NOTE ON DIRICHLET SERIES. (VIII)

ON THE SINGULARITIES OF DIRICHLET SERIES. (V)

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(1) INTRODUCTION. In the previous Note ([1] Theorem 3), we have established the following

THEOREM. Let us put

$$(1.1)\qquad F(s) = \int_0^\infty \exp(-sx)\, d\alpha(x)$$

$$(s=\sigma+it, \quad \alpha(0)=0)$$

where $\alpha(x)$ is of bounded variation in any finite interval $0 \leq x \leq X$, X being an arbitrary positive constant. If (1.1) has the finite simple convergence-abscissa σ_s, then the necessary and sufficient condition for $s=\sigma_s+it$ to be the singular point of (1.1) is that

$$\overline{\lim_{\sigma\to-\infty}} \left\{ \log\log^+|\varphi(\sigma+it)| + \sigma \right\} = \sigma_s ,$$

where $\varphi(s)=\int_0^\infty \exp(-sx)\, d\beta(x)$,

$$\beta(x) = \int_0^x \frac{1}{\Gamma(1+y)}\, d\alpha(y)$$

In this present Note, applying this theorem to Dirichlet series, we shall give a new proof of next G.Polya's theorem. ([2] p.22, [3] pp.85-86, [4])

G.PÓLYA'S THEOREM. Let Dirichlet series $F(s)=\sum a_n \exp(-\lambda_n s)$ with $\lim_{n\to\infty}(\lambda_{n+1}-\lambda_n)=g>0$ have the finite simple convergence-abscissa σ_s. Then, $F(s)$ has at least one singular point on every closed segment on $\Re(s)=\sigma_s$ with length $2\pi\Delta$, where Δ ($\leq 1/g$) is the maximal density of $\{\lambda_n\}$, which is defined as follows:

$$\Delta = \lim_{\xi\to1}\ \overline{\lim_{r\to+\infty}}\ \frac{N(r)-N(r\xi)}{r-r\xi}, \qquad N(r)=\sum_{\lambda_n<r} 1$$

(2) LEMMAS. For its proof, we need some Lemmas:

LEMMA I. Under the assumptions of the theorem, if $\lim_{n\to\infty}\frac{1}{\lambda_n}\log|b_n|=0$, then $\varphi_1(s)=\sum_1^\infty \frac{a_n}{\Gamma(1+\lambda_n)}\exp(-\lambda_n s)$ and

$$\varphi_2(s)=\sum_1^\infty \frac{a_n b_n}{\Gamma(1+\lambda_n)}\exp(-\lambda_n s)$$

are both absolutely convergent in the whole plane, and they have the same order and type in the whole plane, where the order ρ_i and the type k_i of $\varphi_i(s)$ $(i=1,2)$ are defined as follows:

$$(a)\quad \rho_i = \overline{\lim_{\sigma\to-\infty}}\ \frac{1}{(-\sigma)}\log^+\log^+ M_i(\sigma),$$

$$M_i(\sigma)=\sup_{-\infty<t<+\infty} |\varphi_i(\sigma+it)| ,$$

$$(b)\quad k_i = \overline{\lim_{\sigma\to-\infty}}\ \frac{1}{\exp((-\sigma)\rho_i)}\log^+ M_i(\sigma)$$

PROOF. From $\lim_{n\to+\infty}(\lambda_{n+1}-\lambda_n)=g>0$ we have evidently

$$(2.1)\qquad \lim_{n\to+\infty}\frac{1}{\lambda_n}\log n = 0$$

Hence, by G.Valiron's theorem ([2] p.3), the simple (= absolute) convergence-abscissa σ_i of $\varphi_i(s)$ is determined respectively by

$$\sigma_1 = \overline{\lim_{n\to\infty}}\ \frac{1}{\lambda_n}\log\left|\frac{a_n}{\Gamma(1+\lambda_n)}\right|$$

$$= \overline{\lim_{n\to\infty}}\ \frac{1}{\lambda_n}\log|a_n| + \lim_{n\to\infty}\frac{1}{\lambda_n}\log\left|\frac{1}{\Gamma(1+\lambda_n)}\right|$$

