STRUCTURE OF HOMOGENEOUS CHAINS

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In the previous paper [3], the author determined the structure of discrete homogeneous chains. Continuing his study, he will show the structure of general homogeneous chains to some extent, in the present paper.

In the structure theory of homogeneous chains, simple homogeneous chains, which will be defined later, seem to be fundamental, and some attempt of the representation of general homogeneous chains, as lexicographic product of simple homogeneous chains is suggested at the end of this paper.

As to definitions and notations, the same as used in [1] and in the author's previous paper, [3] are employed. But, for convenience, a short note of these definitions and the results gained from these definitions, which are used in the author's previous paper, are stated again in §1.

In the study of the structure of homogenéous chains, the homogeneous intervals play an important rôle, and §1 is devoted to the investigation of the homogeneous intervals in a homogeneous chain.

In §2, homogeneous chains, which have very special type, namely, homogeneous chains with unique automorphisms, are studied.

In §3 the structure of simple homogeneous chains, especially that of conditionally complete homogeneous chains, which belongs to this category, are determined to some extent.

Those are homogeneous chains with very special type, but a general homogeneous chain is embedded in a lexicographic product of these simple homogeneous chains. The fact is shown in the last section, §4, of this paper.

§1. Homogeneous intervals.

The terms used without definitions, such as a partially ordered set, (abbr. a poset), a chain (or a totally ordered set), and an ordinal number (or a well-ordered set), ought to be referred to [1].

(1.1) Definition 1. If a chain X has a transitive automorphism group, we call X homogeneous.

Definition 2. A subchain I of a chain X is called an interval of X. if and only if,

$a, b \in I$ and $a < c < b$

implies $c \in I$.

The whole chain X and a subchain which consists of only one element of X are intervals. The other intervals are called proper.

Especially, for any pair of elements a, b of X, the set of elements between (properly) a and b is an interval of X, which we call an open interval (a, b). The set of upper bounds and the set of lower bounds of an element a of X, excluding the element a, are also called (unbounded) open intervals, and are denoted by (a,-) and (-, a) respectively. When two elements a and b are adjoined to the open interval (a, b), we call it a closed interval [a, b]. [a, b) denotes the interval (a, b) with adjoined a only. (a, b], [a, -), and (-, a] are similarly defined.

(1.2) We define the following two kinds of orders in a family of intervals of a chain X.

P.1) We say that Y_1 contains Y_2, if and only if Y_2 is a subset of Y_1, and denote the fact by $Y_2 \subset Y_1$. (Or, we may say that Y_1 is greater than Y_2 in the meaning of P.1).)

P.2) We say that Y_2 is less than Y_1 (or Y_1 is greater than Y_2) if and only if $a < b$ for any pair of $a \in Y_2$, and $b \in Y_1$, and denote the fact by $Y_2 < Y_1$. (Or, precisely, we say that Y_1 is greater than Y_2, in the meaning of P.2).) Especially the subset of X, which consists of only one element $x \in X$ is an interval of X. If x is less than any element of the other interval Y of X, the fact is denoted by $x < Y$. The sign $x > Y$ is similarly defined.

Y_1 and Y_2 are comparable if and only if either they are disjoint or they coincide entirely with each other.

The adequacy of those orderings and the condition of comparability can be easily verified.

(1.3) Definition 3. When an interval Y of a homogeneous chain X is itself a homogeneous chain, then we call Y a homogeneous interval.

Let Y be an interval (not necessarily homogeneous) of a homogeneous chain X. We denote the automorphism group of X by $\mathcal{G}_X$. The set of all automorphisms of X such that

$$\mathcal{G}_Y = \{\varphi \in \mathcal{G}_X,\ \varphi(x) = x \text{ for any } x \notin Y\}$$

is called the characteristic group of the interval Y. $\mathcal{G}_Y$ is isomorphic to the automorphism group of the chain Y (cf. [3]). If Y is a homogeneous interval of X, then $\mathcal{G}_Y$ is transitive within Y. We shall call an automorphism $\varphi \in \mathcal{G}_Y$, an automorphism of Y, simply.

(1.4) It is well known that the automorphism group of a chain X becomes a lattice ordered group. (cf. [1], pp. 214-217) We shall see some detail of this fact.

For a pair of two automorphisms φ, ψ of X, if

$$\varphi(x) \leq \psi(x) \quad \text{for any } x \in X$$

then φ is said to be less than ψ.

Let φ and ψ be two automorphisms (not necessarily comparable) of X, then by the above ordering, the join and meet of φ and ψ can be defined as following

$$\varphi \smile \psi(x) = \max(\varphi(x), \psi(x))$$

$$\varphi \frown \psi(x) = \min(\varphi(x), \psi(x)).$$

Especially, for the identical mapping 0, the automorphism $\varphi \smile 0$, such that

$$\varphi \smile 0\,(x) = \max(\varphi(x), x)$$

is denoted by φ^+, which is called the positive part of φ.

If $\varphi(x) \geqslant x$ for any $x \in X$, then φ is called positive. If $\varphi(x) \geqslant x$ for any $x \in Y$, Y being an interval of X, then φ is called positive in Y. If $\varphi(x) \geqslant x$ for an $x \in X$, we shall say that φ is positive at x.

The negative part $\varphi^- = \varphi \frown 0$ of an automorphism $\varphi \in \mathcal{G}_X$, and the negativity of an automorphism φ are similarly defined.

(1.5) Lemma 1. Let φ be an automorphism of a homogeneous chain X. For instance, φ is assumed to be positive at $a \in X$. Set $\varphi^n(a) = a_n$, and $Y_n = [a_n, a_{n+1})$ for $n = 0, \pm 1, \pm 2, \dots$. Then $Y = \bigcup_n Y_n$ is a homogeneous interval of X, and φ is positive in Y. When φ is negative at a, the result is all the same.

Remark. If we take a displacement θ of X such that

$$\theta(y) = \varphi(y) \quad \text{for } y \in Y$$

$$\theta(z) = z \quad \text{for } z \notin Y$$

then we get a new automorphism θ of X, and $\theta^n(a) = \varphi^n(a)$, so the set

$$\bigcup_n [\,\theta^n(a),\ \theta^{n+1}(a))$$

entirely coincides to Y. Hence we may take θ in the place of φ in the following consideration.

Proof of the lemma.

We denote by Y* the union of the interval Y and its lower bounds. Y* is not homogeneous in general. First we shall see that for any y, $a_n \leq y < a_{n+1}$, there exists an automorphism τ of Y*, which maps a_n onto y.

If $y = a_n$ for some n, this statement is obvious. Let $y \in (a_n, a_{n+1})$. Take an automorphism ψ of X such that $\psi(a_n) = y$, then $\psi(a_n) < a_{n+1} = \theta(a_n)$ and the following three cases are possible.

i) $\psi(a_m) = \theta(a_m)$ for some $m > n$

ii) $\psi(a_m) < \theta(a_m)$ for any $m > n$

iii) $\psi(a_m) > \theta(a_m)$ for some $m > n$.

Case i) Take the automorphism of X such that,

$$\tau(z) = \psi(z) \quad \text{for } z \leqslant a_m$$

$$\tau(z) = \theta(z) \quad \text{for } z > a_m$$

then $\tau(a_n) = y$, and obviously τ is an automorphism of the interval Y*.

Case ii) We shall prove that

$$\psi^i(a_n) < \theta^i(a_n) \quad \text{for any } i > 0.$$

When $i = 1$, $\psi(a_n) = y < a_{n+1} = \theta(a_n)$.

