NOTE ON DIRICHLET SERIES. (VII)

ON THE DISTRIBUTION OF VALUES OF DIRICHLET SERIES ON THE VERTICAL LINES

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(1) INTRODUCTION. Let us put

$$(1\cdot 1)\qquad F(s)=\sum_{1}^{\infty} a_n \exp(-\lambda_n s)$$

$(s=\sigma+it,\quad 0\leqq\lambda_1<\lambda_2<\cdots<\lambda_n\to+\infty)$

If $F(s)$ has the uniform convergence-abscissa $\sigma_u<+\infty$, then for $\sigma_u<\sigma$, $F(\sigma+it)$ is an almost periodic function of t by H.Bohr's theorems. ([1] pp. 48-49, [2]) If we assume only the existence of the simple convergence-abscissa $\sigma_s<+\infty$, what we can say about the behaviour of Dirichlet series on the vertical line $\mathfrak{R}(s)=\sigma>\sigma_s$? Concerning this problem, we have not any knowledge except K.Ananda-Rau's short note.([3]) In this present Note, we shall establish

THEOREM. Let (1.1) have the simple convergence-abscissa $\sigma_s<+\infty$. Then, on the vertical line $\mathfrak{R}(s)=\sigma>\sigma_s$, next three cases are possible:

(a) On $\mathfrak{R}(s)=\sigma>\sigma_s$, $F(\sigma+it)$ is bounded and it is an almost periodic function of t.

(b) On $\mathfrak{R}(s)=\sigma>\sigma_s$, $F(\sigma+it)$ is bounded and it has no limit as $t\to+\infty$ (or $-\infty$). Furthermore, for any given $\varepsilon\ (>0)$, in the vertical strip: $\sigma_s<\sigma-\varepsilon<\mathfrak{R}(s)<\sigma+\varepsilon$, $F(s)$ assumes every value, except perhaps two (∞ included), infinitely many times.

(c) On $\mathfrak{R}(s)=\sigma>\sigma_s$, $F(\sigma+it)$ is unbounded, but it is impossible that we have simultaneously

$$\lim_{t\to+\infty\ (\text{or}\ -\infty)}|F(\sigma+it)|=+\infty,\quad |\arg F(\sigma+it)-\theta|\leqq\vartheta<\pi/2,$$

where $\theta\ (0\leqq\theta<2\pi)$, $\vartheta\ (0<\vartheta<\pi/2)$: arbitrary but fixed constants.

(2) LEMMAS. For its proof, we need next Lemmas.

LEMMA 1. Let (1.1) have the simple convergence-abscissa $\sigma_s<+\infty$. Then we have

$$F(s)=\sum_{\lambda_n<\omega} a_n \exp(-\lambda_n s)\{1-\exp(\nu(\lambda_n-\omega))\}-\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}\frac{\nu F(z)\exp(\omega(z-s))}{(z-s)(z+\nu-s)}dz.$$

for $\max(\sigma_s,\sigma-\nu)<\beta<\sigma$, $\sigma=\mathfrak{R}(s)$, where ω, ν are arbitrary but fixed positive constants.

PROOF. By Perron's formula ([1] p.9), we have

$$(2\cdot 1)\qquad \sum_{\lambda_n<\omega} a_n\exp(-\lambda_n s)-F(s)=\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}\frac{F(z)\exp(\omega(z-s))}{z-s}dz$$

for $\sigma_s<\beta<\sigma=\mathfrak{R}(s)$;

$$(2\cdot 2)\qquad \sum_{\lambda_n<\omega} a_n\exp(-\lambda_n(s-\nu))=\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}\frac{F(z)\exp(\omega(z-s+\nu))}{(z-s+\nu)}dz$$

for $\sigma-\nu<\beta$. Hence, by (2.1), (2.2),

$$\sum_{\lambda_n<\omega} a_n\exp(-\lambda_n s)\{1-\exp(\nu(\lambda_n-\omega))\}-F(s)$$

$$=\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}\left\{\frac{1}{z-s}-\frac{1}{z+\nu-s}\right\}F(z)\exp(\omega(z-s))dz$$

$$=\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}\frac{\nu F(z)\exp(\omega(z-s))}{(z-s)(z+\nu-s)}dz$$

for $\max(\sigma_s,\sigma-\nu)<\beta<\sigma=\mathfrak{R}(s)$. q.e.d.

LEMMA 2. Let (1.1) have the simple convergence-abscissa $\sigma_s<+\infty$. If $|F(s)|\leqq M$ on $\mathfrak{R}(s)=\sigma_0>\sigma_s$, then $|F(s)|\leqq M$ for $\mathfrak{R}(s)\geqq\sigma_0>\sigma_s$.

