A NOTE ON FOURIER-STIELTJES INTEGRAL

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1. The following theorems concerning Fourier-Stieltjes integral are well known.

Theorem A (Bochner). Let $f(x)$ be a complex-valued, bounded continuous function defined over $(-\infty, \infty)$. Then the necessary and sufficient condition for $f(x)$ to be representable as $f(x) = \int e^{itx} dF(t)$ with a function of bounded variation $F(t)$ is to hold the following condition for every $x_1, \dots, x_n$ and complex numbers $c_1, \dots, c_n$:

$$(1) \quad \left|\sum_{\mu=1}^{n} c_\mu f(x_\mu)\right| \le M \sup_{-\infty<t<\infty} \left|\sum_{\mu=1}^{n} c_\mu e^{itx_\mu}\right|.$$

Theorem B (Schoenberg). In order that the bounded measurable function $f(x)$ can be represented as $f(x) = \int e^{itx} dF(t)$ a.e., it is necessary and sufficient that the following relation for every $\phi(x) \in L_1(-\infty, \infty)$ holds

$$(2) \quad \left|\int_{-\infty}^{\infty} f(x)\phi(x)dx\right| \le M \max_{-\infty<t<\infty} \left|\int_{-\infty}^{\infty} e^{itx}\phi(x)dx\right|$$

Recently, Bochner's theorem was extended over to bounded measurable functions by Phillips [3] . In this note, we shall prove these theorems from the view point of topological group theory. Already Bochner's theorem for locally compact abelian group has been obtained by M.Krein [4] , but the method of our proof is essentially different from that of his own.

2. In what follows, G is a locally compact abelian group, and $\hat G$ its character group, while $x, y, \dots$ are elements of G and $\hat x, \hat y, \dots$ characters of G. $(x, \hat x)$ denotes the value of $\hat x$ at x . $\hat\mu(\hat x)$ is a bounded Radon measure on $\hat G$, and dx , $d\hat x$ are Haar measures on G , $\hat G$ respectively. When a function is measurable or integrable with respect to Haar measure, we say simply measurable or integrable. Then we obtain:

Theorem 1. (Schoenberg). Let $f(x)$ be a measurable function on G , then the necessary and sufficient condition for $f(x) = \int (x,\hat x)\, d\hat\mu(\hat x)$ a.e. is the following relation for every $\phi \in L_1(G)$:

$$(3) \quad \left|\int_G f(x)\phi(x)dx\right| \le M \max_{\hat x\in\hat G} \left|\int_G (x,\hat x)\phi(x)dx\right|.$$

Proof. The necessity is obvious. For $\phi \in L_1(G)$, we define a new norm $\|\phi\| = \max_{\hat x\in\hat G}|\hat\phi(\hat x)|$, where $\hat\phi(\hat x) = \int (x,\hat x)\phi(x)dx$, then L_1 with this norm (denote this by $\mathfrak{L}_1$) is isometrically isomorphic to a subspace $\hat{\mathfrak{L}}_1$ of $L_\infty(\hat G)$.
On the other hand, a bounded linear functional defined on $\hat{\mathfrak{L}}_1$ can be expressed in the form $\int_{\hat G}\hat\phi(\hat x)d\hat\mu(\hat x)$ with a bounded Radon measure $\hat\mu(\hat x)$ and we have

$$\int_{\hat G}\hat\phi(\hat x)d\hat\mu(\hat x) = \int_{\hat G}\int_G (x,\hat x)\phi(x)\,dx\,d\hat\mu(\hat x)$$

$$= \int_G\int_{\hat G}(x,\hat x)\,d\hat\mu(\hat x)\,\phi(x)dx$$

$$= \int_G g(x)\,\phi(x)\,dx$$

where $\quad g(x) = \int_{\hat G}(x,\hat x)\,d\hat\mu(\hat x)$

So the bounded linear functional on $\mathfrak{L}_1$ is always written as $\int_G g(x)\phi(x)dx$, and on the other hand, the condition of the theorem shows that $\int_G f(x)\phi(x)dx$ is a bounded linear functional on $\mathfrak{L}_1$. Therefore, there exists $g(x)$ such as $\int_G f(x)\phi(x)\,dx = \int_G g(x)\phi(x)\,dx$ for all $\phi(x) \in L^1(G)$. Therefore, $f(x) = g(x)$ a.e. q.e.d.