Now, let the statement be proved for an i, then

$$\psi^{i+1}(a_n) < \psi\theta^i(a_n) = \psi(a_{n+i})$$

$< \theta(a_{n+i}) = \theta^{i+1}(a_n)$

Hence, the induction is accomplished, and so every $\gamma^i(a_n)$ is contained in Y^* for any $i > 0$.

Set $Z = \bigcup_i [\gamma^i(a_n), \gamma^{i+1}(a_n))$, $i = 0, \pm 1, \pm 2, \ldots$, then $Z \subset Y^*$, and if we take the displacement τ of X such that

$$\tau(z) = \gamma(z) \quad \text{for } z \in Z$$
$$= z \quad \text{otherwise},$$

then obviously τ is an automorphism of Y^*, since $Z \subset Y^*$, and $\tau(a_n) = y$.

Case iii) Let m be the first integer greater than n such that

$$\gamma(a_m) > \theta(a_m)$$

that is, $\gamma(a_\ell) < \theta(a_\ell)$ for $n \leq \ell < m$. Take a displacement τ of X such that

$$\tau(x) = \gamma(x) \text{ for } x < a_{m-1}$$

$$\tau(x) = \gamma \frown \theta(x) \text{ for } a_{m-1} \leq x < a_m$$

$$\tau(x) = \theta(x) \quad \text{for } x \geqslant a_m.$$

Since $\gamma \frown \theta(a_{m-1}) = \gamma(a_{m-1})$ and $\gamma \frown \theta(a_m) = \theta(a_m)$, τ is an automorphism of Y^*, and $\tau(a_n) = y$.

Second, we shall see that for any $x, y \in Y^*$, there exists an automorphism τ of Y^* such that $\tau(x) = y$.

Take an automorphism τ_1 of Y^* such that $\tau_1(a_m) = x$, and another such that $\tau_2(a_m) = y$, and set $\tau = \tau_2 \theta^{m-n} \tau_1^{-1}$, then $\tau(x) = y$, and since τ_1, τ_2, and θ, are automorphisms of Y^*, so is τ also.

Finally, we shall see that for any $x, y \in Y$, there exists an automorphism π of Y, which maps x to y.

If $Y^\dagger$ denotes the union of the interval Y and its upper bounds, it is all the same as the above that there exists an automorphism π_1 of $Y^\dagger$, which maps x to y. Take an automorphism π_2 of Y^* which maps x to y, and set

$$\pi(z) = \pi_1(z) \quad \text{for } z \leq x$$

$$\pi(z) = \pi_2(z) \quad \text{for } z > x.$$

Then obviously π is an automorphism of X, and $\pi(z) = z$ for any $z \notin Y$ and $\pi(x) = y$.

Hence the proof of the first part of the lemma is established, and the second part is obvious.

2. <u>A homogeneous chain with unique automorphisms.</u>

(2.1) <u>Definition 4.</u> If for some pair of elements x,y of a homogeneous chain X, the automorphism which maps x to y is unique, we call X a <u>homogeneous chain with unique automorphisms, or sometimes we say that the automorphisms of X are unique, or X has unique automorphisms.</u>

If for a pair of elements x, y of a homogeneous chain X, the automorphism φ which maps x to y is unique, then the automorphism which does not displace x is unique, that is, such an automorphism is only the identical mapping.

Indeed, if there exists some automorphism γ which does not displace x and $\gamma(a) = b \neq a$ for some $a \in X$, then $\varphi\gamma(x) = \varphi(x) = y$, and $\varphi\gamma(a) = \varphi(b) \neq \varphi(a)$ since $a \neq b$. Hence the automorphism which maps x to y is not unique.

Conversely, it is similarly verified that if the automorphism which does not displace x is only the identical mapping, then the automorphism which maps x to some y is also unique.

If there exists a non-identical automorphism φ which does not displace x, take an automorphism γ which maps x to y, then the automorphism $\gamma\varphi\gamma^{-1}$ does not displace y, and is not identical. Hence we can conclude that if for some pair of elements $x, y \in X$, the automorphism which maps x to y is unique, then for any pair of elements of X, the automorphism which maps one to the other is unique.

(2.2) <u>Theorem 1.</u> A homogeneous chain X has unique automorphisms if and only if there exists no proper homogeneous interval in X.

Proof. If X has a proper homogeneous interval Y then the non-identical automorphism of Y does not displace any element outside of Y, so the automorphisms of X are not unique.

Conversely if there exists a non-identical automorphism φ of X which does not displace some element x of X, then the set $\bigcup_{n=-\infty}^{\infty} [\varphi^n(a), \varphi^{n+1}(a))$ for some a such that $\varphi(a) \neq a$, is a homogeneous interval by lemma 1, which is proper since it does not contain x.

(2.3) In a homogeneous chain X with unique automorphisms, if, for a pair of automorphisms φ and γ, and for some $x \in X$, $\varphi(x) < \psi(x)$, then for any element y of X, $\varphi(y) < \psi(y)$.

Indeed, $\varphi(y)$ can not be equal to $\psi(y)$. If $\varphi(y) > \psi(y)$ for some $y \in X$, then the automorphism $\theta = \varphi \frown \psi$ is equal to φ at x, but not equal to φ at y, hence the automorphism which maps x to $\varphi(x)$ is not unique.

(2.4) We fix some element a of X, and consider the correspondence between the automorphisms of X and the elements of X such that $\varphi \longleftrightarrow \varphi(a)$. Then this correspondence is one-to-one, and isotone if we regard the automorphism group of X as a lattice group in the meaning of (1.4).

It is easy to see that the lattice group $\mathcal{G}_X$ of automorphisms of X is totally ordered and Archimedean, hence the group is isomorphic to a subgroup of the lattice ordered additive group of real numbers (cf. [1] p.226). Since the lattice group of automorphisms of X corresponds one-to-one and order-preservingly to the chain X, we can state,

Theorem 2. A homogeneous chain with unique automorphisms is isomorphic to a subchain of real numbers. More exactly, a homogeneous chain with unique automorphisms is isomorphic to the ordered set made of a subgroup of the additive group of real numbers.

It is naturally surmised that a homogeneous chain with unique automorphisms is isomorphic to the chain of integers. But the author could not ascertain this expectation. It is easy to see that the only discrete homogeneous chain with unique automorphisms is that of integers. The author thinks that the following problem is an interesting one.

Problem 1. Is there any homogeneous chain with unique automorphisms which is dense-in-itself?

3. Simple homogeneous chains.

(3.1) Now we shall investigate some other special type of homogeneous chains. Simple homogeneous chains, defined later, have also a special type, yet they seem to make themselves a structural foundation of general homogeneous chains, and moreover, many important chains, such as the chain of integers, the chain of rational numbers, that of real numbers, and general conditionally complete homogeneous chains, are contained in this category.

Definition 5. A regular interval Z of a homogeneous chain X is an interval of X which has the following property.

For any pair of elements x, y in Z and for any automorphism φ of X,

$\varphi(x) \in Z$ implies $\varphi(y) \in Z$.

Example. Let a chain X be the ordered set of real numbers from which the integers are taken away, then as easily seen, the set (0, 1) is a regular interval.

Let a chain X be discrete, then the interval of X, such that between two elements of it, only a finite number of elements exist, is a regular interval.

For any homogeneous chain, the whole chain and the interval which consists of only one element are regular intervals.

Definition 6. A homogeneous chain X is called simple, if and only if it contains no proper regular interval.

Examples. The chain of integers, the chain of rational numbers, and the chain of real numbers are simple as easily seen.

A homogeneous chain with unique automorphisms is simple, since it contains no proper homogeneous interval. (A regular interval is always a homogeneous interval, as seen later.)