PROOF. Since $\sigma_s<\sigma_0$, for sufficiently small $\varepsilon\ (>0)$ we have

$$\sigma_1 < \sigma_0 - \varepsilon < \sigma_0 ,$$

so that $F(s)$ is evidently simply convergent for $s = \sigma_0 - \varepsilon$. Hence, by the well-known theorem ([1] p.8), we have

$$(2.3) \qquad |F(\sigma+it)| = o(|t|)$$

uniformly for $\sigma_0 \leq \sigma$. Putting $s = re^{i\varphi} = \sigma + it$, we have by (2.3),

$$\begin{aligned} &|F(s)| \\ &= |F(\sigma+it)| \\ &= o(|t|) \qquad \text{for } |\varphi| \leq \pi/2 \\ &= o(r|\sin\varphi|) \\ &= o(r) \end{aligned}$$

whence, for any given $\varepsilon\ (>0)$,

$$(2.4) \qquad |F(s)| = o(\exp(\varepsilon r)) \quad \text{for } |\varphi| \leq \pi/2$$

Since $|F(s)| \leq M$ on $\Re(s) = \sigma_0$, by (2.4) and Phragmen-Lindelöf's theorem ([4] p.43) we have $|F(s)| \leq M$ for $\Re(s) \geq \sigma_0$. q.e.d.

<u>**LEMMA 3.** Let (1.1) have the simple convergence-abscissa $\sigma_1 < +\infty$. Then we have</u>

$$a_n = \lim_{T\to\infty} \frac{1}{T} \cdot \int_\alpha^{\alpha+T} F(\sigma+it)\exp(\lambda_n(\sigma+it))\,dt$$
$$(n = 1, 2, \cdots\cdots)$$

<u>for $\sigma_1 < \sigma$, where α : an arbitrary but fixed constant.</u>

<u>REMARK.</u> It is well known that this formula is valid for $\mathcal{F} < \sigma$, where $\mathcal{F}$ is the boundedness-abscissa, which is defined as follows: for any given $\varepsilon\ (>0)$, $F(\sigma+it)$ is regular and uniformly bounded for $\mathcal{F}+\varepsilon \leq \sigma$. ([1] p.15)

<u>But the existence of $\mathcal{F}$ is not necessary for the validity of this formula.</u>

<u>PROOF</u>. Let us put

$$\begin{aligned} &\exp(\lambda_n s)\,F(s) \\ &= a_n + \sum_{i=1}^{n-1} a_i \exp((\lambda_n-\lambda_i)s) + \sum_{i=n+1}^{\infty} a_i \exp(-(\lambda_i-\lambda_n)s) \\ &= a_n + g_1(s) + \exp(-\tau_n s)\cdot g_2(s), \end{aligned}$$

where

(i) $g_1(s) = \sum_{i=1}^{n-1} a_i \exp((\lambda_n-\lambda_i)s)$,

(ii) $g_2(s) = \sum_{i=n+1}^{\infty} a_i \exp(-\lambda_i' s)$,

$$\tau_n = \lambda_{n+1} - \lambda_n > 0, \quad \lambda_i' = \lambda_i - \lambda_{n+1} \geq 0 \quad \text{for } i \geq n+1$$

Hence we have

$$\begin{aligned} (2.5) \qquad &\frac{1}{T}\int_\alpha^{\alpha+T} F(\sigma+it)\exp(\lambda_n(\sigma+it))\,dt \\ &= a_n + \frac{1}{T}\int_\alpha^{\alpha+T} g_1(s)\,dt + \frac{1}{T}\int_\alpha^{\alpha+T}\exp(-\tau_n s)\,g_2(s)\,dt \\ &= a_n + I_1 + I_2, \quad \text{say} \end{aligned}$$

With regard to I_1, we have

$$\begin{aligned} &I_1 \\ &= \frac{1}{T}\int_\alpha^{\alpha+T} g_1(s)\,dt \\ &= \frac{1}{T}\cdot\sum_{i=1}^{n-1} a_i \cdot \int_\alpha^{\alpha+T}\exp((\lambda_n-\lambda_i)(\sigma+it))\,dt \\ &= \frac{1}{T}\cdot\sum_{i=1}^{n-1} a_i \exp((\lambda_n-\lambda_i)\sigma)\cdot\left[\frac{\exp((\lambda_n-\lambda_i)it)}{i(\lambda_n-\lambda_i)}\right]_\alpha^{\alpha+T} \end{aligned}$$

so that

$$(2.6) \qquad |I_1| \leq \frac{2}{T}\cdot\sum_{i=1}^{n-1}|a_i|\cdot\frac{\exp((\lambda_n-\lambda_i)\sigma)}{\lambda_n-\lambda_i} = O\left(\frac{1}{T}\right)$$

