# NOTE ON FOURIER-STIELTJES INTEGRAL, II

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1. In the preceding note [5] we intended to prove the following classical theorem concerning Fourier-Stieltjes integral from the stand point of topological group theory.

Theorem I (Bochner-Phillips). Let $f(x)$ be a bounded measurable function defined on a locally compact abelian group $G$ satisfying the following condition:

$$(1)\quad \left|\sum_{\mu=1}^{n} c_\mu f(x_\mu)\right| \leqq M \sup_{\hat{x}\in\hat{G}} \left|\sum_{\mu=1}^{n} c_\mu (x_\mu, \hat{x})\right|$$

for every $x_\mu \in G$ and complex numbers $c_\mu$ $(\mu = 1, 2, \ldots n)$. Then $f(x)$ coincides almost everywhere with the Fourier transform of a bounded Radon measure $\hat{\mu}$ on the dual group $\hat{G}$, i.e.

$$(2)\qquad f(x) = \int_{\hat{G}} (x, \hat{x})\, d\hat{\mu}(\hat{x}) \qquad a.e.$$

If $f(x)$ is continuous, then the equality holds for all points.

However our argument, from the bottom of page 60 to the top of page 61 of the preceding note, requires that $f(x)$ be continuous, (This is pointed out in a letter by Prof. R.S.Phillips, to whom the author express his best thanks for kind advice) and so it is sufficient only for the Bochner Theorem. In this paper we correct the proof for a bounded measurable function and simultaneously give a generalization of H.Cramér's theorem.

2. We shall use the definitions and the notations of the previous note.

$V$ : a compact neighborhood of the unit $0$ of $G$.

$\hat{V}$ : a compact neighborhood of the unit $\hat{0}$ of $\hat{G}$.

$e_V$ : a positive continuous function whose support is in $V$ and

$$\int_G e_V \, dx_V^{\bullet} = 1.$$

$h_V$ : $h_V = e_V * e_V$ (* shows convolution).

$\phi_V(\hat{x})$ : Fourier-transform of $h_V$, i.e.

$$\phi_V(\hat{x}) = \int_G \overline{(x, \hat{x})}\, h_V(x)\, dx.$$

As $h_V(x)$ is positive definite and integrable, we get

$$h_V(x) = \int_{\hat{G}} (x, \hat{x})\, \phi_V(\hat{x})\, d\hat{x}.$$

For corresponding functions on $\hat{G}$, we take out $\hat{V}$ in place of $V$.

We put $f_{V\hat{V}}(x) = (\phi_{\hat{V}}(x) f(x)) * h_V(x)$, then $f_{V\hat{V}}(x)$ is the Fourier-transform of a continuous, integrable function $\psi_{V\hat{V}}(\hat{x})$ on $\hat{G}$ and, moreover, we get always

$$\int_{\hat{G}} |\psi_{V\hat{V}}(x)|\, d\hat{x} \leq M$$

(c.f. [5; Lemma 2]).

Lemma. For every $g(x) \in L_1(G)$, $\int_G g(x) f_{V\hat{V}}(x)\, dx$ converges to $\int_G g(x) f(x)\, dx$ as $V$ and $\hat{V}$ converges to $0$ and $\hat{0}$ respectively.

Proof. For every $g(x) \in L_1(G)$ and $\varepsilon_1 > 0$ there exists a compact neighborhood $U$ of the unit of $G$ such that

$$\int_G |g(yx) - g(x)|\, dx < \varepsilon_1$$

for every $y \in U$.

(Of course we can assume $U = U^{-1}$. c.f. [6; p.41]). Then we can select a sufficiently small $V$ such that $h_V(x) \equiv 0$ for $x \notin U$.