A conditionally complete homogeneous chain is simple. Indeed, any proper interval Y of a conditionally complete homogeneous chain X has either its upper limit or its lower limit. For instance, assume that Y has its upper limit u, then for any pair of elements $a, b \in Y$, $a < b$, a positive automorphism which maps b to u, maps a into Y. (Obviously, u can not be contained in Y.) Hence Y is not a regular interval.

(3.2) Let Y be a proper interval (not necessarily homogeneous) of a homogeneous chain X. We define an equivalence relation between the elements of X in the following way. We say $x \sim y$ $(x, y \in X)$ if and only if for any automorphism φ of X

$\varphi(x) \in Y$ implies $\varphi(y) \in Y$

and $\varphi(y) \in Y$ implies $\varphi(x) \in Y$.

Obviously this relation satisfies the axioms of equivalency.

If $x \sim y$, $x < y$, and $\varphi(x) < Y$, then $\varphi(y) \not< Y$ by definition. If further $\varphi(y) > Y$, then there exists an automorphism γ of X, which maps $\varphi(x)$ into Y. Then the positive automorphism γ^+ maps $\varphi(x)$ into Y, and maps $\varphi(y)$ out of Y, and then $\gamma^+\varphi(x) \in Y$, and $\gamma^+\varphi(y) > Y$. This contradicts the assumption that $x \sim y$. On the other hand, if $x > y$, and $\varphi(x) < Y$, then it is obvious that $\varphi(y) < Y$. Hence if $x \sim y$, and $\varphi(x) < Y$, then $\varphi(y) < Y$. Similarly if $x \sim y$, and $\varphi(x) > Y$, then $\varphi(y) > Y$.

We shall see that a class by the classification induced by the above equivalence relation is a regular interval.

First, the class Z is an interval of X.

Indeed, if $x \sim y$, and $x < u < y$, then $\varphi(x) \in Y$ implies $\varphi(y) \in Y$, and $\varphi(x) < \varphi(u) < \varphi(y)$, and so $\varphi(u) \in Y$. Hence $\varphi(x) \in Y$ implies $\varphi(u) \in Y$.

Conversely, if $\varphi(x) < Y$, then $\varphi(y) < Y$, and if $\varphi(x) > Y$ then $\varphi(y) > Y$, and since $\varphi(x) < \varphi(u) < \varphi(y)$, $\varphi(u) < Y$ or $\varphi(u) > Y$ respectively. Hence $\varphi(x) \notin Y$ implies $\varphi(u) \notin Y$. Hence $x \sim u$, and $u \in Z$.

Second, the interval Z is regular.

Indeed if $x, y \in Z$ and $\varphi(x) \in Z$, $\varphi(y) \notin Z$ for an automorphism φ of X, then $\varphi(x)$ is not equivalent to $\varphi(y)$, so there exists an automorphism γ of X, which maps one of $\varphi(x)$ and $\varphi(y)$ into Y, and the other out of Y. Then the automorphism $\gamma\varphi$ maps one of x and y into Y, and maps the other out of Y. But this contradicts the assumption that $x \sim y$.

(3.3) We shall consider the case that a homogeneous chain X is simple and its automorphisms are not unique.

Since, the automorphisms of X are not unique, there exists a proper homogeneous interval Y in X. (Theorem 1) Since X is simple, for any pair of elements of X, one is not equivalent to the other with respect to the equivalence relation concerning to the interval Y. (3.2) (Otherwise, each of the equivalent classes is a proper regular interval.) In other words, for any pair of elements $x, y \in X$, (except the case $x = y$.) there exists an automorphism φ of X, such that

either $\varphi(x) \notin Y$ and $\varphi(y) \in Y$

or $\varphi(x) \in Y$ and $\varphi(y) \notin Y$.

We consider the following three cases.

Case i) For any pair of elements x,y in Y, there exists an automorphism φ of X such that $\varphi(x) \in Y$ and $\varphi(y) \notin Y$, and exists another automorphism γ of X such that $\gamma(x) \notin Y$ and $\gamma(y) \in Y$.

Case ii) For some pair of elements $x, y \in Y$, $x < y$, and for any automorphism φ of X, $\varphi(y) \in Y$ implies $\varphi(x) \in Y$.

Case iii) For some pair of elements $x, y \in Y$, such that $x < y$, and for any automorphism φ of X, $\varphi(x) \in Y$ implies $\varphi(y) \in Y$.

Y has either upper or lower bounds, since Y is a proper interval. When Y is lower unbounded, the case is ii). When Y is upper unbounded, the case is iii). Even when Y is both upper and lower bounded, we have no reason to exclude the case ii) and iii), for the present.

In the case ii), there exists an automorphism γ of X such that $\gamma(x) \in Y$, and $\gamma(y) > Y$, since X is simple. We shall see that for any element $u, v \in Y$, $u < v$, there exists an automorphism θ of X such that $\theta(u) \in Y$ and $\theta(v) > Y$.

First we can see that if $\pi(x) < Y$, for an automorphism π of X then $\pi(y) < Y$. Indeed, if $\pi(y) > Y$, then there exists an automorphism τ of X, which maps $\pi(y)$ into Y, then the automorphism $\tau^-\pi$, where τ^- is the negative part of τ, maps y into Y and maps x out of Y. This contradicts the condition of case ii).

Take an automorphism χ of X, which maps u to x, and set $w = \chi(v)$.

If $w > y$, then $\gamma\chi(v) = \gamma(w) > \gamma(y) > Y$, and $\gamma\chi(u) = \gamma(x) \in Y$. Hence the automorphism $\theta = \gamma\chi$ is suitable for our purpose.

If $w < y$, then we shall see that for any automorphism τ of X, $\tau(u) \notin Y$ implies $\tau(v) \notin Y$. Indeed, if for an automorphism τ of X, $\tau(u) > Y$, then $\tau(v) > Y$, since $\tau(u) < \tau(v)$. If $\tau(u) < Y$, then $\tau\chi^{-1}(x) < Y$, hence $\tau\chi^{-1}(y) < Y$, and then $\tau\chi^{-1}(w) < \tau\chi^{-1}(y)$, and so $\tau(v) < Y$. Hence, if $w < y$, then $\tau(u) \notin Y$ implies $\tau(v) \notin Y$ for any

$\tau \in \mathcal{G}_x$. In other words, if $\tau(v) \in Y$, then $\tau(u) \in Y$. Hence, there must be an automorphism θ of X, such that $\theta(u) \in Y$, and $\theta(v) \notin Y$.

(3.4) By above consideration, we can assert that, in case i) and ii), for any pair of x, y $\in$ Y, there exists an automorphism $\theta \in \mathcal{G}_x$, which maps the less element into Y and the greater element into the upper bounds of Y. In case iii), by similar consideration, for any pair of elements u, v $\in$ Y, there exists an automorphism which maps the greater element into Y, and the less element into the lower bounds of Y.

Now we shall see that every bounded open interval in Y is isomorphic to another one in Y. For instance, we assume that it is the case i) or ii). The case iii) can be similarly treated.

First of all, for any three elements x,y and z in Y, Y being a homogeneous proper interval of a homogeneous chain X, if $x < y < z$, then the interval (x, z) is isomorphic to the interval (y, z).

Indeed, there exists an automorphism φ of X, which maps z into the upper bounds of Y and maps y and x into Y (we assume that the case is ii) or iii)). Then, since Y is a homogeneous interval of X, there exists an automorphism ψ of Y, (which fixes any element outside of Y), which maps $\varphi(x)$ onto $\varphi(y)$. Hence the automorphism $\theta = \varphi^{-1}\psi\varphi$ maps the interval (x, z) onto the interval (y,z). This shows that the interval (x, z) is isomorphis to the interval (y, z).

