A GENERALIZATION OF ABSOLUTE NEIGHBORHOOD RETRACTS

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1. In this paper we give some generalization of absolute neighborhood retracts [1] [1]. This generalization is not useful on homotopy theory but admits some generalizations on fixed point properties of ANRsets (section 4). In sections 2 and 3 the familiar definitions and theorems of ARsets and ANRsets are described with the slight modifications.

2. In the following a space and a set are always separable metric.

(2.1) DEFINITION. Given the number ε', $\varepsilon' > 0$, and the sets A and B such that $B \subset A$ we say that a map $r_{\varepsilon'}$ is an ε'-retraction provided $r_{\varepsilon'}$ is defined and continuous on A, $r_{\varepsilon'}(B) \subset B$ and $r_{\varepsilon'}(A) \subset B$, and $\rho(b, r_{\varepsilon'}(b)) < \varepsilon'$ [2] for every $b \in B$. If such maps exist for every $\varepsilon' > 0$, then B is called an ε-retract of A.

(2.2) DEFINITION. Given the sets A and B such that $B \subset A$, we say that B is an ε-neighborhood retract of A provided there exists an open set U such that $B \subset U \subset A$ and such that B is an ε-retract of U.

(2.3) DEFINITION. A space, A, is called an ε-absolute neighborhood retract (ε-ANR or ε-ANRset) provided it is a compactum and for every topological image A_1 of A, such that A_1 is contained in a space M, we have A_1 is an ε-neighborhood retract of M.

(2.4) THEOREM. <u>A necessary and sufficient condition for a set to be an ε-ANR is that it be homeomorphic to a closed ε-neighborhood retract of the Hilbert parallelotope Q.</u>

PROOF. Necessity. Let A be an ε-ANR. Since A is a compactum, we can map A topologically into the Hilbert parallelotope Q [5]. Let $h(A) = A_1$, where h is a homeomorphism and A_1 is a subset of Q. Since Q is a compactum, by (2.3) A_1 is an ε-neighborhood retract of Q. In virtue of the continuity of h and the compactness of A, we have A_1 is compact and therefore closed in Q.

Sufficiency. Let $h(A) = A_1$, where h is a homeomorphism and A_1 is a closed ε-neighborhood retract of Q. Consider any other homeomorphic image A_2 of A such that A_2 is contained in a space M. Let $k(A) = A_2$, where k is a homeomorphism. Q is a compactum. Therefore A_2 is a compactum and hence closed in M. We now apply Tietze's extension theorem [5] to the map $hk^{-1} : A_2 \longrightarrow Q$ and obtain an extension f of hk^{-1} over M relative to Q. Since A_1 is an ε-neighborhood retract of Q, there exists an open set $U_1 \supset A_1$ and for each $\varepsilon' > 0$ an ε'-retraction $r_{\varepsilon'}$ such that $r_{\varepsilon'} : U_1 \longrightarrow A_1$. Now $f^{-1}[f(M) \cap U_1]$ is an open subset of M and clearly $f^{-1}[f(M) \cap U_1] \supset A_2$. The map $kh^{-1} r_{\varepsilon'} f$ maps the open set $f^{-1}[f(M) \cap U_1]$ into A_2. Since A_2 is compact, for sufficiently small ε' by uniform continuity of $kh^{-1} r_{\varepsilon'} f$ we have

$$\rho(a, kh^{-1} r_{\varepsilon'} f(a)) < \varepsilon \quad \text{for every } a \in A_2,$$

where ε is any giving positive number. Thus A_2 is an ε-neighborhood retract of M.

(2.5) DEFINITION. A space, A, is called an ε-absolute retract (ε-AR or ε-ARset) provided it is a compactum and for every topological image A_1 of A, such that A_1 is contained in a space M, we have A_1 is an ε-retract of M.

(2.6) THEOREM. <u>A necessary and sufficient condition for A to be an ε-AR is that it be homeomorphic to a closed ε-retract of the Hilbert parallelotope Q.</u>

This result may be verified by the method of (2.4).

(2.7) In (2.4) and (2.6) when the dimension of A is finite we can replace Q by a sufficiently high dimensional Euclidean space. Naturally every ANR(AR) set is an ε-ANR(ε-AR) set.