As $g(x) \in L_1(G)$, for any $\varepsilon_2 > 0$ there exists a compact set $K$ in $G$ such as

$$\int_{G-K} |g(x)|\, dx < \varepsilon_2,$$

while $\phi_{\hat{V}}(x)$ converges to 1 uniformly on every compact subset in G as $\hat{V}$ converges to the unit of $\hat{G}$[2; VII, Lemma $\alpha$ 1], so for every $\varepsilon_3 > 0$ there exists a $\hat{V}$ such that $|\phi_{\hat{V}}(x) - 1| < \varepsilon_3$

on the compact set $\bar{U}K$. Put $\phi_{\hat{V}}(x) = 1 + \eta(x)$, then $|\eta(x)| \leqq 2$ since

$$|\phi_{\hat{V}}(x)| \leqq \int_{\hat{G}} h_v(\hat{x})\, d\hat{x} = 1 .$$

Then

$$\left| \int_G g(x) f_{V\hat{V}}(x)\,dx - \int_G g(x) f(x)\,dx \right|$$

$$= \left| \int_G \int_G g(x) h_V(y) \phi_{\hat{V}}(y^{-1}x) f(y^{-1}x)\,dy\,dx - \int_G\int_G g(x) f(x) h_V(y)\,dy\,dx \right|$$

$$= \left| \int_G \int_U g(x) h_V(y) \phi_{\hat{V}}(y^{-1}x) f(y^{-1}x)\,dy\,dx - \int_G\int_U g(x) f(x) h_V(y)\,dy\,dx \right|$$

$$(\textit{because } h_V(y) = 0 \textit{ for } y \bar{\in} U)$$

$$\leq \left| \int_G\int_U g(x) h_V(y) f(y^{-1}x)\,dy\,dx - \int_G\int_U g(x) f(x) h_V(y)\,dy\,dx \right|$$

$$+ \left| \int_G\int_U g(x) h_V(y)\, \eta(y^{-1}x) f(y^{-1}x)\,dy\,dx \right| .$$

Let us suppose

$$A = \left| \int_G\int_U g(x) h_V(y) f(y^{-1}x)\,dy\,dx - \int_G\int_U g(x) f(x) h_V(y)\,dy\,dx \right|$$

and

$$B = \left| \int_G\int_U g(x) h_V(y)\, \eta(y^{-1}x) f(y^{-1}x)\,dy\,dx \right| .$$

Then

$$A = \left| \int_U\int_G g(yx) h_V(y) f(x)\,dx\,dy - \int_U\int_G g(x) f(x) h_V(y)\,dx\,dy \right|$$

$$\leq M \int_U\int_G |g(yx) - g(x)| h_V(y)\,dx\,dy \leq \varepsilon_1 M \int_U h_V(y)\,dy$$

$$= \varepsilon_1 M ,$$

$$B = \left| \int_K\int_U g(x) h_V(y) \eta(y^{-1}x) f(y^{-1}x)\,dy\,dx \right.$$

$$+ \left. \int_{G-K}\int_U g(x) h_V(y)\, \eta(y^{-1}x) f(y^{-1}x)\,dy\,dx \right|$$

$$\leq \varepsilon_3 M \int_K\int_U |g(x)| h_V(y)\,dy\,dx + 2M \int_{G-K}\int_U |g(x)| h_V(y)\,dy\,dx$$

$$\leq 2\varepsilon_2 M + \varepsilon_3 M \cdot \|g\|_1 .$$

Therefore,

$$\left| \int_G g(x) f_{V\hat{V}}(x)\,dx - \int_G g(x) f(x)\,dx \right| \leq (\varepsilon_1 + 2\varepsilon_2 + \varepsilon_3 \|g\|_1) M .$$

Since $\varepsilon_1$, $\varepsilon_2$ and $\varepsilon_3$ are arbitrary constants, we have

$$\left| \int_G g(x) f_{V\hat{V}}(x)\,dx - \int_G g(x) f(x)\,dx \right| \longrightarrow 0$$

as V and $\hat{V}$ converges to the unit of G and $\hat{G}$ respectively. q.e.d.

Proof of Theorem. As $f_{V\hat{V}}$ is the Fourier-transform of a bounded measure $\psi_{V\hat{V}}(\hat{x})\,d\hat{x}$ and $\int |\psi_{V\hat{V}}(\hat{x})|\, d\hat{x} \leqq M$ for every V and $\hat{V}$, by Schoenberg's theorem [5; Theorem 1]

$$\left| \int_G g(x) f_{V\hat{V}}(x)\,dx \right| \leqq M \max_{\hat{x} \in \hat{G}} \left| \int_G (x, \hat{x}) g(x)\,dx \right|$$

for every $g(x) \in L_1(G)$. Therefore, by the above lemma

$$\left| \int_G g(x) f(x)\,dx \right| \leqq M \max_{\hat{x} \in \hat{G}} \left| \int_G (x, \hat{x}) g(x)\,dx \right| ,$$

whence by Schoenberg's theorem

$$f(x) = \int_{\hat{G}} (x, \hat{x})\, d\hat{\mu} \quad , \quad a.e.$$

q.e.d.