$$= \sigma_s - \infty = -\infty ,$$

$$\sigma_2 = \overline{\lim_{n\to\infty}}\ \frac{1}{\lambda_n}\log\left|\frac{a_n b_n}{\Gamma(1+\lambda_n)}\right|$$

$$= \overline{\lim_{n\to\infty}}\ \frac{1}{\lambda_n}\log|a_n| + \lim_{n\to\infty}\frac{1}{\lambda_n}\log|b_n| + \lim_{n\to\infty}\frac{1}{\lambda_n}\log\left|\frac{1}{\Gamma(1+\lambda_n)}\right|$$

$$= \sigma_s + 0 - \infty = -\infty ,$$

so that $\varphi_i(s)$ $(i=1,2)$ is absolutely convergent everywhere. Thus the first part of Lemma is proved.

By (2.1) and J.Ritt's theorem ([5]), the order ρ_i of $\varphi_i(s)$ ($i=1,2$) is given respectively by

$$-\frac{1}{\rho_1} = \overline{\lim_{n\to\infty}}\ \frac{1}{\lambda_n\log\lambda_n}\log\left|\frac{a_n}{\Gamma(1+\lambda_n)}\right| .$$

$$= \overline{\lim_{n\to\infty}}\ \frac{1}{\lambda_n\log\lambda_n}\log|a_n| + \lim_{n\to\infty}\frac{1}{\lambda_n\log\lambda_n}\log\left|\frac{1}{\Gamma(1+\lambda_n)}\right|$$

$$= 0 - 1 = -1 ,$$

$$-\frac{1}{\rho_2} = \overline{\lim_{n\to\infty}} \frac{1}{\lambda_n \log\lambda_n} \log\left|\frac{a_n b_n}{\Gamma(1+\lambda_n)}\right|$$

$$= \overline{\lim_{n\to\infty}} \frac{1}{\lambda_n\log\lambda_n} \log|a_n|$$

$$+ \lim_{n\to\infty} \frac{1}{\lambda_n\log\lambda_n} \log|b_n|$$

$$+ \lim_{n\to\infty} \frac{1}{\lambda_n\log\lambda_n} \log\left|\frac{1}{\Gamma(1+\lambda_n)}\right|$$

$$= 0 + 0 - 1 = -1 ,$$

so that

$$(2.2) \qquad \rho_1 = \rho_2 = 1$$

Taking account of (2.1), (2.2) and S.Izumi's theorem ([6]), the type k_ι of $\varphi_\iota(s)$ ($\iota=1,2$) is determined respectively by

$$\log(ek_1) = \overline{\lim_{n\to\infty}} \left\{ \frac{1}{\lambda_n} \log\left|\frac{a_n}{\Gamma(1+\lambda_n)}\right| + \log\lambda_n \right\}$$

$$= \overline{\lim_{n\to\infty}} \frac{1}{\lambda_n} \log|a_n|$$

$$+ \lim_{n\to\infty} \left\{ \frac{1}{\lambda_n}\log\left|\frac{1}{\Gamma(1+\lambda_n)}\right| + \lambda_n \right\}$$

$$= \sigma_s + 1 , \quad \text{i.e.} \quad k_1 = e^{\sigma_s}$$

$$\log(ek_2) = \overline{\lim_{n\to\infty}} \left\{ \frac{1}{\lambda_n} \log\left|\frac{a_n b_n}{\Gamma(1+\lambda_n)}\right| + \log\lambda_n \right\}$$

$$= \overline{\lim_{n\to\infty}} \frac{1}{\lambda_n} \log|a_n| + \lim_{n\to\infty} \frac{1}{\lambda_n}\log|b_n|$$

$$+ \lim_{n\to\infty} \left\{ \frac{1}{\lambda_n}\log\left|\frac{1}{\Gamma(1+\lambda_n)}\right| + \log\lambda_n \right\}$$

$$= \sigma_s + 0 + 1 , \quad \text{i.e.} \quad k_2 = e^{\sigma_s},$$

so that

$$(2.3) \qquad k_1 = k_2 = e^{\sigma_s}.$$

By (2.2), (2.3), the second part of Lemma is established. q.e.d.