(2.8) EXAMPLE. In two-dimensional Euclidean space we consider next set A in a rectangle x-y coordinate.

$$\begin{cases} x = 1/2^n, \ 0 \leq y \leq 1, \text{ where } n = 0, 1, 2, \cdots \\ x = 0, \ 0 \leq y \leq 1 \\ 0 \leq x \leq 1, \ y = 0 \end{cases}$$

It is evident that A is an ε-AR and not an AR.

3. The following Lemmas and Theorems are verified by the usual methods [1, 6] with the slight modification, hence we omit their proofs here. But they are useful in the construction of the examples.

(3.1) LEMMA. A necessary and sufficient condition for A be an ε-ANR is that A be a compactum and that for every $\varepsilon > 0$ and for every map f defined on a closed subset P of a space P_1 such that $f(P) \subset A$, there exists a map f_ε defined on some open subset V, where V contains P, such that $\rho(f(x), f_\varepsilon(x)) < \varepsilon$ for every $x \in P$.

(3.2) THEOREM. If the sets $A_1, \ldots, A_n$ are ε-ANRsets, then the topological product $\prod A_i$ is an ε-ANR.

(3.3) LEMMA. A necessary and sufficient condition for A be an ε-AR is that A be a compactum and that for every $\varepsilon > 0$ and for every map f defined on a closed subset P of a space P_1 such that $f(P) \subset A$, there exists a map f_ε defined on P_1 such that $\rho(f(x), f_\varepsilon(x)) < \varepsilon$ for every $x \in P$.

(3.4) THEOREM. If $\{A_\alpha\}$ is a collection of sets where each A_α is an ε-AR, then the topological product $\prod A_\alpha$ is an ε-AR.

(3.5) Sum theorem could not hold in the standard form but from Borsuk's method [1] we have the following.

THEOREM. Let $A = A_1 \cup A_2$ where $A_1 \cap A_2$ is an ANR and A_1 and A_2 are ε-ANRsets of which for sufficiently small, ε', all ε'-retractions fix every point of $A_1 \cap A_2$. Then A is an ε-ANR.

(3.6) EXAMPLE. From (3.5) the following set A is an ε-ANR and not an ANR.

Let $A = A_1 \cup A_2$, where $A_1 = A$ in (2.9) and A_2 is a boundary of an unit square and $A_1 \cap A_2$ is, $x = 0$, $0 \leq y \leq 1$.

4. (4.1) THEOREM. (Borsuk [1]) If A is an ε-AR, then every map which maps A into A has a fixed point.

PROOF. By (2.6), we have $h(A) = A_1$, where h is a homeomorphism and A_1 is an ε-retract of the Hilbert parallelotope Q. Since Q has the fixed point property, for every map f: $A \to A$ and for every ε-retraction r_ε we have a fixed point $a'_\varepsilon \in A$ such that $a'_\varepsilon = fr_\varepsilon(a'_\varepsilon)$. Let $r_\varepsilon(a'_\varepsilon) = a_\varepsilon \in A$, we have

$$\rho(a_\varepsilon, f(a_\varepsilon)) = \rho(r_\varepsilon(a'_\varepsilon), a'_\varepsilon) < \varepsilon.$$

On the other hand if f has not any fixed point, there exists $\varepsilon_0 > 0$ such that

$$\rho(a, f(a)) \geq \varepsilon_0$$

for every $a \in A$.

This is a contradiction. Thus f has a fixed point.

(4.2) LEMMA (Eilenberg [4]). Let A be an ε-ANR. For every $\eta > 0$ there exists $\varepsilon > 0$ such that when a map f is an ε-map[3] then there exists a map φ of f(A) into A and

$$\rho(\varphi f(a), a) < \eta$$

for every $a \in A$.

PROOF. Let $\{f_n\}$ be a sequence of ε_n-maps of A, where $\lim_{n\to\infty}\varepsilon_n = 0$. Then we can embed spaces A, $f_1(A)$, $f_2(A)$, ..., in a compactum $C = A \cup \sum_{n=1}^{\infty} f_n(A)$ such that the sequence $\{f_n\}$ converges uniformly to $f_0(A) = A$ in C [3]. Since A is an ε-ANR there exists an open set U, where $A \subset U \subset C$ and ζ-retraction r_ζ : $U \to A$ for every $\zeta > 0$. For sufficiently large n we have $f_n(A) \subset U$ and the sequence $\{r_\zeta f_n\}$ converges uniformly to $r_\zeta f_0 = r_\zeta$. Hence for sufficiently large n we have

$$\rho(r_\zeta f_n(a), r_\zeta(a)) < \frac{\eta}{2}$$

for every $a \in A$.