Second, we shall prove that an open interval (x, y) in Y is isomorphic to another one (u, v).

Indeed, since the interval Y is homogeneous, there exists an automorphism φ of Y, which maps v onto y. Then $x \leqslant \varphi(u) < y = \varphi(v)$, or $\varphi(u) < x < y = \varphi(v)$. In both case, the open interval $(\varphi(u), \varphi(v))$ is isomorphic to the open interval (x, y). Hence (u, v) is isomorphic to (x, y), and the statement is proved.

(3.5) We have just proved that every bounded open interval in a proper homogeneous interval Y of a simple homogeneous chain X, whose automorphisms are not unique, is isomorphic to one another. In other words, those bounded open intervals in Y is isomorphic to a definite chain T.

The chain T has following properties;

T1) Any (bounded or unbounded) open interval in T is isomorphic to T itself.

T2) If by T^+, the chain $1 \oplus T$, where 1 is an adjoined element, and $\oplus$ is the ordinal sum (cf. [1] p.9), is denoted, then

$$T = T \oplus T^+$$

T3) T is homogeneous.

T1) and T2) are obvious. We shall verify T3)

Since, for any x, and y in T, the open intervals (-, x), (-, y), (x, -), and (y, -) are isomorphic to T, there exist an isomorphism φ from (-, x) to (-, y), and an isomorphism ψ from (x, -) to (y, -). Then the mapping θ, such that

$$\theta(z) = \varphi(z), \quad \text{for } z \in (-, x)$$

$$\theta(x) = y$$

$$\theta(z) = \psi(z), \text{ for } z \in (x,-)$$

is an automorphism of T, which maps x to y.

T4) Every interval U of T, which has neither the greatest element nor the least element in U, is a homogeneous interval of T.

Indeed, for any pair of elements x,y of U, there exist two elements u,v $\in$ U, such that $u < x,y < v$. But since the interval (u,v) is homogeneous ((T3)), there exists an automorphism of the interval (u, v), which maps x to y and this automorphism is also an automorphism of U.

T5) Let ρ be the regular ordinal number (cf. [2] pp.130-135) which is cofinal to the chain T, and let $\hat{\sigma}$ be the dual of a regular ordinal number, which is co-initial to the chain T; then obviously $(\rho, \hat{\sigma})$ is the element-character (cf. [2] pp. 134-137) of T, and following identities hold,

$$T = T \oplus (\rho \circ T^+)$$

$$T = \hat{\sigma} \circ T^+$$

and $$T = (\hat{\sigma} \oplus \rho) \circ T^+$$

where $\oplus$ and $\circ$ denote the ordinal sum and ordinal product, respectively. The proof of T5) is easy.

Definition 7. We call the chain T, which has the property T1), a totally homogeneous chain.

Remark 1. The properties T2) - T5) are induced from the property T1).

Remark 2. Such a chain is what is called a homogeneous chain by F. Hausdorff (cf. [2] p.173) But on account of the change of the definition of homogeneity, we use the new term. When such a chain is also conditionally complete, it is called a homogeneous linear continuum, and we shall make use of this term latter.

(3.6) Now we shall prove that any bounded open interval (x,y) in X is isomorphic to T. (The notations X, Y and T are the same as the previous propositions.)

We define the following equivalence relation: for a pair of elements $x,y \in X$, $x \approx y$, if the open interval (x, y) or (y, x), according to $x < y$, or $x > y$ respectively, is isomorphic to T. Then, assuming that $x \approx x$, this relation satisfies the axioms of equivalency, since the transitive law is valid by the properties T1) and T2) of T. Each equivalent class is an interval by T1). And obviously, one of these classes contains the homogeneous interval Y, and hence this class contains at least two elements.

We shall see that each class is a regular interval. Indeed, if $x,y \in U$, U being one of the equivalent classes, for instance, that which contains Y, and $\varphi(x) \in U$, and $\varphi(y) \notin U$ for some $\varphi \in \mathcal{G}_X$, then the interval $(\varphi(x), \varphi(y))$, (or the interval $(\varphi(y), \varphi(x))$ is isomorphic to the interval (x, y) (or the interval (y,x)), and hence to the chain T, since x is equivalent to y. On the other hand, $(\varphi(x), \varphi(y))$ (or $(\varphi(y), \varphi(x))$) can not be isomorphic to T, since $\varphi(y)$ is not contained in the equivalent class U which contains $\varphi(x)$. This is a contradiction.

Thus we have proved that the equivalent class is a regular interval, but, since X is simple by assumption, each class must be a non-proper interval of X, and especially the class which contains Y must entirely coincide with the whole chain X. This proves that for every pair of elements $x, y \in X$; $x < y$, the interval (x, y) is isomorphic to T.

Hence we have the following Theorem.

Theorem 3. For every pair of elements x, and y in a simple homogeneous chain, whose automorphisms are not unique, the open interval (x, y) (assuming that $x < y$) is isomorphic to a definite totally homogeneous chain.

(3.7) We shall say that the totally homogeneous chain T associates to the simple homogeneous chain X whose automorphisms are not unique.

Let W_1 be the regular ordinal number, which is cofinal to the chain X, and let $\hat{W}_2$ be the dual of a regular ordinal number, which is coinitial to the chain X. Obviously, W_1 is, at the greatest, the first ordinal number ω_μ, with the type of which, no subchain of T can contained in T, and $\hat{W}_2$ is, at the greatest, the dual $\hat{\omega}_\nu$ of the first ordinal number, with the dual type of which, no subchain of T can contained in T, T being the totally homogeneous chain which associates to X.

We consider the following four cases.

C1) W_1 is less than ω_μ, and $\hat{W}_2$ is (dually) less than ω_ν.

C2) W_1 is equal to ω_μ, and $\hat{W}_2$ is (dually) less than $\hat{\omega}_\nu$.

C3) W_1 is less than ω_μ, and $\hat{W}_2$ is equal to $\hat{\omega}_\nu$.

C4) W_1 is equal to ω_μ, and $\hat{W}_2$ is equal to $\hat{\omega}_\nu$.

In the first case C1), X is isomorphic to the chain $(W_1 \oplus \hat{W}_2)\circ T^+$, as easily seen. On the other hand, we can take a subchain S' of T, which is isomorphic to $(W_1 \oplus \hat{W}_2)$. Then the minimal interval S of T, which contain S' (S consists of all elements which exist between in some pair of elements in S') is obviously isomorphic to $(W_1 \oplus \hat{W}_2)\circ T^+$, and hence isomorphic to X. In other words, X is siomorphic to a subchain of T.

In the case C2), the unbounded open interval $(-, x)$ of X, for some $x \in X$, is isomorphic to an interval S of X, similarly as in the case C1). And hence the chain X is isomorphic to the chain $S \oplus (\omega_\mu \cdot T^+)$. In the case C3), the result is similar, and we can get the following Theorem.

Theorem 4. For any simple homogeneous chain X, there exists a totally homogeneous chain T, which associates to X, and X is isomorphic to one of the following four representations.

C1) S

C2) $S \oplus (\omega_\mu \circ T^+)$

C3) $(\hat{\omega}_\nu \circ T^+) \oplus S$

C4) $(\hat{\omega}_\nu \oplus \omega_\mu) \circ T^+$,

where ω_μ is the first ordinal number with the type of which no subchain of T can be contained in T, $\hat{\omega}_\nu$ is the dual of the first ordinal number, with the dual type of which no subchain of T can be contained in T, and S is an interval of T, dependent on X.

