ON DISCRETE HOMOGENEOUS CHAINS

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Introduction. The object of the present paper is to determine some types of discrete homogeneous chains.

Because all discrete homogeneous chains correspond one-to-one to all homogeneous chains, as will be shown later, it is impossible to classify and determine the types of all descrete homogeneous chains, unless the types of all homogeneous chains are determined. So the author only determined the special type of discrete homogeneous chains, that is, absolutely discrete homogeneous chains, which will be defined later on.

The same definitions and notations as in [1] are employed, but concerning the ordinal power, those in the author's paper [2] are employed.

In §1, general homogeneous chains are investigated.

In §2, the construction of general discrete homogeneous chains is studied, and later the absolute discreteness is defined.

In §3, some examples of absolutely discrete homogeneous chains are investigated.

In §4, we shall see that every absolutely discrete homogeneous chain is after all one of examples mentioned in §3, and then the type is determined.

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1. On homogeneous chains.

(1.1) We use the same definitions and notations as in [1], unless otherwise mentioned, but concerning the definition of ordinal power, we use the following one, on which the author has studied in a previous paper [2].

Definition I. Let X and Y be posets, and y_0 be a fixed element of Y. The ordinal power $^X Y\langle y_0\rangle$ consists of all functions $f(x) = y$ from X to Y, such that 'the set $\{x \mid f(x) \neq y_0\}$ satisfies the descending chain condition', where $f \leqslant g$ means that for each $x \in X$ such that $f(x) \nleq g(x)$, there exists an $x' < x$ such that $f(x') < g(x')$.

Based on this definition, we get a poset $^X Y\langle y_0\rangle$ without any restriction on the original posets, such as the descending chain condition on X.

If Y is homogeneous, then the structure of $^X Y\langle y_0\rangle$ does not depend on the choice of y_0, and the sign $\langle y_0\rangle$ can be omitted. In the case when Y is homogeneous, the resultant set $^X Y$ is also homogeneous. In the case when both X and Y are chains, the set $^X Y\langle y_0\rangle$ is also a chain.

About those fact, see the previous paper [2].

(1.2) Definition 2. If a chain X has a transitive automorphism group, we call X homogeneous.

Theorem 1. If X and Y are homogeneous chains, then $X \circ Y$ (ordinal product, cf. [1], p.9) is a homogeneous chain. If X is a chain, and Y is a homogeneous chain, then $^X Y$ is a homogeneous chain.

These propositions are corellaries of Theorem II and III of [2].

Note I. Let $Z = X \circ Y$. The homogeneity of X and Y implies that of Z. But neither the homogeneity of Z and X nor that of Z and Y implies that of the rest. Z may be homogeneous when neither X nor Y is homogeneous. If Z and

Y are homogeneous, then Y is a homogeneous interval (which will be defined later on) of Z. But the existence of a homogeneous interval Y of a homogeneous chain Z does not imply the existence of a chain X such that $Z = X \circ Y$.

Example I. Let S be the homogeneous chain of all real numbers, J be that of all integers, and S^+ be the chain of all real numbers, which are equal to or greater than zero. Then $J \circ S^+$ is a homogeneous chain isomorphic to S, but S^+ is not homogeneous.

Let 2 be the 2nd ordinal number, and R be the homogeneous chain of all rational numbers, then $2 \circ R$ is a homogeneous chain isomorphic to R. But 2 is not homogeneous.

We denote the dual of a poset X by $\hat{X}$ and the chain of all positive integers by ω, then $\hat{\omega} \circ S^+$ is a homogeneous chain isomorphic to S, but neither $\hat{\omega}$ nor S^+ is homogeneous.

We denote the first uncountable ordinal by ω_1. Then the set $S \oplus (\omega_1 \circ S^+) = T$ is a homogeneous chain, and contains a homogeneous interval isomorphic to S, but there is no chain X such that $T = X \circ S$. (The last fact can be proved from the conditional completeness of T.) The last example shows the existence of a homogeneous chain which is not self-dual.

