A NOTE ON THE MAXIMAL IDEALS OF ANALYTIC FUNCTIONS

By Sin HITOTUMATU

1. Introduction. At the meeting of the Mathematical Society of Japan on 21 Feb., 1952, Mr. J. Igusa has remarked that the algebraic structure of the ideals of analytic functions in a domain of regularity determines perfectly the analytic structure of the domain D, using the ideal-theory of analytic functions due to Messrs. H.Cartan and K.Oka [2]. Suggested by him, we shall consider the maximal ideals of analytic functions of several complex variables in this Note. From the atomic structure of the space, or suggested by the theory of normed-ring, one can easily conjecture that the maximal ideals will correspond to the points of the domain. In the case of the domain of regularity, this is certainly true.

On the other hand, the author has once been asked by Mr. O.Kôta, whether a closed ideal in a domain of regularity has finite basis or not. I do not know yet its answer. Here we give a result concerning this for maximal ideals, even though it were but a very special case.

2. Notations and Lemmas used later. We denote by $\mathcal{O}$ or more precisely by $\mathcal{O}_D$ the domain of integrity consisting of all the holomorphic functions in a domain D in the space of n complex variables $z_1, \dots, z_n$. For simplicity's sake, we assume that the domain D is always finite and univalent.

We shall use a few terminologies concerning the ideals such as the ponctual ideal generated by the ideal $\mathcal{J}$ at a point a, the closure of an ideal, etc., after mainly H.Cartan [2]. An ideal $\mathcal{J}$ in $\mathcal{O}$ is said to be maximal if $\mathcal{J} \neq \mathcal{O}$ and if there is no ideal $\mathcal{O}\!\!\!/$ such that $\mathcal{J} \subsetneqq \mathcal{O}\!\!\!/ \subsetneqq \mathcal{O}$.

We use the symbol $[f_1, f_2, \dots]$ for the ideal generated by the functions $f_1, f_2, \dots$. Further we denote by $\mathfrak{p}_a$ the ideal consisting of all the functions in $\mathcal{O}_D$ vanishing at a point $a \in D$. It is evident that $\mathfrak{p}_a$ is a closed maximal ideal in $\mathcal{O}_D$.

For later use, we assume the following:

Theorem 1. A domain D is a domain of regularity if and only if every closed ideal (especially ideal with finite basis) [3] $\mathcal{J}$ in $\mathcal{O}_D$ satisfies the following condition $\mathfrak{G}$:

$\mathfrak{G}$: If a function $f \in \mathcal{O}_D$ belongs to every ponctual ideal $(\mathcal{J})_a$ generated by $\mathcal{J}$ at every point $a \in D$, then f belongs to $\mathcal{J}$ itself.

The necessity is one of the main Theorems in the theory of ideals of several complex variables.[4] The sufficiency is evident, because if D is not a domain of regularity, there exists a point $b = (b_1, \dots, b_n)$ out of D, satisfying that every function of $\mathcal{O}_D$ is also holomorphic at b. Then the function $f = 1$ and the ideal $\mathfrak{p}_b$ do not satisfy the condition $\mathfrak{G}$.

Corollary 1. Suppose that D is a domain of regularity. If a system of finite number of functions $f_1, \dots, f_m \in \mathcal{O}_D$ have no common zero-point in D, we have $[f_1, \cdots, f_m] = \mathcal{O}_D$.

Proof. Use the condition $\mathfrak{G}$ for the ideal $\mathcal{J} = [f_1, \dots, f_m]$ and $f = 1$.

Corollary 2. (Weil-Hefer)[5] Suppose that D is a domain of regularity. For every point $a = (a_1, \dots, a_n) \in D$, we have

$$\mathfrak{p}_a = [z_1 - a_1, \cdots, z_n - a_n].$$

Proof. Use the condition $\mathfrak{G}$ for the ideal $\mathcal{J} = [z_1 - a_1, \dots, z_n - a_n]$ and f taken arbitrary from the ideal $\mathfrak{p}_a$.