3. Next, we introduce the following notations

V : a compact neighborhood of the unit of G.

$\hat V$: a compact neighborhood of the unit of $\hat G$.

e_V : a positive continuous function on whose support is in V and $\int e_V\,dx = 1$.

k_V : $k_V = e_V * e_V$ ($*$ shows convolution)

$\phi_V(\hat x)$: Fourier-transform of k_V, i.e. $\phi_V(\hat x) = \int (x,\hat x)\, k_V(x)\,dx$ so $k_V(x) = \int (x,\hat x)\,\phi_V(\hat x)\,d\hat x$

For corresponding functions on $\hat G$, we take out $\hat V$ in place of V.

Lemma 1. If $f(x)$ satisfies the following condition,

$$(4)\quad \Big|\sum_{\mu=1}^{n} c_\mu f(x_\mu)\Big| = M \sup_{\hat x \in \hat G} \Big|\sum_{\mu=1}^{n} c_\mu (x_\mu, \hat x)\Big|$$

for every $x_\mu \in G$ and complex number c_μ ($\mu = 1, 2, \cdots, n$), then $k_{\hat V}(x) \equiv \phi_{\hat V}(x)\cdot f(x)$ and $g_V \equiv k_V * f(x)$ satisfy it also.

Proof.

$$\Big|\sum_{\mu=1}^{n} c_\mu k_{\hat V}(x_\mu)\Big| = \Big|\sum_{\mu=1}^{n} c_\mu \int_{\hat G} (x_\mu, \hat y)\, k_{\hat V}(\hat y)\, d\hat y\, f(x_\mu)\Big|$$

$$\leq M \sup_{\hat x \in \hat G} \Big|\sum_{\mu=1}^{n} c_\mu \int_{\hat G} (x_\mu, \hat y)\, k_{\hat V}(\hat y)\, d\hat y\,(x_\mu, \hat x)\Big|$$

$$\leq M \sup_{\hat x \in \hat G} \Big|\sum_{\mu=1}^{n} c_\mu (x_\mu, \hat x)\Big|$$

Alike, $e_V * f(x)$ satisfies (4), so $k_V * f(x)$ and $f_{V\hat V}(x) \equiv k_V(x) * \phi_{\hat V} f(x)$ satisfy (4), too. q.e.d.

If $f(x)$ is measurable, we have

$$f_{V\hat V}(x) \equiv (\phi_{\hat V}\cdot f) * k_V$$

$$= \int_G \phi_{\hat V}\cdot f(y)\, k_V(y^{-1}x)\, dy$$

$$= \int_G \phi_{\hat V}\cdot f(y) \int_{\hat G} (y^{-1}x, \hat x)\,\phi_V(\hat x)\, d\hat x\, dy$$

$$= \int_{\hat G} (x, \hat x) \Big[\int_G \phi_{\hat V}\, f(y)\,(y^{-1}, \hat x)\, dy\Big]\phi_V(\hat x)\, d\hat x$$

$$= \int_{\hat G} (x, \hat x)\,\psi_{V\hat V}(\hat x)\, d\hat x$$

where

$$\psi_{V\hat V}(\hat x) \equiv \int_G \phi_{\hat V}\, f(y)\,(y^{-1}, \hat x)\, dy \cdot \phi_V(\hat x).$$

Then $\psi_{V\hat V}(\hat x)$ is a continuous and integrable function, so $\psi_{V\hat V}(\hat x)\,d\hat x$ is a bounded Radon measure on $\hat G$.

Lemma 2. $\Big|\int_{\hat G} \psi_{V\hat V}(\hat x)\, d\hat x\Big| \leq M$ for every V, $\hat V$.

Proof. From the above relation, we obtain

$$\Big|\sum_{\mu=1}^{n} c_\mu f_{V\hat V}(x_\mu)\Big| = \Big|\int_{\hat G} \sum_{\mu=1}^{n} c_\mu (x_\mu, \hat x)\,\psi_{V\hat V}(\hat x)\, d\hat x\Big|$$

$$\leq M \sup_{\hat x \in \hat G} \Big|\sum_{\mu=1}^{n} c_\mu (x_\mu, \hat x)\Big|.$$

So, for a continuous almost periodic function $p(\hat x)$ on $\hat G$, we have

$$\Big|\int_{\hat G} p(\hat x)\,\psi_{V\hat V}(\hat x)\, d\hat x\Big| \leq M \sup_{\hat x \in \hat G} |p(\hat x)|$$