Now let us define the angular domain $\mathcal{D}$ as follows:

$$\begin{cases} \text{(i)} & |\arg(s-\sigma_0)| \leq \vartheta < \pi/2, \quad \sigma_1 < \sigma_0 < \sigma \\ \text{(ii)} & \sigma + i\alpha \in \mathcal{D} \end{cases}$$

As regards I_2, by Cauchy's theorem we have

$$\begin{aligned} (2.7) \qquad I_2 &= \frac{1}{iT}\int_{\sigma+i\alpha}^{\sigma+i(\alpha+T)} \exp(-\tau_n s)\,g_2(s)\,ds \quad (\text{on } \Re(s)=\sigma) \\ &= \frac{1}{iT}\int_{\sigma+i\alpha}^{s_1} + \frac{1}{iT}\int_{s_1}^{s_2} + \frac{1}{iT}\int_{s_2}^{\sigma+i(\alpha+T)} \\ &\quad (\text{on } \Im(s)=\alpha) \quad (\text{on } \Re(s)=\Re(s_1)) \quad (\text{on } \Im(s)=\alpha+T) \\ &= I_1' + I_2' + I_3', \quad \text{say}, \end{aligned}$$

where (i) s_2 is the intersecting point of two straight lines:

$$\Im(s) = \alpha + T, \qquad \arg(s-\sigma_0) = \vartheta,$$

(ii) $s_1 = \sigma_1' + i\alpha, \quad \sigma_1' = \Re(s_2)$

By the well-known theorem ([1] p.5), $g_2(s)$ converges uniformly in $\mathcal{D}$, so that there exists a constant K such that

$$(2.8) \qquad |g_2(s)| < K \quad \text{for } s \in \mathcal{D}.$$

Since $\Im(s) = \alpha$ is contained in $\mathcal{D}$, by (2.8) we have

$$\begin{aligned} (2.9) \qquad |I_1'| &\leq \frac{1}{T}\int_\sigma^{\sigma_1'}\exp(-\tau_n\sigma)K\,d\sigma \\ &< \frac{K}{T}\int_\sigma^{+\infty}\exp(-\tau_n\sigma)\,d\sigma = O(1/T) \end{aligned}$$

Since the segment: $\alpha \leq \mathcal{J}(s) \leq \alpha+T$, $\mathcal{R}(s) = s_1$ is contained in $\mathcal{D}$, by (2.8) we get

$$|I_2'| \leq \frac{1}{T}\int_{\alpha}^{\alpha+T} \exp(-\tau_n \sigma_1)\, K\, dt = K \exp(-\tau_n \sigma_1)$$

Since $\sigma_1 = O(T)$, we have evidently

(2.10) $|I_2'| \leq K \exp(-\tau_n O(T)) = o(1)$ as $T \to +\infty$

Since $g_2(s)$ is simply convergent for $\sigma_s < \sigma$, by the above-mentioned theorem ([1] p.8) we have

$$g_2(\sigma + it) = o(|t|)$$

uniformly for $\sigma_0 \leq \sigma$. Hence,

(2.11) $$|I_3'| \leq \int_{\sigma}^{\sigma_2} \exp(-\tau_n \sigma) \left| \frac{g_2(\sigma + i(\alpha+T))}{T} \right| d\sigma < o(1) \int_{\sigma}^{\infty} \exp(-\tau_n \sigma)\, d\sigma = o(1)$$

as $T \to +\infty$

By (2.7), (2.9), (2.10), (2.11),

$$|I_2| < o(1) \quad \text{as } T \to +\infty,$$

so that by (2.5), (2.6) we get

$$\lim_{T\to\infty} \frac{1}{T} \int_{\alpha}^{\alpha+T} F(\sigma+it) \exp(\lambda_n(\sigma+it))\, dt = a_n$$

q.e.d.

(3.) PROOF OF THEOREM. We distinguish three cases:

(a) On $\mathcal{R}(s) = \sigma > \sigma_s$, $F(s)$ is bounded and it is also bounded on $\mathcal{R}(s) = \sigma'$ $(\sigma_s < \sigma' < \sigma)$, where σ' is a suitable constant.

(b) On $\mathcal{R}(s) = \sigma > \sigma_s$, $F(s)$ is bounded, but it is not so on $\mathcal{R}(s) = \sigma'$, for every σ' $(\sigma_s < \sigma' < \sigma)$.

(c) On $\mathcal{R}(s) = \sigma > \sigma_s$, $F(s)$ is unbounded.