3. Next we give a generalization of H. Cramér's theorem [3][4].

Theorem 2. Let $\phi_w(x)$ ($w \in W$) be a directed set of continuous, positive definite, integrable function which converges to 1 uniformly on every compact set in G and $\phi_w(0) = 1$ for all $w \in W$. Then a measurable function f(x) is representable as (2) if and only if $\psi_w(\hat{x}) = \int_G \overline{(x, \hat{x})}\, \phi_w(x) f(x)\,dx$

can be defined and

$$\int_{\hat{G}} |\psi_w(\hat{x})| \, d\hat{x} \leqq M$$

for every w, where M is a constant.

Proof. Necessity. As $\phi_w(x)$ is a continuous, positive definite and integrable function, its Fourier-transform $h_w(\hat{x})$ is a non-negative, continuous and integrable function and moreover we obtain

$$\phi_w(x) = \int_{\hat{G}} (x,\hat{x}) h_w(\hat{x}) d\hat{x}, \quad \phi(0) = \int_{\hat{G}} h_w(\hat{x}) d\hat{x} = 1.$$

If f(x) be representable as (2), we can assume without loss of generality that f(x) be continuous and satisfies (1). Then $\phi_w(x)f(x)$ satisfies (1) too, because for every $x_\mu \in G$ and complex numbers $c_\mu$ ($\mu = 1,2,\dots,n$)

$$\left| \sum_{\mu=1}^{n} c_\mu \phi_w(x_\mu) f(x_\mu) \right| = \left| \sum_{\mu=1}^{n} c_\mu \int_{\hat{G}} (x_\mu, \hat{y}) h_w(\hat{y}) d\hat{y} f(x_\mu) \right|$$

$$\leqq M \sup_{\hat{x} \in \hat{G}} \left| \sum_{\mu=1}^{n} c_\mu \int_{\hat{G}} (x,\hat{y}) h_w(\hat{y}) d\hat{y} (x_\mu, \hat{x}) \right|$$

$$\leqq M \int_{\hat{G}} \sup_{\hat{x} \in \hat{G}} \left| \sum_{\mu=1}^{n} c_\mu (x_\mu, \hat{x}\hat{y}) \right| h_w(\hat{y}) dy$$

$$\leqq M \sup_{\hat{x} \in \hat{G}} \left| \sum_{\mu=1}^{n} c_\mu (x_\mu, \hat{x}) \right| \left| \int_{\hat{G}} h_w(\hat{y}) d\hat{y} \right|$$

$$= M \sup_{\hat{x} \in \hat{G}} \left| \sum_{\mu=1}^{n} c_\mu (x_\mu, \hat{x}) \right| .$$

Therefore, $\phi_w(x)f(x)$ is integrable and coincides everywhere with the Fourier-transform of a bounded Radon measure on $\hat{G}$, hence $\phi_w(x)f(x)$ is the Fourier-transform of a integrable, continuous function $\psi_w(\hat{x})$ and by the same argument as the proof of [5; Lemma 2], we can conclude

$$\int_{\hat{G}} |\psi_w(\hat{x})| d\hat{x} \leqq M.$$

Sufficiency. By assumption $\psi_w \in L_1(\hat{G})$ and $\|\psi_w\|_1 \leqq M$, whence $\phi_w(x)f(x)$ coincides almost everywhere with the Fourier transform of $\psi_w(\hat{x})$, therefore, $|\phi_w(x)f(x)| \leqq M$ a.e. While $\phi_w(x)$ converges to 1 uniformly on every compact set in G, hence $|f(x)| \leqq M$ a.e.

