LINEARIZATION OF TOPOLOGICAL GROUPS AND ORDERED RINGS

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It has recently been solved by K.Iwasawa [1] the problem of linearization of an abstract groups. In this note we shall investigate the problem of linearization of topological groups. We say that a topological group G can be linearized, if we can introduce such a linear order in G that G becomes a linearly ordered group and that the topology of G is equivalent to the intrinsic topology by this order. Then, using the concept of an ordered ring we characterize the field of real numbers and the ring of integers.

1. Since the linearly ordered groups are at most one-dimensional, we restrict ourselves to topological groups whose dimensions are at most one.

Theorem 1. A one-dimensional topological group G can be linearized if and only if G satisfies the following conditions:

1) the component C of the unit is open and isomorphic to the additive group of real numbers with the usual topology,

2) the set corresponding to the positive numbers in C is self-conjugate in G,

3) G/C can be linearized as an abstract group.

Remark. In the above conditions we can replace 1) by the following: G is a non-compact, locally connected, locally compact and connected topological group satisfing the first countability axiom [Cf. [2] Theorem 3.3] . If G satisfies the condition 1) and G is an abelian group without elements of finite order, then G/C satisfies the condition 3). In fact it is sufficient to prove that G/C has no element of finite order [3] and if there exists some integer n such that $nX = E$ for an element X, where E is a unit element and X an element of G/C, then, since X is a coset of G by C, any element of X can be represented in the form $x_1 + c_1$, $x_1 \notin C$ and $c_1 \in$ C. By the assumption, we have $nx_1 + nc_1 \in C$, i.e., $nx_1 \in C$. On the other hand C is isomorphic to the additive group of real numbers and so for any integer n and any element x there exists always only one element y such that $ny = x$. Hence by the assumption that G has no element of finite order, x_1 belongs to C. This is a contradiction.

Proof of Theorem 1. Necessity. The condition 1) and 2) follows from Theorem 3.3 in [2] and C is isomorphic to the additive group R of real numbers. Hence all elements corresponding to the positive numbers of R are positive in the order in G (if negative, we have only to invert the order in G). Therefore, by the property of linearly ordered group the condition 3) is satisfied.

Sufficiency. Since the topological consideration is not necessary by the condition 1), it follows from Lemma 5 in [1] .

Using Iwasawa's result for an abstract group, we immediately see that a zero-dimensional topological group G can be linearized, if G satisfies the conditions of Theorem 2 in [1] and every element of the family of subgroups $\{G_\lambda ; \lambda \in L\}$ of G indexed by the linearly ordered set L in that theorem is open and $\{G_\lambda\}$ is a base of the neighbourhoods of unit. (This is a necessary and sufficient condition).

2. Ordered rings.

Definition. A set G is called an ordered ring if

1) G is a ring with the unit element e with respect to the multiplication,

2) G is a linearly ordered group with respect to the addition,

3) $a > 0$, $b > 0 \longrightarrow ab > 0$ (0 is the unit in the addition).

In particular, if G is commutative, it is called a commutative ordered ring. An ordered ring G is called an ordered field if it is a field. We have the following properties for an ordered ring G [Cf. [4]].

i. G has no null-factor.

ii. $e > 0$

iii. If b has the left inverse x, then x is also the right inverse of b and conversely. (x is called the inverse of b and denoted by b^{-1})

iv. $a > c,\ b > d \longrightarrow ab > cd$

v. (An element a of G is called positive atomic if

α) $a > 0$,

β) $a \geq b \geq 0 \longrightarrow b = 0$ or $a = b$.)

If G has a positive atomic element a, then $a = e$.

Theorem 2. If an ordered ring G with a strong unit (Cf. [3]) has a positive atomic element, G is isomorphic to the ring of integers.

Proof. Since e is at once a strong unit and a positive atomic element, G is archimedean. If $ke \leq x \leq (k+1)e$ where k is any positive integer, then we have $x = ke$ or $x = (k+1)e$. Therefore any element x can be written in the form ke, thus our assertion is proved.

Remark. The assertion of Theorem 2 is not true when G has not a strong unit. Let G be the ring of polynomials with integral coefficients (in one variable x and with the usual rule of the addition and multiplication). The order in G is defined in the following manner: $f(x) = \sum_{i=0}^{n} a_i x^i \geq 0 \ (a_n \neq 0) \longleftrightarrow a_n \geq 0$. Then 1 is a unit but not a strong unit and G is not isomorphic to the ring of integers.

Next we consider an ordered ring G as a topological group with the intrinsic topology [Cf. [2]].

Theorem 3. If in a one-dimensional ordered ring G, the product operation is continuous in each of variables, then G is (topologically) isomorphic to the field of real numbers with the usual topology.