We shall use the notations in Example I throughout the present paper.

(1.3) A void set, and the set which consists of only one element are homogeneous chains. We shall call them the trivial homogeneous chains.

A non-trivial homogeneous chain contains a subchain which is isomorphic to the chain J of all integers.

A non-trivial, dence-in-itself and homogeneous chain contains a subchain isomorphic to the chain R of all rational numbers.

A non-trivial, dense-in-itself, conditionally complete and homogeneous chain must contain a subchain isomorphic to the chain S of all real numbers.

These propositions are obvious.

(1.4) Definition 3. A subchain I of a chain X is called an interval of X, if and only if

$a, b \in I$ and $a \leq c \leq b$ implies $c \in I$.

The fact that I is an interval of X, is denoted by $I \mathrel{\dot{\in}} X$.

Especially, for any pair of $a, b \in X$, the set of elements between (properly) a and b is an interval. We call it an open interval and denote it by (a,b). When two elements a and b are adjoined to the open interval (a,b), we call it a closed interval and denote it by $[a,b]$.

(1.5) We define the following three kinds of orders in a family of intervals of X.

i) We say that I_2 contains I_1, if and only if I_1 is a subset of I_2, and denote the fact by $I_1 \mathrel{\dot{\subset}} I_2$.

ii) We say that I_1 is less than I_2, if and only if $a < b$ for every pair of $a \in I_1$ and $b \in I_2$, and denote the fact by $I_1 < I_2$.

I_1 and I_2 are comparable if and only if either they are disjoint or they coincide entirely with each other.

iii) If and only if for any $a \in I_1$ there exists a $b \in I_2$ such that $a \leq b$, and for any $b \in I_2$ there exists an $a \in I_1$ such that $a \leq b$, we say that I_1 is lower than I_2 and denote the fact by $I_1 \propto I_2$.

A necessary and sufficient condition that I_1 is comparable with I_2, is that, if one of them contains the other, either their upper bounds or their lower bounds conncide entirely with each other. (Of cause, this condition admits the case when neither I_1 nor I_2 contains the other.).

The adequacy of those orderings and the conditions of comparablity can easily be proved.

(1.6) If neither of two intervals contains the other, there are two cases.

1) They are disjoint.

2) One is lower than the other, but their intersection is non-void.

In the case 2), let $I_1 \propto I_2$, and if we denote the complement of I by I', $I_1 \frown I_2'$, $I_1 \frown I_2$, $I_1' \frown I_2$ are also intervals which are disjoint with one another, and $I_1 \frown I_2' < I_1 \frown I_2 < I_1' \frown I_2$ in the meaning of the order ii) (1.5).

Proofs are evident.

(1.7) Let X be a homogeneous chain. Let $\mathfrak{G}_X$ be the automorphism group of X, and let I be a homogeneous interval of X.

If $a, b \in I$, then there is an automorphism φ of I such that $\varphi(a) = b$. Consider the following inner mapping θ of X:

$$\theta(x) = \varphi(x) \qquad \text{for all } x \in I,$$

$$\theta(x) = x \qquad \text{for all } x \notin I.$$

Then, obviously θ is an automorphism of X.

The set of automorphisms of X such as θ, that is,

$$\{\theta \in \mathfrak{G}_X \mid \theta(x) = x \quad \text{for any } x \notin I\}$$

is a subgroup of $\mathfrak{G}_X$, which is isomorphic to the automorphism group of I. We denote this subgroup of $\mathfrak{G}_X$ by $\mathfrak{G}_I$, and call it the <u>characteristic group</u> of I.

(1.8) Let X be a homogeneous chain, and I_1 and I_2 be its homogeneous intervals whose intersection is non-void. Then both $I_1 \smile I_2$ and $I_1 \frown I_2$ are homogeneous intervals of X.

Proof. If one contains the other, this proposition is obvious.

