MAXIMAL CONTINUATION OF A RIEMANN SURFACE

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Let F be a Riemann surface. If F can be mapped one-to-one conformally on a proper part F' of another Riemann surface Φ, then Φ is called a continuation of F. If there exists no such a continuation, then F is called maximal.

Bochner[1] proved:

<u>Theorem. For any Riemann surface, there exists a maximal continuation.</u>

Bochner uses the selection axiom in his proof. Heins[2] proved Bochner's theorem without using the selection axiom. I shall simplify a little Heins' proof in the following lines.

<u>Proof.</u> In the beginning, we remark the following. If Φ is a continuation of F, then we enlarge Φ to a Riemann surface Φ^* in the following way. Namely Φ^* is a covering surface of Φ, such that a closed curve on Φ^* is (on Φ) homotop null or homotop to a closed curve on F'. Then $\Phi^* \supset \Phi$ and Φ^* is a continuation of F. In the following, a continuation Φ of F means always the thus enlarged surface. Let F be a Riemann surface spread over the z-plane and $z = 0$ be contained in F and different from its branch point. Let Φ be a continuation of F, spread over the w-plane and let $z=0$ be mapped on $w=0$ by $w=f(z)$ $(f(0)=0)$. We map the universal covering surface of F on $|\zeta_0| < 1$. Then we obtain a Fuchsian group $\mathfrak{G}(F)$ in $|\zeta_0| < 1$. We map the universal covering surface of Φ on $|\zeta| < R$ by $\zeta = h(w)$ $(h(0)=0)$, where R is determined by the condition:

$$\varphi(0) = 0, \quad \varphi'(0) = 1, \tag{1}$$

where $\varphi(z) = h(f(z))$.

Let $\mathfrak{G}(\Phi)$ be the Fuchsian group corresponding to $h^{-1}(\zeta)$ in $|\zeta| < R$. Then by the remark in the beginning, $\mathfrak{G}(F)$ is mapped on $\mathfrak{G}(\Phi)$ homomorphically, such that to an element of $\mathfrak{G}(F)$, there corresponds an element of $\mathfrak{G}(\Phi)$, but an element of $\mathfrak{G}(F)$, which is the identity, may correspond to the identity of $\mathfrak{G}(\Phi)$. By $\zeta = \varphi(z)$, F is mapped conformally on a proper part of the fundamental domain of $\mathfrak{G}(\Phi)$.

Let

$$S_\nu^0 : \zeta_0' = \frac{e^{i\theta_\nu^0}(\zeta_0 - a_\nu^0)}{1 - \bar{a}_\nu^0 \zeta_0} \quad (\nu = 1, 2, \cdots) \tag{2}$$

be an element of $\mathfrak{G}(F)$, then by homomorphism, S_ν^0 corresponds to an element S_ν of $\mathfrak{G}(\Phi)$:

$$S_\nu : \zeta' = e^{i\theta_\nu} \frac{R^2(\zeta - a_\nu)}{R^2 - \bar{a}_\nu \zeta} \quad (|a_\nu| < R)$$
$$(\nu = 1, 2, \cdots). \tag{3}$$

We assume that F does not admit a Riemann sphere or a closed Riemann surface of genus 1 as its continuation. We consider all continuations Φ of F and let

$$\sup R = R_0 . \tag{4}$$

Then there exist continuations Φ_n $(n = 1, 2, \ldots)$ of F, such that $R_n \to R_0$ and let $\varphi_n(z)$ ($\varphi_n(0) = 0$, $\varphi_n'(0) = 1$) be the corresponding functions.

Let a schlicht disc $|z| < \rho$ be contained in F. Since $\varphi_n(z)$ is schlicht in $|z| < \rho$, by Koebe's theorem, $\varphi_n(z)$, $\varphi_n'(z)$, $1/\varphi_n'(z)$ are uniformly bounded in $|z| \leqq \rho_1 < \rho$. Since $\varphi_n(z)$ is locally schlicht on F, we see easily that $\varphi_n(z)$ is uniformly bounded in any compact domain on F.

Hence we can find a partial sequence, which we denote again $\varphi_n(z)$, such that

$$\lim_n \varphi_n(z) = \varphi(z), \; \varphi(0) = 0, \; \varphi'(0) = 1 \tag{4}$$

converges uniformly in the wider sense on F. $\varphi(z)$ is schlicht on F.