Proof. The component C of G is an open invariant subgroup isomorphic to the additive group of real numbers with the usual topology. Moreover C is an ideal. To prove this, we consider ga, $g \in G$, $a \in C$, if we take an element $b \in C$, sufficiently near to 0 small, we have $gb \in U(0)$ for some neighbourhood U(0) of 0 (since ga is continuous with respect to a) and $U(0) \subset C$. Since C is connected, a is written in the form $a = b_1 + \cdots + b_n$, $b_i \in U(0)$, $i = 1, 2, \cdots, n$. Hence $ga = gb_1 + gb_2 + \cdots + gb_n \in U(0) + U(0) + \cdots + U(0) \subset C$. Thus C is an ideal. Next we consider the mapping $f: x \longrightarrow ax$ for a fixed element $a \in C$, $a > 0$, $x \in C$. First of all, f is 1-1 mapping. For $ax = ay$ implies $a(x - y) = 0$, i.e. we have $x = y$ by i). f preserves the order (by iv)). Moreover, since f is continuous, $f(C)$ is connected and contained in C. Next we shall prove that f is a mapping from C onto itself. For assume that $p \in C$ and $p \notin f(C)$, then $p < q$ implies $q \notin f(C)$, because if $q \in f(C)$ $f(C)$ is not connected, but it is impossible. Thus the mapping is "onto". Therefore there exists such an element x that $ax = a$, $x \in C$. On the other hand $ae = a$, hence $ax - ae = a(x-e) = 0$ i.e., by i) $x = e$. Thus C contains the unit e, so $C \supset G$ and G is a field. Thus G is a connected, locally compact and metric separable topological field, and by the Pontrjagin's Theorem G is (topologically) isomorphic to the field of real numbers [5]. (The continuity of the product xy in both variables x and y, and of x^{-1} follows from [6]).

Since a commutative ordered ring has no null-factor, we can consider the quotient field G^* of G. Now we shall introduce a linear order in it. First, we define the symbol ε^p, $p \in G$, in the following way:

$$\varepsilon^p a = \begin{cases} a & \text{for } p > 0 \\ -a & \text{for } p < 0 \end{cases} \quad (a \in G)$$

Definition.

$$(a, b) > (c, d) \longleftrightarrow \varepsilon^{ab}|a||d| > \varepsilon^{cd}|c||b|$$

In this definition, since $\varepsilon^{ab} = \varepsilon^{(-a)(-b)}$ and $\varepsilon^{ab} = \varepsilon^a \varepsilon^b$, we can assume, without the loss of generality, that $b > 0$ and $d > 0$. Hence we have $(a, b) > (c, d) \longleftrightarrow \varepsilon^a |a| d > \varepsilon^c |c| b$. This order relation satisfies the transitivity. For if $(a, b) > (c, d)$ and $(c, d) > (p, q)$, then $(a, b) > (p, q)$, since $\varepsilon^c |c| q > \varepsilon^p |p| d$ and $\varepsilon^a |a| d > \varepsilon^c |c| b$ imply $\varepsilon^a |a| q > \varepsilon^p |p| b$. Moreover, if $(a, b) > (p, q)$, we have for any element (c, d) of an equivalent class to (a, b), $(c, d) > (p, q)$ and $(a, b) = (c, d)$ (the latter part is obvious). For since $\varepsilon^a |a| q > \varepsilon^c |p| b$ and $ad = bc$ ($b > 0$, $d > 0$), we have $\varepsilon^a = \varepsilon^c$, $|a| d = b |c|$ and $\varepsilon^c |c| q b = \varepsilon^a |c| b q = \varepsilon^a |a| d q > \varepsilon^p |p| d b$, therefore $\varepsilon^c |c| q > \varepsilon^p |p| d$. By this order G^* becomes a commutative ordered field. In fact, for $(a, b) > (c, d)$ and any element (x, y) of G, we shall have $(a, b) - (c, d) = (a, b) - (x, y) - (c, d) + (x, y) > 0$ if the following relation can be proved:

$$(*) \qquad (a, b) > (c, d) \longleftrightarrow (a, b) - (c, d) = \frac{ad - bc}{bd} > 0$$

Next, for $(a, b) > 0$ and $(c, d) > 0$, we have $(a, b)(c, d) = (ac, bc) > 0$; in fact, $\varepsilon^a |a| b > 0$ and $\varepsilon^c |c| d > 0$ imply $\varepsilon^{ac} |ac| bd = \varepsilon^a \varepsilon^c |a||c| bd > 0$.

The proof of ($*$). It is sufficient to prove that $\varepsilon^{ad-bc} |ad - bc| > 0$ is equivalent to $\varepsilon^a |a| d > \varepsilon^c |c| b$. If $\varepsilon^{ad-bc} |ad - bc| > 0$, then $ad > bc$ and $b > 0$, $d > 0$. Hence the following cases are considered:

1) if $a > 0$, $c > 0$, then
$$\varepsilon^a |a| d > \varepsilon^c |c| b,$$
2) if $a < 0$, $c > 0$, then since $|a| d > b |c|$ we have
$$\varepsilon^a |a| d > \varepsilon^c |c| b,$$
3) if $a > 0$, $c < 0$, then we have $\varepsilon^a |a| d > \varepsilon^c |c| b$.

Conversely, if $\varepsilon^a |a| d > \varepsilon^c |c| b$, we have the following cases:

1) $a < 0$, $c < 0$ and $ad > bc$

2) $a > 0$, $c > 0$ and $|a| d > |c| d$

3) $a > 0$, $c < 0$ and $ad > bc$.

We can easily prove for each case that $ad - bc > 0$. Thus we have the following theorem

Theorem 4. A commutative ordered ring G is embedded in the ordered field G^*, the order relation being preserved.

Corollary. A commutative ordered ring G is archimedean, if and only if G^* has a strong unit.

Proof. If the unit (e, e) of G^* is a strong unit, then for any $(a, c) > 0$, there exists an integer n such that $(a, c) < n(e, e) = (ne, e)$. This shows that $ne \cdot c > ae$, i.e. $nc > a$. Hence G is archimedean. Conversely, if G is archimedean, for any element a and c, $a > c > 0$, there exists an integer n such that $nc > a$, this shows that $(a, c) > 0$ and $(a, c) < n(e, e)$.

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