Now, let $I_1 \propto I_2$ (1.6). Then both $I_1 \smile I_2$ and $I_1 \frown I_2$ are intervals. We shall only prove that they are homogeneous.

Let $a, b \in I_1 \smile I_2$. If these two elements are contained together in I_1, then the automorphism $\varphi \in \mathfrak{G}_{I_1}$, which maps a to b, displaces no element outside of $I_1 \smile I_2$. The case when both elements are contained together in I_2, is similar.

Let $a \in I_1 \frown I_2'$ and $b \in I_1' \frown I_2$. Take an element c from $I_1 \frown I_2$. Then there exist a $\varphi \in \mathfrak{G}_{I_1}$ which maps a to c, and a $\psi \in \mathfrak{G}_{I_2}$ which maps c to b. The automorphism $\psi\varphi$ of X does not displace any element outside of $I_1 \smile I_2$, and maps a to b. This shows that any two elements of $I_1 \smile I_2$ are mutually transitive within $I_1 \smile I_2$.

Let $a, b \in I_1 \frown I_2$. Then there is a $\varphi \in \mathfrak{G}_{I_1}$, which maps a to b, and a $\psi \in \mathfrak{G}_{I_2}$ which maps a to b. Consider the following mapping θ of X:

$$\theta(x) = \psi(x) \qquad \text{for } x \leqq a,$$

$$\theta(x) = \varphi(x) \qquad \text{for } x > a.$$

Then θ is an automorphism of X, and maps a to b, and moves no element outside of $I_1 \frown I_2$ (remember that I_1 is lower than I_2). This means that any two elements of $I_1 \frown I_2$ are mutually transitive within $I_1 \frown I_2$.

We state this fact in the following form.

<u>Theorem 2</u>. Let $\mathfrak{J}$ be a <u>maximal subset</u> of the set of homogeneous intervals of a homogeneous chain X such that any two intervals in $\mathfrak{J}$ have their non-void intersection, then $\mathfrak{J}$ is a sublattice of the Boolean algebra of all subset of X.

(If we make use of Zorn's lemma, we see that such a maximal subset always exists, but the axiom of choice is not assumed throughout this paper.)

2. The discrete homogeneous chains.

(2.1) <u>Definition 4</u>. A homogeneous chain X is called <u>discrete</u>, if and only if there exists a pair of elements of X which have the covering relation (cf. [1] p.5).

This definition requires only the existence of some pair of elements which have the covering relation, but on account of the homogeneity of X, this condition is equivalent to the fact that any element of X has an element which covers it and an element which is covered by it.

A non-discrete homogeneous chain is dense-in-itself.

(2.2) Let X be a non-trivial descrete homogeneous chain. We define an equivalence relation in X such that $a \sim b$ in X means that there exists only finite elements between a and b.

Then this equivalence relation induces a classification of X, and each class is isomorphic to the chain J of all integers. These classes are mutually disjoint intervals. So, if we introduce the order in the meaning of ii) (1.5) into the set of classes, then the set is a chain.

Now, let $\mathfrak{M}$ be the set of all classes. We shall see that the chain $\mathfrak{M}$ is homogeneous.

In fact, an automorphism φ of X, which maps an element a in a class I_1 to an element b in a class I_2, maps any element c in I_1 into I_2, for, if $\varphi(c) = d$, then the number of the elements between b and d is equal to that of the elements between a and c, which is finite.

So an automorphism φ of X induces an inner mapping of $\mathfrak{M}$, which is an automorphism of $\mathfrak{M}$ as easily seen, and the transitivity of $\mathcal{G}_X$ implies the transitivity of $\mathcal{G}_{\mathfrak{M}}$, that is, $\mathfrak{M}$ is a homogeneous chain.

The previous fact implies that

$$X = \mathfrak{M} \circ J.$$

On the other hand, $\mathfrak{M} \circ J$ is always a discrete homogeneous chain for any homogeneous chain $\mathfrak{M}$. So we obtain the following theorem.