As $\psi_{V\hat V} \in L_1(\hat G)$, for any $\varepsilon > 0$, there exists such a compact set $\hat K$ in $\hat G$ that $\int_{\hat G - \hat K} |\psi_{V\hat V}(\hat x)|\, d\hat x < \varepsilon$. Let $g(\hat x)$ be a continuous almost periodic function $\hat G$ such that $g(\hat x) = \overline{\psi_{V\hat V}(\hat x)}/|\psi_{V\hat V}(\hat x)|$ for $\hat x \in \hat K$ and $|g(\hat x)| \leq 1$, then the following relation holds:

$$\Big|\int_{\hat G} \psi_{V\hat V}(\hat x)\, d\hat x\Big| \leq \int_{\hat G} |\psi_{V\hat V}(\hat x)|\, d\hat x$$

$$\leq \int_{\hat K} |\psi_{V\hat V}(\hat x)|\, d\hat x + \varepsilon = \int_{\hat K} g(\hat x)\,\psi_{V\hat V}(\hat x)\, d\hat x + \varepsilon$$

$$\leq \Big|\int_{\hat G} g(\hat x)\,\psi_{V\hat V}(\hat x)\, d\hat x\Big| + \Big|\int_{\hat G - \hat K} g(\hat x)\,\psi_{V\hat V}(\hat x)\, d\hat x\Big| + \varepsilon$$

$$\leq M + 2\varepsilon$$

So the norm of bounded measure $\psi_{V\hat V}(\hat x)\,d\hat x$ is smaller than M. q.e.d.

Moreover, we obtain

$$|f_{V\hat V}(x) - f(x)|$$

$$= \left| \int_G h_V(y)\,\phi_{\vartheta}(y^{-1}x) f(y^{-1}x)\,dy - \phi_{\vartheta}(x) f(x) + \phi_{\vartheta}(x) f(x) - f(x) \right|$$

$$\leq \int_G h_V(y) \left| \phi_{\vartheta}(y^{-1}x) f(y^{-1}x) - \phi_{\vartheta}(x) f(x) \right| dy + \left| \phi_{\vartheta}(x) f(x) - f(x) \right| \to 0$$

(uniformly on compact sets as $V \to e$, $\vartheta \to \hat{e}$).

As $f_{V\vartheta}(x)$ is Fourier-transform of a bounded measure, by Theorem 1,

$$\left| \int_G g(x) f_{V\vartheta}(x)\,dx \right| \leq M \max_{\hat{x} \in \hat{G}} \left| \int_G (x, \hat{x})\, g(x)\,dx \right|$$

for every $g(x) \in L_1(G)$ Therefore,

$$\left| \int_G g(x) f(x)\,dx \right| \leq M \max_{\hat{x} \in \hat{G}} \left| \int_G (x, \hat{x})\, g(x)\,dx \right|$$

so by Theorem 1, $f(x) = \int_{\hat{G}} (x, \hat{x})\, d\hat{\mu}(\hat{x})$ a.e.

Then we obtain the following theorem.

Theorem 2 (Bochner-Phillips).

Every bounded measurable function $f(x)$ satisfying (4) is representable with a bounded Radon measure $\hat{\mu}$ on $\hat{G}$ in the following form :

$$f(x) = \int (x, \hat{x})\, d\hat{\mu}(\hat{x}) \quad a.e.$$

If $f(x)$ is continuous, the equality holds for all points.

(*) Received May 12, 1952.

1 S.Bochner, A Theorem on Fourier-Stieltjes integrals. Bull. Amer. Math. Soc., 40(1943), pp.271-276.

2 I.J.Schoenberg, A Remark on the preceding note by Bochner, Bull. Amer. Math. Soc., 40(1943), pp.277-278.

3 R.S.Phillips, On Fourier-Stieltjes integrals, Trans. Amer. Math. Soc., 69(1950), pp.312-323.

4 M.Krein, A Ring of Functions on a topological space, C. R.(Doklady), 29(1940), pp. 275-280.

5 H.Cartan-R.Godement, Analyse harmonique et théorie de la dualité dans les groupes abéliens localement compacts., Ann. Ecole Norm. (1947).

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These reports. No.1 (1952), p. 25.

The five lines in the bracket at the end of § 1 should be deleted, since the axiom of choice is necessary in the last two theorems.