(3.8) If a chain X is a conditionally complete homogeneous chain, then the problem stated at the last of chapter 2 is easily resolved.

Theorem 5. A conditionally complete homogeneous chain X with unique automorphisms is isomorphic to the chain of integers.

Proof. We embed the chain X into the additive group of real numbers (Theorem 2). Take an element x of X and a non-identical automorphism φ of X, then the set $J = \{\varphi^n(x)\}$ is isomorphic to the chain of integers.

If for any $y \in X$, and the unique automorphism γ such that $\gamma(x) = y$, there exists an integer m such that $\gamma^m(x) \in J$, then it is easy to see that X is isomorphic to either the chain of integers or the chain of rational numbers, the later can not be complete.

If for some element y of X, and the unique automorphism γ such that $\gamma(x) = y$, any $\gamma^m(x)$ is not contained in J, then it is easy to see that the set $K = \{\gamma^m \varphi^n(x)\}$ where m, n are integers, is dense in the chain of real numbers. Hence the chain X is also dence in the ordered additive group of real numbers, but since X is conditionally complete, X must entirely coincide with the chain of real numbers, whose automorphisms are not unique. Hence the proof is accomplished.

(3.9) If the conditionally complete homogeneous chain X is dense-in-itself, and hence its automorphisms are not unique, then X is always simple (cf. the Example of Definition 6). Hence X is associated with a totally homogeneous chain T, which must be also conditionally complete. A conditionally complete totally homogeneous chain is so-called a homogeneous linear continuum (cf. the Remark of Definition 7).

In a homogeneous chain, every point has the same character (cf. [2] pp. 142-147) as that of another. In a conditionally complete homogeneous chain, which is dense-in-itself, the definite point-character is obviously $(\omega, \hat{\omega})$ where ω is the first infinite ordinal number, and $\hat{\omega}$ is its dual. Since there exists no gap in T, the first ordinal number, with the type of which no subchain of T can be contained in T, and the dual of the first ordinal number, with the dual type of which, no subchain can be contained in T, are Ω and $\hat{\Omega}$, respectively, where Ω and $\hat{\Omega}$ are the first uncountable ordinal number and its dual respectively. Moreover, every interval in T, which has no greatest element nor least element in it, is an open interval. Hence, in this case, Theorem 4 can be stated as the following

Theorem 6. A conditionally complete homogeneous chain X is either isomorphic to the chain of integers, or its every bounded open interval is isomorphic to a definite homogeneous linear continuum T.

In the later case, the conditionally complete homogeneous chain X is isomorphic to one of the following four representations.

C1) T

C2) $T \oplus (\Omega \cdot T^+)$

C3) $\hat{\Omega} \cdot T^+$

C4) $(\hat{\Omega} \oplus \Omega) \circ T^+$

where T is the homogeneous linear continuum associating to the chain X, and Ω and $\hat{\Omega}$ are the first uncountable ordinal number and its dual, respectively.

4. General homogeneous chains.

(4.1) To investigate the structure of general homogeneous chains, we shall study the behaviours of regular intervals in a general homogeneous chain.

Let Y be a regular interval in a general homogeneous chain X. We define an equivalence relation between elements of X. We say $x \sim y$, if and only if, for any automorphism φ of X,

$\varphi(x) \in Y$ implies $\varphi(y) \in Y$.

Then that $x \sim x$ and that $x \sim z$ follows from $x \sim y$ and $y \sim z$ are obvious. Let $x \sim y$, and assume that for some automorphism φ of X, $\varphi(x) \notin Y$ and $\varphi(y) \in Y$. Take an automorphism ψ of X, which maps $\varphi(x)$ into Y. Then $\psi\varphi(x) = u \in Y$, and so $\psi\varphi(y) = v \in Y$, by the definition of $x \sim y$. But then $\psi^{-1}(u) = \varphi(x) \notin Y$, $\psi^{-1}(v) = \varphi(y) \in Y$, and $u, v \in Y$. This contradicts the regularity of Y. Hence $x \sim y$ implies $y \sim x$.

(4.2) We shall call the each equivalent class, a <u>co-class</u> of Y. <u>Y itself is a co-class of Y.</u>

Indeed, any element in Y is equivalent to the other element of Y by the regularity of Y. On the other hand an element in Y is not equivalent to an element out of Y, since the condition of equivalency does not hold for the identical mapping.

Let Y_1 and Y_2 be co-classes of Y. We shall show that if an automorphism φ of X maps an element x of Y_1 into Y_2, then φ maps Y_1 entirely onto Y_2.

Indeed, if $\varphi(x) = u \in Y_2$, and $\varphi(y) = v \notin Y_2$ for some $x, y \in Y_1$, then u is not equivalent to v, hence there exists an automorphism ψ of X such that $\psi(u) \in Y$, and $\psi(v) \notin Y$. Then $\psi\varphi(x) \in Y$, and $\psi\varphi(y) \notin Y$. This contradicts the assumption; $x \sim y$.

Hence φ maps Y_1 into Y_2, but since $\varphi^{-1}(u) = x \in Y_1$, $u \in Y_2$, φ^{-1} maps Y_2 into Y_1. Hence φ maps Y_1 entirely onto Y_2. This proves the statement.

Since φ gives an isomorphism from Y_1 to Y_2, the co-classes are isomorphic to one another.

Especially if $Y_1 = Y_2$, then we see that for any pair of elements $x, y \in Y_1$, and for any automorphism φ of X, $\varphi(x) \in Y_1$ implies $\varphi(y) \in Y_1$. In other words, each co-class of Y is a regular interval.

Moreover, for any pair of elements $x, y \in Y_1$, there exists an automorphism φ of X, which maps x to y. But then φ maps Y_1 onto Y_1 itself, hence Y_1 is itself a homogeneous interval of X. Especially Y is so. Hence a regular interval of X is a homogeneous interval.

(4.3) The co-classes of Y are mutually disjoint, and the union of all co-classes agrees with the whole chain X.

If we define an ordering in the meaning of P2), (1.2), between those co-classes, then the set $\alpha = \{Y_\rho\}$ of all co-classes Y_ρ of Y, becomes a chain (1.2).

Let φ be an automorphism of X, then φ maps one co-class Y_1 onto another co-class Y_2. Hence naturally, this mapping φ is regarded as an one-valued mapping of the chain α of co-classes, and this mapping $\varphi: Y_\rho \to Y_\sigma = \varphi(Y_\rho)$ is obviously one-to-one and order-preserving in α, hence φ is regarded as an automorphism of α. Moreover, the homogenuity of α follows from the homogeneity of X. Hence we can assert

<u>Theorem 7.</u> If Y is a regular <u>interval of X</u>, then Y is a homogeneous interval of X, and there exists a homogeneous chain α with which X is represented as a ordinal product:

$$X = \alpha \circ Y.$$

(4.4) If a regular interval Y of X intersects some other homogeneous interval Z of X, then one of them contains the other.

Indeed, if neither of them contains the other, then there exists three elements $x \in Y \frown Z'$, $y \in Y \frown Z$, and $z \in Y' \frown Z$, where the prime denotes the complement set. Then since Z is homogeneous, there exists a $\varphi \in \mathfrak{G}_Z$ such that $\varphi(y) = z$, and $\varphi(x) = x$. But this contradicts the regularity of Y. Hence the statement is proved.

If a regular interval Y contains another regular interval Z of X, then every co-class of Z is contained by some co-class of Y, and every co-class of Y contains some co-class of Z.