Case (a): Applying Lemma 1, in which we put $\sigma_s < \sigma' < \sigma - \nu < \beta < \sigma$, we have

(3.1) $$F(s) = \sum_{\lambda_n < \omega} a_n \exp(-\lambda_n s)\{1 - \exp(\nu(\lambda_n - \omega))\} - \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} \frac{\nu F(z) \exp(\omega(z-s))}{(z-s)(z+\nu-s)}\, dz$$

By Lemma 2, there exists a constant M such that

$$|F(s)| \leq M \quad \text{for } \mathcal{R}(s) \geq \sigma'$$

Then we have

$$\left| \frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} \frac{\nu F(z) \exp(\omega(z-s))}{(z-s)(z+\nu-s)}\, dz \right|$$

$$\leq \frac{M\nu}{2\pi} \exp(-\omega(\sigma-\beta)) \int_{-\infty}^{+\infty} \left| \frac{1}{(z-s)(z+\nu-s)} \right| |dz|$$

$$= \frac{M\nu}{\pi} \exp(-\omega(\sigma-\beta)) \int_0^{\infty} \frac{1}{\sqrt{[\tau^2 + (\beta-\sigma)^2][\tau^2 + (\beta+\nu-\sigma)^2]}}\, d\tau$$

$$= O(1) \exp(-\omega(\sigma-\beta)),$$

where O(1): a constant independent upon t. Hence by (3.1) we get

$$F(\sigma+it) = \sum_{\lambda_n<\omega} a_n \exp(-\lambda_n(\sigma+it))\{1 - \exp(\nu(\lambda_n - \omega))\} + O(\exp(-\omega(\sigma-\beta)))$$

Since $\sigma - \beta > 0$, $O(\exp(-\omega(\sigma-\beta)))$ tends to 0 uniformly for $-\infty < t < +\infty$, so that

(3.2) $$F(\sigma+it) = \lim_{\omega\to+\infty} \sum_{\lambda_n<\omega} a_n \exp(-\lambda_n(\sigma+it))\{1 - \exp(\nu(\lambda_n-\omega))\}$$

for $-\infty < t < +\infty$. On the other hand, $\sum_{\lambda_n<\omega} a_n \exp(-\lambda_n(\sigma+it))\{1 - \exp(\nu(\lambda_n-\omega))\}$ is an almost periodic function of t, so that by (3.2) $F(\sigma+it)$ is also an almost periodic function of t ([5] p.186), which proves (a) of our theorem.

Case (b): Since $F(s)$ is bounded on $\mathcal{R}(s) = \sigma$, by Lemma 2 there exists a constant M such that

(3.3) $|F(s)| \leq M$ for $\mathcal{R}(s) \geq \sigma$

Let us define the rectangle $R(\varepsilon)$ as follows:

$$|\mathcal{R}(s) - \sigma| \leq \sigma, \quad |\mathcal{J}(s)| \leq 1 + \varepsilon, \quad (\varepsilon > 0)$$

On account of the hypothesis, on $\mathcal{R}(s) = \sigma - \frac{\varepsilon}{2}$ $F(s)$ is unbounded. Then without any loss of generality we can assume that $F(s)$ is unbounded for $\mathcal{R}(s) = \sigma - \frac{\varepsilon}{2}$, $\mathcal{J}(s) \geq 0$, so that there exists a sequence $\{s_k\}$ such that

(3.4) $$\begin{cases} \text{(i)} & \mathcal{R}(s_k) = \sigma - \frac{\varepsilon}{2}, \quad \mathcal{J}(s_k) \to +\infty \\ \text{(ii)} & \lim_{k\to\infty} |F(s_k)| = +\infty \end{cases}$$

Now in $R(\varepsilon/2)$ we consider the functions-family $\{F_n(s)\} = \{F(s + i2n)\}$

($n = 1,2,\ldots$). By (3.4) in $R(\varepsilon/2)$ there exists a sequence $\{s_k\}$ such that

$$(3.5)\quad \begin{cases} \text{(i)}\ \ \Re(s_k) = \sigma - \frac{\varepsilon}{2}, \\ \text{(ii)}\ \ S_k = s_k + i2n_k, \\ \text{(iii)}\ \ |F_{n_k}(s_k)| \to +\infty \quad as\ k \to +\infty \end{cases}$$

Then $\{F_n(s)\}$ is not normal in $R(\varepsilon/2)$. In fact, by (3.3) and (3.5), any partial sequence of $\{F_{n_k}(s)\}$ neither tends to ∞ uniformly nor tends to the finite analytic function in $R(\varepsilon/2)$. Hence, there exists at least one not-normal point of $\{F_n(s)\}$ in $R(\varepsilon/2)$, so that in $|\Re(s)-\sigma|<\varepsilon$, $\Im(s) > -1-\varepsilon$, a fortiori in the vertical strip: $|\Re(s)-\sigma|<\varepsilon$ $F(s)$ assumes every value, except perhaps two (∞ included), infinitely many times, which proves the second part of (b) of our theorem.