If we put $\zeta < \frac{\eta}{2}$ then we have

$$\rho(r_\zeta f_n(a), a) \leq \rho(r_\zeta f_n(a), r_\zeta(a)) + \rho(r_\zeta(a), a) < \frac{\eta}{2} + \frac{\eta}{2} = \eta$$

for every $a \in A$.

(4.3) THEOREM (Borusk [2]). Let A be an ε-ANR and has a map f which maps A into itself without the fixed point. Then there exists

$\varepsilon > 0$ such that every image of ε-map of A has also a map which maps it into itself without the fixed point.

PROOF. Since A is compact, there exists $\eta > 0$ such that

(1) $\rho(\psi$

for every $a \in A$.

Since A is an ε-ANR, Lemma (4.2) holds good, that is, for $\eta > 0$ there exists $\varepsilon > 0$ such that a map φ is an ε-map then there exists a map ψ of $\varphi(A)$ into A and

(2) $\rho(\psi\varphi(a), a) < \eta$

for every $a \in A$.

Put $g = \varphi f \psi$ then g is a required map. If g has not a required property, there exists $b_0 \in \varphi(A)$ such that

$$\varphi f \psi(b_0) = b_0.$$

Hence we have

(3) $\psi\varphi f \psi(b_0) = \psi(b_0).$

Substitute $f\psi(b_0)$ for a in (2), we have

$$\rho(\psi\varphi f\psi(b_0), f\psi(b_0)) < \eta.$$

From (3) we have

$$\rho(\psi(b_0), f\psi(b_0)) < \eta.$$

This contradicts (1).

(4.4) THEOREM (Borusk [1]). If A is a finite dimensional ε-ANR, and f is a null-homotopic map of A into A, then f has a fixed point.

PROOF. Since A is an ε-ANR and finite dimensional, we may assume that by (2.7) and (2.4) A is in certain dimensional Euclidean space E and there exists a complex K such that $U \supset K$ and the interior of K involves A, where U is an open set of E of which there are ε-retractions $r_\varepsilon : U \to A$, for every $\varepsilon > 0$. Since $f \sim 0$ in A, we have $fr_\varepsilon : K \to A$ is null-homotopic in A. Since K is an ANR, then there exists an $a_\varepsilon \in A$ such that $fr_\varepsilon(a_\varepsilon) = a_\varepsilon$ for every $\varepsilon > 0$ [1]. On the other hand if f has not any fixed point, there exists $\delta > 0$ such that

$$\rho(a, f(a)) \geqq \delta$$

for every $a \in A$.

For sufficiently small ε we have $\delta/2 \geqq \rho(f(a), fr_\varepsilon(a))$ for every $a \in A$ and we have

$$0 = \rho(fr_\varepsilon(a_\varepsilon), a_\varepsilon) \geqq \rho(a_\varepsilon, f(a_\varepsilon)) - \rho(f(a_\varepsilon), fr_\varepsilon(a_\varepsilon)) \geqq \delta - \frac{\delta}{2} = \frac{\delta}{2}$$

This is a contradiction. Thus f has a fixed point.

[1] K.Borsuk; Ueber eine Klasse von lokal zusammenhängenden Räumen, Fund. Math., Vol.19 (1932), pp.220-242.

[2] K.Borsuk; Sur un problème de MM. Kuratowski et Ulam. Fund. Math., Vol.31 (1938), pp. 154-159.

[3] S.Eilenberg; Sur l'invariance par rapport aux petites transformations. C.R.Acad. Sci. Paris, Vol.200 (1935), pp. 1003-1005.

[4] S.Eilenberg; Sur les transformations à petites tranches. Fund. Math., Vol.30 (1938), pp.92-95.

[5] S.Lefschetz; Algebraic Topology. New York. 1942.

[6] C.W.Saalfrank; Retraction properties for normal Hausdorff spaces. Fund. Math., Vol. 36 (1949), pp.93-108.

Notes

1) We abbreviate "absolute neighborhood retract" by ANR or ANRset.
2) $\rho(a, b)$ is the distance from a to b.
3) $f: A \to M$ is called an ε-map provided the diameter of inverse image of each point of $f(A) \subset M$ is less than ε.

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