<u>LEMMA II.</u> <u>Let us put</u>

$$\sigma(t) = \overline{\lim_{\sigma\to-\infty}} \frac{1}{\exp((-\sigma))} \log^+|\varphi_1(\sigma+it)|,$$

$$k_1(\mathcal{P}) = \overline{\lim_{\sigma\to-\infty}} \frac{1}{\exp((-\sigma))} \log^+ M_1(\sigma;\mathcal{P})$$

<u>where the horizontal strip $\mathcal{P}$</u> : $|\mathcal{J}(s)| = |t| \leq \alpha$,

$$M_1(\sigma;\mathcal{P}) = \max_{|t|\leq\alpha} |\varphi_1(\sigma+it)| .$$

<u>Then we have</u>

$$\max_{|t|\leq\alpha} \sigma(t) = k_1(\mathcal{P})$$

<u>REMARK.</u> By (2.3), we have evidently

$$\left.\begin{matrix}\sigma(t)\\ k_1(\mathcal{P})\end{matrix}\right\} \leq e^{\sigma_s} < +\infty.$$

<u>PROOF.</u> Since $|\varphi_1(\sigma+it)| \leq \max_{|t|\leq\alpha} |\varphi_1(\sigma+it)| = M_1(\sigma;\mathcal{P})$ we have easily

$$\sigma(t) \leq k_1(\mathcal{P}) ,$$

so that

$$(2.4) \qquad \max_{|t|\leq\alpha} \sigma(t) \leq k_1(\mathcal{P})$$

Since $\varphi_1(s)$ is of exponential type on account of (2.3), by well-known Phragmén-Lindelöf's theorem, for any given $\varepsilon(>0)$ we have

$$|\varphi_1(\sigma+it)| < \exp\{e^{-\sigma}(\sigma(t)+\varepsilon)\}$$

$$\text{for } |\sigma| > K(\varepsilon)$$

uniformly with respect to t ($|t|\leq\alpha$). Hence, for $|\sigma| > K(\varepsilon)$

$$M_1(\sigma,\mathcal{P}) = |\varphi_1(\sigma+it_\sigma)|$$

$$< \exp\{e^{-\sigma}(\sigma(t_\sigma)+\varepsilon)\}$$

$$< \exp\{e^{-\sigma}(\max_{|t|\leq\alpha}\sigma(t)+\varepsilon)\},$$

so that

$$k_1(\mathcal{P}) \leq \max_{|t|\leq\alpha} \sigma(t) + \varepsilon$$

Letting $\varepsilon \to 0$,

$$(2.5) \qquad k_1(\mathcal{P}) \leq \max_{|t|\leq\alpha} \sigma(t).$$

By (2.4), (2.5) Lemma 2 is established. q.e.d.

<u>LEMMA III.</u> <u>In the horizontal strip $\mathcal{P}$</u> : $|\mathcal{J}(s)| \leq \pi\Delta + \varepsilon$ $(\varepsilon>0)$, <u>$\varphi_1(s)$ has the same type as in the whole plane, i.e.</u>

$$k_1(\mathcal{P}) = k_1 = e^{\sigma_s}.$$

<u>PROOF.</u> Since $M_1(\sigma,\mathcal{P}) = \max_{|t|\leq\pi\Delta+\varepsilon} |\varphi_1(\sigma+it)| \leq \sup_{-\infty<t<+\infty} |\varphi_1(\sigma+it)| = M_1(\sigma)$, we have evidently

$$(2.6) \qquad k_1(\mathcal{P}) \leq k_1 = e^{\sigma_s}$$

Since $\overline{\varphi}_1(s) = \sum_1^\infty \frac{\overline{a}_n}{\Gamma(1+\lambda_n)} \exp(-\lambda_n s)$ has evidently the same order and type in the whole plane as $\varphi_1(s) = \sum_1^\infty \frac{a_n}{\Gamma(1+\lambda_n)} \exp(-\lambda_n s)$, either 1/2.($\varphi_1(s) + \overline{\varphi}_1(s)$) or 1/2. ($\varphi_1(s) - \overline{\varphi}_1(s)$) has the same order and type in the whole plane as $\varphi_1(s)$. Without any loss of generality, we can assume that 1/2.($\varphi_1(s) + \overline{\varphi}_1(s)$) has the same order and type as $\varphi_1(s)$. Putting $a_n = \alpha_n + i\beta_n$ $(n=1,2,\dots)$, we have