Theorem 3. All discrete homogeneous chains correspond one-to-one to all general homogeneous chains, and a necessary and sufficient condition that X is a discrete homogeneous chain is that there exists a homogeneous chain $\mathfrak{M}$ such that

$$X = \mathfrak{M} \circ J.$$

By this theorem, the type of discrete homogeneous chains is determined to some extent. But, unless the type of general homogeneous chains is determined, we can't yet establish complete determination theorem for discrete homogeneous chains.

However, if $\mathfrak{M}$ is also discrete in the previous identity, then $\mathfrak{M}$ is decomposed into $\mathfrak{M}_1 \circ J$ samely, and if $\mathfrak{M}_1$ is discrete, it is all the same, and so on. What type of discrete chains admits such transfinite decompositions without any non-discrete multiplier set? What about ρ_J, where ρ is a dual set of an ordinal number? (See Definition 1.) To answer these questions, we introduce the following definition.

(2.3) Definition 5. A homogeneous chain X is called absolutely discrete, if and only if there exists no non-discrete homogeneous subchain (including the non-proper one also), except the interval consisting of only one point.

(A homogeneous chain consisting of only one point is regarded as a non-discrete chain in the present paper (see Definition 4).)

A discrete homogeneous chain X is absolutely discrete if and only if the $\mathfrak{M}$ is absolutely discrete, when we decompose X into $\mathfrak{M} \circ J$ in the form of Theorem 3.

This is a corollary of the next proposition.

(2.4) Let $X = Y \circ Z$, Y and Z being homogeneous chains. X is an absolutely discrete homogeneous chain, if and only if both Y and Z are absolutely discrete (except the case when Y or Z is trivial).

Proof. The necessity is obvious. Hence, we now give a proof for sufficiency.

We consider that U is a non-trivial, non-discrete homogeneous subchain of $Y \circ Z$. Especially, U can be regarded as a chain isomorphic to the chain R of all rational numbers (1.3). An element of $Y \circ Z$ has the form (y, z), where $y \in Y$ and $z \in Z$. We call y the Y-coordinate and z the Z-coordinate.

Any two elements of U must have the different Y-coordinates. Indeed, if both $u = (y, z_1)$ and $v = (y, z_2)$ have the same Y-coordinates, then the open interval $V = (u, v)$ in U is also isomorphic to R, and each its element has the same Y-coordinate. So V can be imbedded in Z. This contradicts the absolute discreteness of Z.

So any two elements of U have the different Y-coordinate, and U is isomorphic to the subchain of Y, whose element are the Y-coordinates of the elements of U. But this contradicts the absolute discreteness of Y.

Example 2. J is an absolutely discrete homogeneous chain, then also is $J \circ J$. In general, ${}^{n}J$, where n is a finite ordinal, is absolutely discrete.

$R \circ J$, and $S \circ J$ are discrete but not absolutely discrete.

3. Examples of absolutely discrete homogeneous chains.

We shall show an important series of absolutely discrete homogeneous chains. It is remarkable that these examples exhaust all of absolutely discrete homogeneous chains, as we shall see later on.

(3.1) Example 3. Let Γ be an ordinal number.

Set $H_\Gamma = {}^{\hat{\Gamma}}J$ ($\hat{\Gamma}$ implies the dual of Γ) (see Definition 1).

That is, an element of H_Γ is a series of integers arranged in the form of a dually well-ordered set $\hat{\Gamma}$, such that only a finite number of these integers are not zero, where the order is defined in the meaning of Definition 1.

H_Γ is a homogeneous chain by the note on Definition 1. H_Γ is an absolutely descrete homogeneous chain. To prove the fact, we shall study the construction of H_Γ .

(3.2) If $h \in H_\Gamma$, then for a $\gamma \in \Gamma$, $h(\gamma) \in J$. We say the value of $h(\gamma)$, the γ-coordinate of h, and denote it by h_γ . For only a finite number of γ , the γ-coordinates of h are not zero.