Indeed, $Z \subset Y$ implies $\varphi(Z) \subset \varphi(Y)$ for any $\varphi \in \mathfrak{G}_X$, (4.3), and this verifies both statements.

Moreover, for any pair of regular intervals Y and Z, some co-class of Y must intersect with Z. Hence some co-class of Y either contains Z or is contained in Z.

If we regard the set $\alpha = \{Y_\rho\}$ of all co-classes of Y, as a division of X, and call α a <u>regular division</u> of X, then for any <u>pair of regular</u> division of X, one of them must be a refinement of the other.

(4.5) <u>Definition 8.</u> The <u>hyper-index chain $\mathfrak{A}^*$ of a homogeneous chain X is</u> the set of all regular divisions of X. The <u>index chain</u> $\mathfrak{A}$ of X is the set <u>of all regular</u> divisions which have the next finer regular sub-division.

Of course, $\mathfrak{A}$ is a subset of $\mathfrak{A}^*$.

We introduce an ordering into $\mathfrak{A}^*$. We say $\alpha < \beta$, $\alpha, \beta \in \mathfrak{A}^*$, if and only if β is finer than α. Then, by (4.4), $\mathfrak{A}^*$ becomes a chain. $\mathfrak{A}$ is naturally a subchain of $\mathfrak{A}^*$.

(4.6) We denote the set of all regular intervals which contains some $x \in X$, by $\mathfrak{A}_x^*$ defining the order in the meaning of P.1) (1.2).

For any regular division $\alpha \in \mathfrak{A}^*$, there exists a regular interval $Y \in \alpha$, which contains x. Hence the correspondence $\Delta_x : \alpha \to Y \in \mathfrak{A}_x^*$

is one-to-one, and it is easily seen that this correspondence Δ_x is a dual-order-isomorphism from $\mathfrak{a}^*$ to $\mathfrak{a}^*_x$. Hence the order-type of $\mathfrak{a}^*_x$ does not depend on the choice of $x \in X$.

(4.7) We shall see that for any subset $\mathfrak{L}^*_x$ of $\mathfrak{a}^*_x$,

$$\bigcap Y_\alpha \ ; \quad Y_\alpha \in \mathfrak{L}^*_x \quad \text{and}$$

$$\bigcup Y_\alpha \ ; \quad Y_\alpha \in \mathfrak{L}^*_x$$

are also regular intervals.

Indeed, if $u, v \in \bigcap Y_\alpha$, $Y_\alpha \in \mathfrak{L}^*$ and $\varphi(u) \in \bigcap Y_\alpha$, $\varphi \in \mathfrak{G}_x$, then, since each interval Y_α is regular, $\varphi(v) \in Y_\alpha$ for any $Y_\alpha \in \mathfrak{L}^*_x$, hence $\varphi(v) \in \bigcap Y_\alpha$. This shows that $\bigcap Y_\alpha$, $Y_\alpha \in \mathfrak{L}^*_x$ is regular.

If $u, v \in \bigcup Y_\alpha$, $Y_\alpha \in \mathfrak{L}^*_x$, then there exists a $Y_\beta \in \mathfrak{L}^*_x$, which contains u, and exists a $Y_\gamma \in \mathfrak{L}^*_x$, which contains v. But since $x \in Y_\beta \frown Y_\gamma$, one of Y_β and Y_γ contains the other (4.4). Assume $Y_\gamma \subset Y_\beta$, then $u, v \in Y_\beta$.

We shall see that if $\varphi(u) \in \bigcup Y_\alpha$, then $\varphi(v) \in \bigcup Y_\alpha$. Indeed, if $x \in \varphi(Y_\beta)$ then $\varphi(Y_\beta) = Y_\beta$, since $\varphi(Y_\beta)$ is a co-class of Y_β. In this case $\varphi(v) \in \varphi(Y_\beta) = Y_\beta \in \bigcup Y_\beta$. If $x \notin \varphi(Y_\beta)$, then there exists a $Y_\delta \in \mathfrak{L}^*_x$, which contains $\varphi(u)$. But then, $\varphi(u) \in Y_\delta \frown \varphi(Y_\beta)$, and $Y_\delta \not\subset \varphi(Y_\beta)$, since $x \notin \varphi(Y_\beta)$. Hence $\varphi(Y_\beta) \subset Y_\delta$, and so $\varphi(v) \in \varphi(Y_\beta) \subset Y_\delta \subset \bigcup Y_\alpha$. Hence $\varphi(u) \in \bigcup Y_\alpha$ implies $\varphi(v) \in \bigcup Y_\alpha$, and so $\bigcup Y_\alpha$ is a regular interval.

(4.8) For any pair of elements $x, y \in X$, the intersection $\mathfrak{m}_{x,y}$ of all regular intervals, which contain both x and y, is a regular interval, and this is the minimal regular interval which contains both x and y.

On the other hand, the union $\mathfrak{n}_{x,y}$ of all regular intervals which contain x and do not contain y, is a regular interval, and this is the maximal regular interval, which contains x and does not contain y.

It is easily seen that $\mathfrak{n}_{x,y}$ is properly contained in $\mathfrak{m}_{x,y}$, and that there exists no regular interval which contains $\mathfrak{n}_{x,y}$ and is contained in $\mathfrak{m}_{x,y}$, properly.

<u>Definition 9.</u> For a pair of regular intervals $\mathfrak{m}$, $\mathfrak{n}$ such that $\mathfrak{m} \supset \mathfrak{n}$, if there exists no regular interval which properly contains $\mathfrak{n}$, and is properly contained in $\mathfrak{m}$, then we call the pair $[\mathfrak{m}, \mathfrak{n}]$ <u>the simple pair of regular intervals.</u> Especially, the pair $[\mathfrak{m}_{x,y}, \mathfrak{n}_{x,y}]$ where $\mathfrak{m}_{x,y}$ is the minimal regular interval which contains both x and y, and $\mathfrak{n}_{x,y}$ is the maximal regular interval which contains x and does not contain y, is a simple pair, which we call the <u>simple pair defined by the pair $[x, y]$ of elements</u> x,y of X.

(4.9) The set of all regular intervals $\mathfrak{m} \in \mathfrak{a}^*_x$, which cover some $\mathfrak{n} \subset \mathfrak{a}^*_x$ is denoted by $\mathfrak{a}_x$. Since the correspondence Δ_x from $\mathfrak{a}^*$ to $\mathfrak{a}^*_x$ is a dual isomorphism, Δ_x gives also a dual isomorphism from $\mathfrak{a}$ to $\mathfrak{a}_x$.

If $[\mathfrak{m}, \mathfrak{n}]$, $\mathfrak{m}, \mathfrak{n} \in \mathfrak{a}^*_x$, is a simple pair, then $\mathfrak{n}$ is a regular interval of the homogeneous chain $\mathfrak{m}$. Hence there exists a homogeneous chain M, by which $\mathfrak{m}$ is represented as an ordinal product:

$$\mathfrak{m} = M \circ \mathfrak{n} \qquad \text{(Theorem 7)}$$

If $\mathfrak{m}_1$ is a co-class of $\mathfrak{m}$, and $\mathfrak{n}_1$ is a co-class of $\mathfrak{n}$ such that $\mathfrak{n}_1 \subset \mathfrak{m}_1$, then for an automorphism φ of X, such that $\varphi(\mathfrak{n}) = \mathfrak{n}_1$, $\varphi(\mathfrak{m}) = \mathfrak{m}_1$. Hence the automorphism φ gives an isomorphism from the chain of all co-classes of $\mathfrak{n}$ in $\mathfrak{m}$, to the chain of all co-classes of $\mathfrak{n}_1$ in $\mathfrak{m}_1$. In other words, if $\mathfrak{m}_1$ is represented as $\mathfrak{m}_1 = N \circ \mathfrak{n}_1$, then N is isomorphic to M. Hence if we denote the chain M by $\mathfrak{m}/\mathfrak{n}$, then we can assert:

If α is a regular division in the index-chain $\mathfrak{a}$ of X, then for any regular interval $\mathfrak{m} \in \alpha$, and any regular interval $\mathfrak{n}$ which is covered by $\mathfrak{m}$, the homogeneous chain $\mathfrak{m}/\mathfrak{n}$ is isomorphic to a definite chain X_α.