$$\frac{1}{2}(\varphi_1(s) + \overline{\varphi}_1(s))$$

$$= \sum_1^\infty \frac{\alpha_n}{\Gamma(1+\lambda_n)} \exp(-\lambda_n s)$$

Then, without any restriction of generality, we can assume

(2·7) $\qquad \alpha_1 > 0$

Now we classify $(\lambda_n, \lambda_{n+1})$ $(n=1,2,\cdots)$ in two classes as follows:

(a) if $\alpha_n \alpha_{n+1} > 0$, $(\lambda_n, \lambda_{n+1})$ belongs to 2-class.

(b) if $\alpha_n \alpha_{n+1} < 0$, $(\lambda_n, \lambda_{n+1})$ belongs to 1-class.

(c) when $\alpha_{n+1} \neq 0$, $\alpha_n = \alpha_{n-1} = \cdots\cdots = \alpha_{n-\nu+1} = 0$, $\alpha_{n-\nu} \neq 0$ if $\alpha_{n-\nu}\alpha_{n+1} > 0$, $(\lambda_i, \lambda_{i+1})$ $(i = n-\nu, \cdots, n)$ belongs to 2-class, and if $\alpha_{n-\nu}\alpha_{n+1} < 0$, $(\lambda_i, \lambda_{i+1})$ $(i = n-\nu, \cdots, n-1)$ belongs to 2-class, but $(\lambda_n, \lambda_{n+1})$ belongs to 1-class.

Then, by G.Polya's theorem ([7] p.610), there exists an integral function $G(z)$ of exponential type such that

$$(2.8)\quad \begin{cases} \text{(a)} & G(\lambda_1) > 0, \quad G(\lambda_n) \ : \text{real} \quad (n = 1,2,\ldots) \\ \text{(b)} & \text{if } (\lambda_n, \lambda_{n+1}) \text{ belongs to } \nu\text{-class, then } (-1)^\nu G(\lambda_n) G(\lambda_{n+1}) > 0 . \\ \text{(c)} & \lim_{n\to\infty} \frac{1}{\lambda_n} \log |G(\lambda_n)| = 0 \\ \text{(d)} & \text{the indicator-diagram of } G(z) \text{ is the segment: } -\pi\Delta \leq \mathscr{J}(z) \leq \pi\Delta . \end{cases}$$

Then we have evidently

$$(2\cdot9)\qquad \alpha_n G(\lambda_n) \geq 0 \qquad (n = 1,2,\cdots)$$

By Cramer-Ostrowski's theorem ([3] pp.51-52), and (d) of (2.8), we have

$$(2\cdot10)\qquad \varphi_2(s) = \sum_1^\infty a_n G(\lambda_n) \frac{1}{\Gamma(1+\lambda_n)} \exp(-\lambda_n s) = \frac{1}{2\pi i} \oint_L \varphi_1(s-u)\, \Phi(u)\, du,$$

where (a) $G(z) = \sum_0^\infty c_m z^m$, $\Phi(u) = \sum_0^\infty m!\, c_m / u^{m+1}$

(b) the path of integration L is any closed curve surrounding the segment $-\pi\Delta \leq \mathscr{J}(u) \leq \pi\Delta$.

Hence, putting $\varphi_3(s) = \sum_1^\infty \frac{\alpha_n G(\lambda_n)}{\Gamma(1+\lambda_n)} \cdot \exp(-\lambda_n s)$, by (2.9), (2.10) we get

$$(2\cdot11)\qquad \varphi_3(\sigma) = \sum_1^\infty \frac{\alpha_n G(\lambda_n)}{\Gamma(1+\lambda_n)} \exp(-\lambda_n \sigma) = |\Re\, \varphi_2(\sigma)| \leq |\varphi_2(\sigma)| \leq \frac{1}{2\pi} \oint_L |\varphi_1(\sigma-u)\, \Phi(u)|\, |du|$$