The set of elements of H_Γ , whose γ-coodinates are zero for any ordinal γ equal to or larger than some ordinal $\Delta < \Gamma$, is an interval of H_Γ. We denote the intervals by $I_{\Delta,\Gamma}$, but $I_{\Delta,\Gamma}$ is isomorphic to H_Δ regardless of the ordinal Γ . So we may omit the subscript Γ from $I_{\Delta,\Gamma}$.

When Γ is a discrete ordinal and $\Gamma = \Delta \oplus 1$, then obviously

$$H_\Gamma = J \circ H_\Delta$$

When Γ is a limit ordinal, then

$$H_\Gamma = \bigcup_{\Delta<\Gamma} I_\Delta .$$

But because each I_Δ is isomorphic to H_Δ, we can write

$$H_\Gamma = \bigcup_{\Delta<\Gamma} H_\Delta$$

in the above meaning.

(3.3) Let all H_Δ , $\Delta < \Gamma$ be absolutely discrete.

When $H_\Gamma = J \circ H_\Delta$, then the absolute discreteness of H_Γ follows from the proposition (2.4).

Let $H_\Gamma = \bigcup_{\Delta<\Gamma} H_\Delta$ (when Γ is a limit ordinal).

If H_Γ should contain a subchain U isomorphic to R (see (1.3)), and a, $b \in U$, then the open interval V = (a, b) of U would also be isomorphic to R. But there exists a $\Delta < \Gamma$ such that a, $b \in H_\Delta$, and so $V \subset H_\Delta$; this contradicts the absolute discreteness of H_Δ .

So, every H_Γ is absolutely discrete.

(3.4) We define an additive operation on H_Γ .

Let h, $k \in H_\Gamma$, then the sum of h and k, $h \dotplus k$, is such an element of H_Γ that its every γ-coordinate is the arithmetic sum of the γ-coordinates of h and k.

Based on this definition, H_Γ becomes a group. Furthermore, it is an ordered group as easily seen. The mapping

$$\varphi_k(h) = h \dotplus k$$

is order preserving and one-to-one, that is, φ_k is an automorphism of H_Γ .

We make use of this group character of H_Γ , only for denoting the automorphism φ_k of this type by $\dotplus k$, and do not use it essentially in the following.

(3.4) I_Δ is an interval of H_Γ. For any $k \in H_\Gamma$, the set

$$I_\Delta \dot{+} k = \{ l \mid l = h \dot{+} k,\ h \in I_\Delta \}$$

is also an interval of H_Γ, which is isomorphic to H_Δ, and contains k. The elements of $I_\Delta \dot{+} k$ have fixed γ-coordinates for any $\gamma \geq \Delta$, which are equal to the γ-coordinate of k. So the interval $I_\Delta \dot{+} k$ has upper bounds and lower bounds in H_Γ.

Lemma

1) Every proper homogeneous interval (non-trivial) I of H_Γ is $I_\Delta \dot{+} k$ for some $\Delta < \Gamma$ and a $k \in H_\Gamma$.

2) H_Γ can not be isomorphic to its proper interval.

Proof. Those are obvious for $H_1 = J$. Let the above two statement be proved for H_Δ for all $\Delta < \Gamma$.

The proof of 1).

i) The case that the Γ is a discrete ordinal, and $H_\Gamma = J \circ H_\Lambda$.

Let I be a homogeneous, non-trivial, proper interval of H_Γ, and $I \ni k$. Then, I intersects with $I_\Lambda \dot{+} k$. and so, the following three cases are conceivable.

Case 1) I contains $I_\Lambda \dot{+} k$.

Case 2) $I_\Lambda \dot{+} k$ contains I.

Case 3) Neither of them contains the other.