<u>Definition 10.</u> We shall call the <u>above chain X_α</u>, the <u>factor chain</u> of X, which corresponds <u>to a regular</u> division α in the index chain $\mathfrak{a}$ of X.

(4.10) We shall see that every factor chain of X is a simple homogeneous chain.

We have already seen that every factor chain is homogeneous. (Theorem 7)

We shall prove that the chain $M = \mathfrak{m}/\mathfrak{n}$, where $\mathfrak{n} \subset \mathfrak{m} \in \alpha \in \mathfrak{a}$ and $\mathfrak{n}$ is covered by $\mathfrak{m}$, is a simple chain.

Indeed M is the chain of co-classes of $\mathfrak{n}$ in $\mathfrak{m}$. If there should exist a proper regular interval L in M, then the union $\mathfrak{L}$ of co-classes in L, would become an interval of $\mathfrak{m}$. If for an automorphism φ of $\mathfrak{m}$, an element x of $\mathfrak{L}$ is mapped in $\mathfrak{L}$, then the co-class $\mathfrak{n}_1$ of $\mathfrak{n}_\beta$, which contains x is mapped entirely onto a co-class $\mathfrak{n}_2$, which is contained in $\mathfrak{L}$. But since φ induces an automorphism of M, any co-classes $\mathfrak{n}_\rho \in$ L of $\mathfrak{n}$ must be mapped in $\mathfrak{L}$, as the set L of these co-classes $\mathfrak{n}_\rho$ is a regular interval of M. Hence any element in $\mathfrak{L}$ is mapped in $\mathfrak{L}$, that is, $\mathfrak{L}$ is a regular interval of $\mathfrak{m}$. This is obviously a proper interval of $\mathfrak{m}$, and properly contains $\mathfrak{n}$, since L is proper interval of M. This contradicts the assumption that $\mathfrak{m}$ covers $\mathfrak{n}$. Hence the chain M must be simple.

The factor chain X_α is isomorphic to this M, by the definition, hence every factor chain is a simple homogeneous chain.

(4.11) We number the elements of X with ordinal numbers $\tau < \omega_\xi$, where ω_ξ is a suitable ordinal number, for instance, the initial ordinal number with the power of the chain X, and denote the element numbered with τ by $x(\tau)$.

We have already seen that for any simple pair $[\mathfrak{m}, \mathfrak{n}]$ the chain $M = \mathfrak{m}/\mathfrak{n}$ is isomorphic to the factor chain X_α, where α is the regular division which consists of all co-classes of $\mathfrak{m}$. We choose from each factor chain X_α an element $\gamma_{\alpha,0}$, and define an isomorphism $\Theta_{\mathfrak{m}}$ from $\mathfrak{m}/\mathfrak{n}$ to X_α in such a way that the co-class of $\mathfrak{n}$, which contains the element of X, numbered with the least ordinal number within $\mathfrak{m}$, is mapped onto the chosen element $\gamma_{\alpha,0}$ of X_α. Since the factor chain X_α is homogeneous, such an isomorphism can be always selected.

(4.12) For each $x \in X$, we shall define a function f_x of $\mathfrak{A}$, which selects for each $\alpha \in \mathfrak{A}$, an element $\gamma_\alpha = f_x(\alpha) \in X_\alpha$. For each $\alpha \in \mathfrak{A}$, there exists a regular interval $\mathfrak{m}_\alpha \in \alpha$, which contains x, and a regular interval $\mathfrak{n}_\alpha$ which is covered by $\mathfrak{m}_\alpha$, and contains x. Set

$$f_x(\alpha) = \Theta_{\mathfrak{m}_\alpha}(\mathfrak{n}_\alpha).$$

We shall see that the set $D_{f_x} = \{\alpha \in \mathfrak{A} \mid f_x(\alpha) \neq \gamma_{\alpha,0}\}$ ($\{\alpha \mid P\}$ denotes the set of all elements which satisfy the condition P) satisfies the descending chain condition.

Take any subset E of D_{f_x}, and set $\Delta_x(E) = \{\Delta_x(\alpha) \mid \alpha \in E\}$ where Δ_x is the dual isomorphism from $\mathfrak{A}^*$ to $\mathfrak{A}^*_x$, which was defined in (4.6). We denote the union of the regular intervals $\mathfrak{m}_\alpha$ in $\Delta_x(E)$, by $\mathcal{E}$, then there exists the least ordinal number τ with which an element x in $\mathcal{E}$ is numbered. $x(\tau)$ is contained in some $\mathfrak{m}_\alpha \in \Delta_x(E)$. If another regular interval $\mathfrak{m}_\beta = \Delta_x(\beta)$, $\beta \in E$, contains $\mathfrak{m}_\alpha$, then since $\mathfrak{m}_\beta$ covers some $\mathfrak{n}_\beta \in \mathfrak{A}^*_x$, $\mathfrak{m}_\alpha$ must be contained also in $\mathfrak{n}_\beta$. Hence $\mathfrak{n}_\beta$ contains $x(\tau)$. But τ is the least ordinal number with which an element in $\mathcal{E}$ is numbered, hence by the definition of function f_x, $f_x(\beta) = \gamma_{\beta,0}$. This contradicts the assumption that $\beta \in E \subset D_{f_x}$. Hence $\mathfrak{m}_\alpha$ is the maximal regular interval in $\Delta_x(E)$, that is, $\alpha = \Delta_x^{-1}(\mathfrak{m}_\alpha)$ is the least regular division in E. Hence any subset of D_{f_x} has the least element.

Now let $x, y \in X$, $x < y$, and let f_x and f_y be functions defined above, corresponding respectively to x and to y. Then there exists a simple pair $[\mathfrak{m}, \mathfrak{n}]$ defined by the pair $[x, y]$ (cf. Definition 7) and for $\beta = \Delta_x^{-1}(\mathfrak{m})$, $f_x(\beta) < f_y(\beta)$, since the co-class of $\mathfrak{n}$ which contains y, is greater than the co-class of $\mathfrak{n}$, which contains x, in the chain $M = \mathfrak{m}/\mathfrak{n}$. Moreover, β is the least element in the set $D_{f_x, f_y} = \{\alpha \mid f_x(\alpha) \neq f_y(\alpha)\}$. Indeed, if for a simple pair $[\mathfrak{m}_1, \mathfrak{n}_1]$, $\mathfrak{m}_1, \mathfrak{n}_1 \in \mathfrak{A}^*_x$, $\mathfrak{m}_1$ contains $\mathfrak{m}$ properly, then obviously $\mathfrak{n}_1$ contains $\mathfrak{n}$ properly. But since $\mathfrak{n}$ is the maximal regular interval which contains x and does not contain y, $\mathfrak{n}_1$ contains both x and y. Hence for $\gamma = \Delta_x^{-1}(\mathfrak{m}_1)$, $f_x(\gamma) = f_y(\gamma)$ that is, for any $\gamma < \beta$, $f_x(\gamma) = f_y(\gamma)$. This shows that β is the least element in the set D_{f_x, f_y}. It is similarly seen that the correspondence $x \longrightarrow f_x$ is one-to-one.

