direct integral over the real line has been shown in the previous note [3] using the reduction theory of von Neumann. Recently I.E.Segal has been published his decomposition theory "Decomposition of Operator Algebras. I and II, Mem. Amer. Math. Soc., 1951". If we apply his theory, Th. 1 of [4] may be shown in a most general form (in separable case). The precise discussion may be stated in the following in which we may prove that, in Th.1 of [4] all $\omega \in \Omega$ are characters of A which is not always separable.

(2) For any $A \in \mathcal{L}^{a}$, let the corresponding bounded element ($\in$ B) denote A^{θ} .

(3) It is known that for semi-trace or trace of a D*-algebra being pure, it is NASC that the corresponding two-sided representation is irreducible respectively (cf. [3]).

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