But, the last case is impossible. Because, if I does not contain $I_\Lambda \dot{+} k$, then their intersection is a proper interval of $I_\Lambda \dot{+} k$, which is homogeneous by (1.8). Since $I_\Lambda \dot{+} k$ is isomorphic to H_Λ, its proper homogeneous interval has the form $I_{\Lambda'} \dot{+} h \dot{+} k$, where $\Lambda' < \Lambda$, and $h \in I_\Lambda$, but such an interval has its upper bound and lower bound in $I_\Lambda \dot{+} k$ (3.4). So I is contained in $I_\Lambda \dot{+} k$, because I is an interval.

In the case 2), I has the form $I_{\Lambda'} \dot{+} h \dot{+} k$, $\Lambda' < \Lambda$, $h \in I_\Lambda$, by the assumption.

In the case 1), if $I = I_\Lambda \dot{+} k$, then the proposition is true.

The interval $I_\Lambda \dot{+} k$ consists of all elements whose Γ-coordinates are equal to the Γ-coordinate k_Γ of k. So, if I has an element h outside $I_\Lambda \dot{+} k$, then the Γ-coordinate h_Γ of h is not equal to k_Γ, and I contains $I_\Lambda \dot{+} h$ all the same, because I is not contained in $I_\Lambda \dot{+} h$. Still more, I contains all the elements whose Γ-coordinates are between h_Γ and k_Γ, since I is an interval.

If the Γ-coordinates of the elements of I have neither their upper bound nor their lower bound, then the interval is not proper.

Consider that the Γ-coodinates of the elements of I have their upper bound, for instance, then some $k \in I$ has the least upper bound m as its Γ-coodinate, since the values of Γ-coodinates of I are integers. If another element $h \in I$ has not the Γ-coodinate m, then the automorphism φ of I, which maps h to k, maps $I_\Lambda \dot{+} k$ into $I_\Lambda \dot{+} k$ itself as a proper interval of $I_\Lambda \dot{+} k$. But this contradicts the assumption of induction 2).

So, if the Γ-coordinates of the elements of I ranges at least over two integers, then they ranges over all J, and so I is not proper in H_Γ.

So, if I is proper and contains k, then either $I \subset I_\Lambda \dot{+} k$, and so $I = I_\Delta \dot{+} h$ for some $\Delta < \Lambda$, or I coincides with $I_\Lambda \dot{+} k$.

ii) The case that Γ is a limit ordinal.

Consider that I is a homogeneous interval of H_Γ, and contains k. If I contains $I_\Delta \dot{+} k$ for any $\Delta < \Gamma$, then I contains all elements of H_Γ, that is, I is non-proper interval.

Now, assume that I does not contain $I_{\Delta'} \dot{+} k$ for some $\Delta' < \Gamma$, then we see that I is contained in $I_{\Delta'} \dot{+} k$, quite similarly as in the previous proof. So I has the form $I_\Delta \dot{+} h \dot{+} k$ where $\Delta < \Delta'$, $h \in I_{\Delta'}$ and $k \in I$ by the assumption of induction.

The proof of 2).

If H_Γ is isomorphic to its proper interval I, then I is homogeneous, and so I has the form $I_\Delta \dot{+} k$ where $\Delta < \Gamma$, $k \in H_\Gamma$. Then the isomorphism which maps H_Γ on $I_\Delta \dot{+} k$, maps the interval $I_\Delta \dot{+} k$ onto its proper interval.

But $I_\Delta \dotplus k$ is isomorphic to H_Δ, and this contradicts the assumption of induction.

(3.5) We call the element of H_Γ, whose every coordinate is zero, the <u>original element.</u>

It follows from the previous lemma that the interval of H_Γ, which is isomorphic to H_Δ, $\Delta < \Gamma$, and contains the original element, coincides with I_Δ. We say I_Δ, the <u>Δ-original interval.</u>

But, on account of the homogeneity of H_Γ, neither the original element nor the original intervals have any distinguished character, unless the group character of H_Γ is concerned.