Let

$$S_\nu^{(n)} : \zeta' = e^{i\theta_\nu^{(n)}} \frac{R_n^2(\zeta - a_\nu^{(n)})}{R_n^2 - \bar{a}_\nu^{(n)}\zeta}$$

$$= e^{i\theta_\nu^{(n)}} \frac{(\zeta - a_\nu^{(n)})}{1 - \frac{\bar{a}_\nu^{(n)}}{R_n} \cdot \frac{\zeta}{R_n}} \quad (|a_\nu^{(n)}| < R_n) \quad (\nu = 1, 2, \cdots) \tag{5}$$

be the element of $\mathfrak{G}(\Phi_n)$, which corresponds to (2) by homomorphism. By (4), se see that

$$\lim_n a_\nu^{(n)} = a_\nu \quad (\nu = 1, 2, \cdots) \tag{6}$$

exists and we may assume, by taking a suitable partial sequence, that

$$\lim_n \theta_\nu^{(n)} = \theta_\nu \quad (\nu = 1, 2, \cdots) \tag{7}$$

exists.

If $R_0 = \infty$ and $R_n \to \infty$, then by (5), (6), (7),

$$S_\nu^{(n)} \to S_\nu : \zeta' = e^{i\theta_\nu}(\zeta - a_\nu) \quad (\nu = 1, 2, \cdots)$$

Since $\mathfrak{G}(F)$ has no fixed points, $\theta_\nu = 0$, so that

$$S_\nu : \zeta' = \zeta - a_\nu \quad (\nu = 1, 2, \cdots). \tag{8}$$

By Koebe's theorem, the image of $|z| < \rho$ by $\varphi_n(z)$ contains a disc $|\zeta| < \rho/4$, hence the group $\mathfrak{G}$ generated by S_ν is properly discontinuous. Hence $\mathfrak{G}$ is either the identity, or a simply periodic group of translations of a doubly periodic group of translations. Since F is mapped conformally on a part of the fundamental domain of $\mathfrak{G}$, F admits the Riemann sphere or a closed Riemann surface of genus 1 as its continuation, which contradicts the hypothesis. Hence $R_0 < \infty$, so that by (5),

$$S_\nu^{(n)} \to S_\nu : \zeta' = e^{i\theta_\nu} \frac{R_0^2(\zeta - a_\nu)}{R_0^2 - \bar{a}_\nu \zeta}$$

$$(|a_\nu| < R_0) \quad (\nu = 1, 2, \cdots). \tag{9}$$

Since the group $\mathfrak{G}$ generated by S_ν is properly discontinuous, let D be its fundamental domain, then F is mapped by $\zeta = \varphi(z)$ conformally on a part of D. D can be considered as a Riemann surface Φ, so that Φ is a continuation of F.

We shall prove that Φ is maximal. Suppose that Φ is not maximal and can be mapped conformally on a proper part of another Riemann surface Φ_1. As before, we map the universal covering surface of Φ_1 on $|\zeta_1| < R_1$ and let $\varphi_1(z)$ ($\varphi_1(0) = 0$, $\varphi_1'(0) = 1$) be the corresponding function defined by (1) for Φ_1. Then $|\zeta| < R_0$ is mapped on a proper part of $|\zeta_1| < R_1$, which is the image of Φ. Let $\zeta_1 = h(\zeta)$ ($h(0) = 0$) be the mapping function, then by Schwarz's lemma, $|h'(0)| < R_1/R_0$. Since $\varphi_1(z) = h(\varphi(z))$ and $\varphi'(0) = h'(0)\varphi'(0)$, $\varphi_1'(0) = 1$, $\varphi'(0) = 1$, we have $h'(0) = 1$, so that $1 < R_1/R_0$, or $R_1 < R_0$, which contradicts the definition of R_0. Hence Φ is maximal.

(*) Received May 17, 1952.

1) S.Bochner: Fortsetzung Riemannscher Flachen. Math. Ann. 98 (1927).
2) M.H.Heins: On the continuation of a Riemann surface. Annals of Math. 43 (1942).

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