If $F(\sigma+it)$ should tend to the finite value β as $t\to+\infty$, then without any loss of generality, we could assume that $\beta = 0$. In fact, it suffices to consider $F(s)-\beta$ instead of $F(s)$. Hence there exists a constant $T_0(\varepsilon)$ such that

$$|F(\sigma+it)|<\varepsilon \quad for \quad t > T_0(\varepsilon)$$

Therefore by Lemma 3 we have

$$(3.6)\quad a_n = \lim_{T\to\infty} \frac{1}{T}\cdot\int_\alpha^{\alpha+T} F(\sigma+it)\exp(\lambda_n(\sigma+it))\,dt$$
$$= \lim_{T\to\infty}\left\{\frac{1}{T}\int_\alpha^{T_0} + \frac{1}{T}\int_{T_0}^{\alpha+T}\right\}$$
$$= \lim_{T\to\infty}(I_1 + I_2), \quad say$$

Then we get

$$|I_1| \leq \frac{1}{T}\cdot\max_{\alpha\leq t\leq T_0}|F(\sigma+it)|\ (T_0-\alpha)\cdot\exp(\lambda_n\sigma) = O(\tfrac{1}{T}),$$

$$|I_2| \leq (T+\alpha-T_0)\cdot\frac{1}{T}\ \varepsilon\ \exp(\lambda_n\sigma) = O(\varepsilon)$$
$$as\quad T\to+\infty$$

Hence, by (3.6) we have

$$|a_n| \leq o(1) + O(\varepsilon)$$

Letting $\varepsilon\to 0$, $a_n = 0$ ($n = 1,2,\ldots$), which is impossible. Thus the first part of (b) is proved.

Case (c): If we should have simultaneously

$$\lim_{t\to+\infty\,(or\,-\infty)} |F(\sigma+it)| = +\infty,$$

$$|\arg F(\sigma+it) - \theta| \leq \vartheta < \pi/2,$$

then without any loss of generality we could assume that $\lambda_1 = 0$, $\theta = 0$. In fact, it suffices to consider $F(s)\exp(-i\theta) + \alpha$ instead of $F(s)$, where

$$\alpha = 0, \quad if\ a_1 \neq 0,\ \lambda_1 = 0$$

$$\alpha > 0, \quad if\ a_1 \neq 0,\ \lambda_1 > 0$$

Then, by Lemma 3, we get

$$(3.7)\quad a_1 = \lim_{T\to\infty}\frac{1}{T}\cdot\int_\alpha^{\alpha+T} F(\sigma+it)\,dt$$

Since $\lim_{t\to+\infty}|F(\sigma+it)| = +\infty$, $|\arg F(\sigma+it)| \leq \vartheta < \pi/2$, we have

$$(3.8)\quad \begin{cases} \text{(i)}\ \ |F(\sigma+it)| > K \quad for\ t > T_0(K) \\ \quad\text{where } K\text{ : an arbitrary positive constant.} \\ \text{(ii)}\ \ \Re\{F(\sigma+it)\} \geq \cos\vartheta\ |F(\sigma+it)| \end{cases}$$

By (3.7), (3.8)

$$\Re(a_1) = \lim_{T\to\infty}\frac{1}{T}\int_\alpha^{\alpha+T}\Re\{F(\sigma+it)\}\,dt$$
$$\geq \varliminf_{T\to\infty}\frac{\cos\vartheta}{T}\int_\alpha^{\alpha+T}|F(\sigma+it)|\,dt$$
$$= \cos\vartheta\ \varliminf_{T\to\infty}\left\{\frac{1}{T}\cdot\int_\alpha^{T_0} + \frac{1}{T}\int_{T_0}^{\alpha+T}\right\}$$
$$\geq \cos\vartheta\left\{\varliminf_{T\to\infty}\frac{(T_0-\alpha)}{T}\min_{\alpha\leq t\leq T_0}|F(\sigma+it)| + \lim_{T\to\infty}\frac{T+\alpha-T_0}{T}\,K\right\} = K\cos\vartheta$$

Letting $K\to+\infty$, $\Re(a_1) = +\infty$, which is impossible. Thus (c) of our theorem is proved.

(*) Received September 25, 1951.

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