Put $f_{VW}(x) = (\phi_w \cdot f) * h_V$ where $h_V(x)$ is the function defined at the top of §2. Then $f_{VW}(x)$ is the Fourier transform of $\psi_{VW}(\hat{x})$ defined by $\psi_{VW}(\hat{x}) = \psi_w(\hat{x})\phi_V(\hat{x})$, while

$$\int_{\hat{G}} |\psi_{VW}(\hat{x})| d\hat{x} \leqq \int_{\hat{G}} |\psi_w(\hat{x})| d\hat{x} \leqq M$$

since $|\phi_V(\hat{x})| \leqq 1$.

Hence by the same argument as §2, we get

$$\int_G g(x) f_{VW}(x) dx \to \int_G g(x) f(x) dx$$

and

$$\left| \int_G g(x) f(x) dx \right| \leqq M \max_{\hat{x} \in \hat{G}} \left| \int_G (x,\hat{x}) g(x) dx \right|$$

for every $g(x) \in L_1(G)$. So the theorem is clear. q.e.d.

Corollary (H.Cramér). Let $\phi(x)$ be a continuous, positive definite, integrable function defined on the real line and $\psi(t)$ be a non-negative function which satisfies the following two conditions:

$$(3) \qquad \phi(x) = \int_{-\infty}^{\infty} e^{itx} \psi(t) dt$$

$$(4) \qquad \phi(0) = \int_{-\infty}^{\infty} \psi(t) dt = 1.$$

Then a measurable function f(x) is representable as

$$f(x) = \int_{-\infty}^{\infty} e^{itx} dF(t) \qquad a.e.$$

by a function F(t) of bounded variation if and only if

$$\psi_\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \phi(\delta x) f(x) dx$$

can be defined and

$$\int_{-\infty}^{\infty} |\psi_\delta(t)| dt \leqq M$$

for all $0 < \delta < 1$.

Proof. Clearly $\phi(\delta x)$ is a continuous, positive definite, integrable function. We show that $\phi(\delta x)$ converges to 1 uniformly on every closed interval $[-l, l]$ as $\delta \to 0$. As $\psi(t)$ is integrable, for any $\varepsilon > 0$ there exist a $m > 0$ such that

$$\int_{-\infty}^{-m} |\psi(t)| dt + \int_{m}^{\infty} |\psi(t)| dt \leqq \frac{\varepsilon}{4},$$

while

$$|\phi(\delta x) - 1| = \left| \int_{-\infty}^{\infty} e^{it\delta x} \psi(t) dt - \int_{-\infty}^{\infty} \psi(t) dt \right|$$

$$\leqq \int_{-\infty}^{\infty} |e^{it\delta x} - 1| \, \psi(t) dt.$$

If we define $\delta$ so small that $|e^{ilm\delta} - 1| < \frac{\varepsilon}{2}$, $|lm\delta| < \frac{\pi}{2}$, then

$$|\phi(\delta x) - 1| \leqq \int_{-\infty}^{-m} + \int_{-m}^{m} + \int_{m}^{\infty} |e^{it\delta x} - 1| \, \psi(t) dt$$

$$\leqq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \text{for } x \in [-l, l]$$

Hence $\phi(\delta x)$ converges to 1 uniformly on the interval $[-l, l]$ as $\delta \to 0$. Then the conclusion of the corollary follows from the above theorem.

4. In this last section we give a proof of a proposition which is used in the proof of [5; Lemma 2].

Proposition. Let f be a continuous function on a compact subset K in a locally compact abelian group G, then f can be extended to a continuous almost periodic function on G.

Proof. Let H be the universal Bohr compactification of G in the sence of [1] (c.f. also [6; pp.137-138]), then G is represented in a dense subgroup in H. We denote this representation by $\theta$. Then $\theta(K)$ is a closed compact set in H and f defines a continuous function g on $\theta(K)$ such that $g(\theta(x)) = f(x)$ for $x \in K$.

As $\bar{H}$ is a compact space, H is a normal space, so g can be extended continuously over H. Let h be such a extended function, then the function $k(x)$ defined by $k(x) = h(\theta(x))$ $(x \in G)$ gives a desired almost periodic function. q.e.d.


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(*) Received Feb. 4, 1953.