3. We shall prove the following:

Theorem 2. Suppose that D is a domain of regularity. A maximal ideal $\mathfrak{J}$ in $\mathcal{O}_D$ with at most countable basis [6] is closed.

Proof. If it were not so, the closure $\bar{\mathfrak{J}}$ of $\mathfrak{J}$ must be $\mathcal{O}$, for otherwise, we have $\mathfrak{J} \subsetneq \bar{\mathfrak{J}} \subsetneq \mathcal{O}$ contradicting the maximality of $\mathfrak{J}$. Hence we may assume that $\mathfrak{J} \subsetneq \bar{\mathfrak{J}} = \mathcal{O}$. Take a countable basis $f_1, f_2, \ldots$ of $\mathfrak{J}$. Let F_j be the collection of all the zero-points of f_j in D. F_j are relatively closed sets in D, and they have the finite-intersection-property, for otherwise, we have $\mathfrak{J} = \mathcal{O}$ by Corollary 1. On the other hand, we have $\bigcap_{j=1}^{\infty} F_j = \emptyset$, for otherwise, we have $\bar{\mathfrak{J}} \neq \mathcal{O}$ by Theorem 1. Then the closed sets $F_1 \cap \ldots \cap F_m$ $(m = 1,2,\ldots)$ never stay in a compact subset of D. Hence we have a sequence of points $p_1, p_2, \ldots$ which has no limit-point in the interior of D and such that $p_m \in F_1 \cap \cdots \cap F_m$. The set $\bigcup_{m=1}^{\infty} \{p_m\}$ is then, an analytic variety in D, and so, we have a holomorphic function $g \in \mathcal{O}_D$ such that $g(p_{2k}) = 0$ and $g(p_{2k-1}) = 1$ $(k = 1,2,\ldots)$ by the first Cousin problem for an analytic variety.[7] Now, since an arbitrary element h of $\mathfrak{J}$ is a linear combination of finite number of functions taken from $f_1, f_2, \ldots$ over $\mathcal{O}$, h must vanish on $\{p_m\}$ up to a finite number of exceptions, which means that $g \notin \mathfrak{J}$. Putting $\mathfrak{A} \equiv [\mathfrak{J}, g]$, we have $\mathfrak{J} \subsetneq \mathfrak{A}$. On the other hand, every element of $\mathfrak{A}$ must vanish at $\{p_{2k}\}$ up to a finite number of exceptions which implies that $1 \notin \mathfrak{A}$. This means that $\mathfrak{J} \subsetneq \mathfrak{A} \subsetneq \mathcal{O}$ contradicting the maximality of $\mathfrak{J}$. Thus our Theorem is proved.

4. We now reach our finil result:

Theorem 3. The following four conditions are equivalent for an ideal $\mathfrak{J}$ in $\mathcal{O}_D$, when D is a domain of regularity.

A) $\mathcal{O}/\mathfrak{J} = \mathbb{C}$, where $\mathbb{C}$ means the complex-number-field.

B) $\mathfrak{J}$ is maximal with at most countable basis.

B') $\mathfrak{J}$ is maximal and closed.

C) There exists (one and only one) point $a \in D$ such that $\mathfrak{J} = \mathfrak{p}_a$.

Proof. It is evident that C) implies A), B), and B'). We have already proved in Theorem 2 that B) implies B').

A) implies C) (by Mr. J.Igusa). By our assumption, we have constants $a_j \in \mathbb{C}$ such that $z_j - a_j \in \mathfrak{J}$ $(j = 1,\ldots,n)$. The point $a \equiv (a_1,\ldots,a_n)$ belongs to D, for otherwise, the ideal $\mathfrak{p} \equiv [z_1 - a_1, \ldots, z_n - a_n] \subseteq \mathfrak{J}$ is $\mathcal{O}$ by Corollary 1, contradicting the assumption A). By Corollary 2, we have $\mathfrak{p} = \mathfrak{p}_a \subseteq \mathfrak{J} \subsetneq \mathcal{O}$, and then, by the maximality of $\mathfrak{p}_a$, we have $\mathfrak{p}_a = \mathfrak{J}$ which proves C).