(4.13) Definition 11. Let $\mathfrak{A}$ be a poset, and for each $\alpha \in \mathfrak{A}$, there be a corresponding poset X_α. Let $\gamma_{\alpha,0}$ be a fixed element in X_α. The lexicographic product $\prod_{\alpha \in \mathfrak{A}} X_\alpha \langle \gamma_{\alpha,0} \rangle$ is defined as the set of all functions f which select for each $\alpha \in \mathfrak{A}$, a $\gamma_\alpha = f(\alpha) \in X_\alpha$ and make the sets $D_f = \{\alpha \in \mathfrak{A} \mid f(\alpha) \neq \gamma_{\alpha,0}\}$ satisfy the descending chain condition, where $f \leqslant g$ means that for every $\alpha \in \mathfrak{A}$ such that $f(\alpha) \neq g(\alpha)$, there exists an $\beta \leqslant \alpha$ such that $f(\beta) < g(\beta)$.

This definition is a slight extention of that of the maximal product by F. Hausdorff (cf. [2] pp.147-161), and the following statements are easily proved.

The axioms of order are satisfied without any restrictive condition on the posets concerning, such as the descending chain condition on the index set $\mathfrak{A}$.

If we denote the set $\{\alpha \mid f(\alpha) \neq g(\alpha)\}$ by $D_{f,g}$, and the set of all minimal element of D by $\min(D)$, then $f \leqslant g$ is equivalent to that $f(\alpha) < g(\alpha)$ for any $\alpha \in \min(D_{f,g})$.

If any factor set X_α, in the lexicographic product $X = \prod_{\alpha \in \mathfrak{A}} X_\alpha \langle \gamma_{\alpha,0} \rangle$ is homogeneous, then the order type of the resultant system X does not depend on the choice of the fixed element $\gamma_{\alpha,0}$ in X, and in this case the sign $\langle \gamma_{\alpha,0} \rangle$ in the lexicographic product can be omitted. Moreover in this case that each factor set is homogeneous, the resultant poset is also homogeneous.

If any factor set X_α is a homogeneous chain, and the index set $\mathfrak{A}$ is a chain, then the resultant poset is also a homogeneous chain.

(4.14) The following theorem follows from the propositions in (4.12), directly.

Theorem 8. For any homogeneous chain X, there exists a homogeneous chain $\bar{X}$ which is represented as a lexicographic product $\prod_{\alpha \in \mathfrak{A}} X_\alpha$ of simple homogeneous chains X_α, $\alpha \in \mathfrak{A}$, and X can be embedded in it as a subchain:

$$X \subset \overline{X} = \Pi_{\alpha \in \mathfrak{a}} \; X_\alpha$$

where $\mathfrak{a}$ is a chain isomorphic to the index chain of X, and each X_α is isomorphic to a factor chain of X.

In this decomposition it is easily seen that the regular interval in X corresponds to the set of functions $f \in X$, such that the values of $f(\alpha)$, $\alpha \in \mathfrak{b}$, are definite, where $\mathfrak{b}$ is a segment of $\mathfrak{a}$.

(4.15) At first the author expected that the conjecture in the following problem would be correct.

Problem 2. Does the identity

$$X = \Pi_{\alpha \in \mathfrak{a}} \; X_\alpha$$

hold, for any homogeneous chain X, and for its index chain $\mathfrak{a}$, and the factor chains X_α?

This problem seems to be equivalent to the following one.

Problem 3. For any homogeneous chain, does there exist a minimal proper regular interval?

But the author could not decided whether those expectations would be correct or false, and the identity in the Problem 2 is confirmed only in the following case.

Theorem 9. If in a homogeneous chain X, for any family of regular intervals with the finite-intersection-property, the regular intervals in it have a non-void intersection, then the chain X is a lexicographic product of its factor chains:

$$X = \Pi_{\alpha \in \mathfrak{a}} \; X_\alpha$$

Proof. We need only to prove that for any function f in the lexicographic product $\Pi_{\alpha \in \mathfrak{a}} X_\alpha$, there exists an $x \in X$, which corresponds to f.

Let $f \in \Pi_{\alpha \in \mathfrak{a}} X_\alpha$. For a $\beta \in D_f = \{\alpha \mid f(\alpha) \neq \gamma_{\alpha,0}\}$ we shall define a regular interval $\mathfrak{m}_\beta$ in the regular division β. Let β_1 be the minimal element of D_f (remark that the D_f satisfies the descending chain condition.), and select as $\mathfrak{m}_1$, the regular interval $\mathfrak{m}_1 \in \beta_1$, which contains x(1) (4.11), and let $\mathfrak{n}_1$ be the regular interval covered by $\mathfrak{m}_1$, such that $\Theta_{\mathfrak{m}_1}(\mathfrak{n}_1) = f(\beta_1)$ (4.11). If for any element β_ν in D_f less than a $\beta_\mu \in D_f$, the simple pair $[\mathfrak{m}_{\beta_\nu}, \mathfrak{n}_{\beta_\nu}]$ is defined in such a way that $\mathfrak{m}_\beta \subset \mathfrak{n}_\gamma$ for $\gamma < \beta$, then $\bigcap \mathfrak{n}_{\beta_\nu}$ is non-void regular interval by assumption, hence $\bigcap \mathfrak{n}_{\beta_\nu}$ intersects some co-class $\mathfrak{m}_{\beta_\mu}$ in $\beta_\mu \in D_f$. But since any $\mathfrak{n}_{\beta_\nu}$ contains $\mathfrak{m}_{\beta_\mu}$, $\mathfrak{n}_{\beta_\nu}$ contains $\mathfrak{m}_{\beta_\mu}$. We select as the regular interval in $\mathfrak{n}_{\beta_\nu}$ which $\bigcap \mathfrak{n}_{\beta_\nu}$ contains the element numbered with the least ordinal number in $\bigcap \mathfrak{n}_{\beta_\nu}$, and set $\mathfrak{n}_{\beta_\mu} = \Theta^{-1}_{\mathfrak{m}_{\beta_\mu}}(f(\beta_\mu))$ Thus we can inductively select a simple pair $[\mathfrak{m}_{\beta_\mu}, \mathfrak{n}_{\beta_\mu}]$ for any $\beta_\mu \in D_f$. $\bigcap \mathfrak{n}_{\beta_\mu}$, $\beta_\mu \in D_f$ is non-void. If $\bigcap \mathfrak{n}_{\beta_\mu}$ consists of only one element x, then set $x(f) = x$. If $\bigcap \mathfrak{n}_{\beta_\mu}$ is yet a proper interval, then set $x(f) = x(\tau)$, where τ is the least ordinal number, with which an element in $\bigcap \mathfrak{n}_{\beta_\mu}$ is numbered. (It is easily seen that the former or the latter case occurs, when D_f is cofinal to $\mathfrak{a}$ or not, respectively.)

Then, as easily seen, this correspondence $f \longrightarrow x(f) \in X$, is the converse one of the correspondence $x \longrightarrow f_x$, defined in (4.12). Hence the mapping $x \longrightarrow f_x$ ranges all over $\Pi_{\alpha \in \mathfrak{a}} X_\alpha$, and the proof is accomplished.

[1] G.Birkhoff; Lattice Theory. (1949).

[2] F.Hausdorff; Grundzüge der Mengenlehre. (1914).

[3] T.Ohkuma; On Discrete Homogeneous Chains. These Rep. pp.23-30 (1952).

(*) Received Oct. 6, 1952.