4. General absolutely discrete homogeneous chains.

(4.1) Let K be an absolutely discrete homogeneous chain. If K has an interval H isomorphic to H_Γ, then it follows from the homogeneity of K that for any element $k \in K$, there exists an interval isomorphic to H_Γ, which contains the element k.

In general, if two homogeneous intervals, one of which is isomorphic to H_Γ for some ordinal Γ, have their intersection, then one of them must contain the other. The proof is quite the same as in the previous lemma.

Let $\mathfrak{M}$ be the set of intervals of K, which are isomorphic to H_Γ. Then the join of the intervals in $\mathfrak{M}$ covers K entirely, and these intervals are mutually disjoint, because of the above statement and previous lemma. (Any interval of $\mathfrak{M}$ can not contain another as a proper interval.)

So, if we define on $\mathfrak{M}$ the order in the meaning of ii) (1.5), then $\mathfrak{M}$ becomes a chain. Let $\varphi \in \mathfrak{G}_K$, and $M \in \mathfrak{M}$, then $\varphi(M)$ is also an interval of K, and is isomorphic to H_Γ; that is, $\varphi(M) \in \mathfrak{M}$. So an automorphism of K induces an automorphism of $\mathfrak{M}$, and the transitivity of the automorphism group of K implies the transitivity of automorphism group of $\mathfrak{M}$. That is, $\mathfrak{M}$ is a homogeneous chain, and

$$K = \mathfrak{M} \circ H_\Gamma$$

(4.2) If an absolutely discrete homogeneous chain K contains an interval isomorphic to H_Δ for any $\Delta < \Gamma$, where Γ is a limit ordinal, then for an element $k \in K$, there exists an interval K_Δ isomorphic to H_Δ which has k as its original element (considering that the same coordinates as those of the corresponding elements in H_Δ, are introduced in the elements in K_Δ). Then, if $\Delta' < \Delta$, $K_{\Delta'}$ is the original interval of K_Δ, because of the lemma and the notations in (4.1) and in (3.5).

It is seen that the set $K_\Gamma = \bigcup_{\Delta<\Gamma} K_\Delta$ is an interval of K, and further verified K_Γ is isomorphic to H_Γ. So K contains an interval isomorphic to H_Γ.

<u>Theorem 3.</u> (<u>The main theorem</u>).

Every absolutely discrete homogeneous chain is isomorphic to H_Γ for some ordinal Γ.

Proof. Let K be an absolutely discrete homogeneous chain. K does not contain an interval isomorphic to H_Λ for an ordinal Λ whose power (cardinal number) is beyond that of K. So there exists a least ordinal Γ such that K has not the proper interval isomorphic to H_Γ, that is, K has no proper interval isomorphic to H_Γ, but K contains a proper interval isomorphic to H_Δ for any ordinal $\Delta < \Gamma$.

i) The case when Γ is a discrete ordinal and $\Gamma = \Delta \oplus 1$:

It follows from (4.1) that $K = \mathfrak{M} \circ H_\Delta$. But $\mathfrak{M}$ is an absolutely discrete homogeneous chain (2.4). So $\mathfrak{M} = \mathfrak{N} \circ J$, and $K = \mathfrak{N} \circ J \circ H_\Delta = \mathfrak{N} \circ H_\Gamma$. But K contains no proper interval isomorphic to H_Γ, so $\mathfrak{N} = 1$, and $K = H_\Gamma$.

ii) The case when Γ is a limit ordinal:

Then K must contain an interval isomorphic to H_Γ (4.2). But such an interval can not be proper. So $K = H_\Gamma$.

<u>Theorem 4.</u> Any discrete homogeneous chain K can be decomposed into an ordinal product

$$K = \mathfrak{M} \circ H_\Gamma ,$$

where $H_{\Gamma} = \hat{\Gamma}J$, and $\mathfrak{M}$ is a 'non-discrete' homogeneous chain (this may consist of only one element).

The proof is similar as that of the previous theorem.

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