B') implies A). Since $\mathfrak{J}$ is closed and is not $\mathcal{O}$, there must exist at least one point a on which all the functions of $\mathfrak{J}$ vanish. The zero-point of $\mathfrak{J}$ must be only one point by the maximality, which reduces the condition C). But we shall prove A) directly.

Since $\mathfrak{J}$ does not contain any constant other than 0, we have $\mathcal{O}/\mathfrak{J} \supseteq \mathbb{C}$. If $\mathcal{O}/\mathfrak{J} \supsetneq \mathbb{C}$, we have a function $g \in \mathfrak{J}$ such that $g - c \notin \mathfrak{J}$ for every $c \in \mathbb{C}$. Hence the ideal $\mathfrak{A} \equiv [\mathfrak{J}, (g - g(a))]$ is different from $\mathfrak{J}$, but it does not coincide with $\mathcal{O}$, since every element of $\mathfrak{A}$ must vanish at a. Therefore we have $\mathfrak{J} \subsetneq \mathfrak{A} \subsetneq \mathcal{O}$ contradicting the maximality of $\mathfrak{J}$. Thus our Theorem is perfectly proved.

Corollary. If D is a domain of regularity, every closed and maximal ideal in $\mathcal{O}_D$ has finite basis.

Remark. The condition A) implies the maximality of $\mathfrak{J}$ for an arbitrary domain. Because, if there exists an ideal such that $\mathfrak{J} \subsetneq \mathfrak{A} \subseteq \mathcal{O}$, we have a function $g \in \mathfrak{A}$ and a constant $c \in \mathbb{C}$, such that $g \notin \mathfrak{J}$, $g - c \in \mathfrak{J} \subset \mathfrak{A}$, which implies that $0 \neq c \in \mathfrak{A}$, and then $\mathfrak{A} = \mathcal{O}$.

Added in Proof. In a paper of L.Schwartz [1], similar results are described for the ideals of finitely or infinitely differentiable functions.

(*) Received Feb. 27, 1952.

1) Theorem 3 later, and further considerations.
2) H.Cartan [1], [2] ; K.Oka [1], [2].
3) Cf. H.Cartan [2], Theorem 11.
4) H.Cartan [2], Theorem 4 ter. See also K.Oka [1], [2], Problem C_1.
5) Described in Weil [1] and proved first by Hefer [1]. See also Hitotumatu [1].
6) I believe that this additional condition will be inessential.
7) H.Cartan [2], Theorem 8 ter.

Cartan, H. [1]. Idéaux de fonctions analytiques de n variables complexes, Ann. École Norm. Sup. (3) 61 (1944), 149-197.

- - -, [2]. Idéaux et modules de fonctions analytiques de variables complexes, Bull. Soc. Math. France 78 (1950), 29-64.

Hefer, H. [1]. Ueber die Zerlegung analytischer Funktionen und die Weilsche Integraldarstellung, Math. Ann. 122 (1950), 276-278.

Hitotumatu, S. [1]. On the possibility of the Weil's integral representation. Proc. Japan Acad. 27 (1951), 279-281.

Oka, K. [1]. Sur les fonctions analytiques de plusieurs variables complexes, VII. Sur quelque notions arithmétiques. Bull. Soc. Math. France 78 (1950), 1-27.

- - -, [2]. - - - - -, VIII. Lemma fundamental, Jour. of Math. Soc. Japan 3 (1951), 204-214; ibid. 259-278.

Schwartz, L. [1]. Analyse et synthèse harmoniques dans les espaces de distributions, Canadian Jour. of Math. 3 (1951), 503-512.

Weil, A. [1]. L'intégrale de Cauchy et les fonctions de plusieurs variables, Math. Ann. 111 (1935), 178-182.

Department of Mathematics,
Tokyo University.