Now by the definition of $k_1(\mathcal{D})$, for any given ε (>0), we have

$$(2\cdot12)\qquad \log^+ M_1(\sigma, \mathcal{D}) < e^{-\sigma}\{k_1(\mathcal{D}) + \varepsilon'\} \quad \text{for } -\sigma > C(\varepsilon').$$

Let us take the contour of the rectangle: $\Re(u) = \pm\varepsilon''$, $\mathscr{J}(u) = \pm i(\pi\Delta + \varepsilon'')$ $(\varepsilon > \varepsilon'' > 0)$ as the path of integration L. Then since $\Re(\sigma - u) = \sigma - \Re(u) \leq \sigma + \varepsilon''$ on L, for $-\sigma > \varepsilon'' + C(\varepsilon')$, by (2.12) we have

$$\log^+ |\varphi_1(\sigma - u)| < \exp(-\Re(\sigma-u))\, (k_1(\mathcal{D}) + \varepsilon') \leq \exp(-\sigma + \varepsilon'')\, (k_1(\mathcal{D}) + \varepsilon')$$

Hence, by (2.11)

$$(2\cdot13)\qquad \varphi_3(\sigma) \leq \exp\{ e^{-\sigma+\varepsilon''} (k_1(\mathcal{D}) + \varepsilon')\} \cdot \frac{1}{2\pi} \oint |\Phi(u)|\, |du|$$

On the other hand, $\sum_1^\infty \frac{\alpha_n}{\Gamma(1+\lambda_n)} \cdot \exp(-\lambda_n s)$ has the same order and type in the whole plane as $\varphi_1(s) = \sum_1^\infty a_n / \Gamma(1+\lambda_n) \cdot \exp(-\lambda_n s)$, so that by (c) of (2.8) and Lemma 1, it is true of $\varphi_3(s) = \sum_1^\infty \alpha_n G(\lambda_n) \cdot \frac{1}{\Gamma(1+\lambda_n)} \cdot \exp(-\lambda_n s)$. Hence, taking account of $\varphi_3(\sigma) = \sup_{-\infty<t<+\infty} |\varphi_3(\sigma+it)|$ and (2.13) we have

$$k_1 = k_3 = \overline{\lim_{\sigma\to-\infty}} \frac{1}{\exp((-\sigma))} \log^+ \varphi_3(\sigma) \leq (k_1(\mathcal{D}) + \varepsilon') \exp(\varepsilon'')$$

Letting $\varepsilon', \varepsilon'' \to 0$,

$$(2\cdot14)\qquad k_1 \leq k_1(\mathcal{D})$$

Thus, by (2.6) and (2.14), Lemma 3 is established. q.e.d.

(3) PROOF OF G.PÓLYA'S THEOREM. Let us denote by $\mathcal{D}$ the horizontal strip: $|\mathscr{J}(s)| \leq \pi\Delta + \varepsilon$, $(\varepsilon > 0)$. Then, by Lemma 2,3, we have

$$\max_{|t| \leq \pi\Delta+\varepsilon} \sigma(t) = k_1(\mathcal{D}) = k_1 = e^{\sigma_s},$$

so that there exists at least one $\bar{t}$ such that

$$\begin{cases} \text{(i)} & |\bar{t}| \leq \pi\Delta + \varepsilon \\ \text{(ii)} & \sigma(\bar{t}) = e^{\sigma_s} \end{cases}$$

Hence,

$$\log \sigma(\bar{t}) = \overline{\lim_{\sigma\to-\infty}} \{ \log\log^+ |\varphi_1(\sigma+it)| + \sigma \} = \sigma_s$$

Accordingly, by the theorem mentioned in (1), $s = \sigma_s + i\xi$ $(|\xi| \leqq \pi\Delta + \varepsilon)$, is the singular point for $F(s) = \sum_{1}^{\infty} a_n \exp(-\lambda_n s)$. Letting $\varepsilon \to 0$ $F(s)$ has at least one singular point on the segment: $\Re(s) = \sigma_s$, $|\Im(s)| \leqq \pi\Delta$.

By the transformation: $s = s' + (\sigma_s + it)$, and the similar arguments as above, we can prove the existence of the singular point on the segment: $\Re(s) = \sigma_s$, $|\Im(s) - t| \leqq \pi\Delta$, q.e.d.

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