NOTE ON LAPLACE-TRANSFORMS (VI)

ON THE DISTRIBUTION OF ZEROS OF PARTIAL SUMS OF LAPLACE-TRANSFORMS

By Chuji TANAKA

(Communicated by Y. Komatu)

(1) INTRODUCTION. Concerning the distribution of zeros of partial sums of Taylor series, the following theorems are known:

JENTZSCH'S THEOREM ([1] p.p.92-95). Let $f(z)=\sum_{0}^{\infty} a_n z^n$ be convergent for $|z|<1$. Then, every point on $|z|=1$ is an accumulation point of zeros of partial sums

$$S_n(z)=\sum_{\nu=0}^{n} a_\nu z^\nu \quad (n=1,2,\cdots).$$

SZEGÖ'S THEOREM. Let $f(z)=\sum_{0}^{\infty} a_n z^n$ be convergent for $|z|<1$. Then, every point on $|z|=1$ is an accumulation point of zeros of partial sums

$$S_{n_k}(z)=\sum_{\nu=0}^{n_k} a_\nu z^\nu \quad (k=1,2,\cdots)$$

with $\quad \lim_{k\to\infty} n_{k+1}/n_k = 1$

In this Note, we shall establish analogous theorems in Laplace-transforms.

(2) THEOREM I. Let us put

$$(2.1) \qquad F(s)=\int_0^\infty \exp(-sx)\, d\alpha(x)$$

$$(s=\sigma+it, \quad \alpha(0)=0),$$

where $\alpha(x)$ is of bounded variation in any finite interval $0 \leqq x \leqq X$, X being an arbitrary positive number. As an extension of Jentzsch's theorem, we can prove

THEOREM I. Let (2.1) be simply convergent for $\sigma>0$. Then, every point on $\sigma=0$ is an accumulation point of zeros of partial sums

$$S_y(s)=\int_0^y \exp(-sx)\, d\alpha(x)$$

$$[x_\nu] \leqq y \leqq x_\nu \quad (\nu=1,2,\cdots)$$

where

$$(2.2) \qquad 0=\sigma_s=\overline{\lim_{x\to\infty}} \frac{1}{x} \log\left|\int_{[x]}^{x} d\alpha(x)\right|$$

$$=\lim_{\nu\to\infty} \frac{1}{x_\nu} \log\left|\int_{[x_\nu]}^{x_\nu} d\alpha(x)\right|$$

REMARK. By T. Ugaheri's theorem ([2]), the simple convergence-abscissa σ_s of (2.1) is determined by

$$\sigma_s=\overline{\lim_{x\to\infty}} \frac{1}{x} \log\left|\int_{[x]}^{x} d\alpha(x)\right|$$

Hence, the sequence $\{x_\nu\}$ satisfying (2.2) certainly exists.

Next corollary 1 immediately follows.

COROLLARY I. Let (2.1) be simply convergent for $\sigma>0$. Then, every point on $\sigma=0$ is an accumulation point of zeros of partial sums

$$S_y(s)=\int_0^y \exp(-sx)\, d\alpha(x)$$

$$(0<y<+\infty)$$

Putting $\alpha(x)=\sum_{\lambda_n<x} a_n \; (0\leq\lambda_1<\cdots<\lambda_n\to+\infty)$, we have easily

COROLLARY II. Let Dirichlet series $F(s)=\sum_1^\infty a_n \exp(-\lambda_n s)$ be simply convergent for $\sigma>0$. Then, every point on $\sigma=0$ is an accumulation point of zeros of partial sums

$$S_n(s)=\sum_{i=1}^{n} a_i \exp(-\lambda_i s) \quad (n=1,2,\cdots).$$

Putting $\alpha(x)=\int_0^x f(y)\,dy$, where $f(y)$ is R-integrable in any finite interval $0\leqq y\leqq Y$, Y being an arbitrary positive number, we get

COROLLARY III. Let Laplace-transform $F(s)=\int_0^\infty \exp(-sx) f(x)\,dx$ be simply convergent for $\sigma>0$. Then, every point on $\sigma=0$ is an accumulation point of zeros of partial sums

$$\int_0^y \exp(-sx) f(x)\,dx \quad (0<y<+\infty)$$

(3) LEMMAS. For the proof of Theorem I, we need next Lemma.

LEMMA 1. Let (2.1) be simply convergent for $\sigma>0$. Then, we have

$$(3.1) \qquad \frac{1}{y} \log|S_y(s)|$$

$$s\in P, \quad \sigma\leqq\eta<\sigma_0 \quad (\eta>0)$$

$$< -\sigma + \sigma_0 + \frac{1}{y}\, C(\sigma_0, \eta, \mathcal{D})$$

where (i) $S_y(s) = \int_0^y \exp(-sx)\, d\alpha(x)$, (ii) $\mathcal{D}$ is a bounded domain, (iii) $C(\sigma_0, \eta, \mathcal{D})$ is a constant depending upon only σ_0, η, $\mathcal{D}$.

<u>Proof.</u> Since $F(\sigma_0)$ $(\sigma_0 > 0)$ is convergent, there exists a constant $K(\sigma_0)$ such that

$$(3.2) \quad |S_y(\sigma_0)| < K(\sigma_0) \qquad (0 \leqq y < +\infty)$$

By integration by parts.

$$\begin{aligned}
S_y(s) &= \int_0^y \exp(-sx)\, d\alpha(x) \\
&= \int_0^y \exp(-(s-\sigma_0)x)\, \exp(-\sigma_0 x)\, d\alpha(x) \\
&= \left[S_x(\sigma_0) \exp(-(s-\sigma_0)x) \right]_0^y \\
&\quad + (s-\sigma_0) \int_0^y S_x(\sigma_0) \exp(-(s-\sigma_0)x)\, dx \\
&= S_y(\sigma_0) \exp(-(s-\sigma_0)y) \\
&\quad + (s-\sigma_0) \int_0^y S_x(\sigma_0) \exp(-(s-\sigma_0)x)\, dx
\end{aligned}$$

Hence, by (3.2), we get, for $s \in \mathcal{D}$ and $\sigma \leqq \eta < \sigma_0$ $(\eta > 0)$,

$$\begin{aligned}
&|S_y(s)| \\
&\leqq K(\sigma_0) \exp(-(\sigma-\sigma_0)y) + |s-\sigma_0|\, K(\sigma_0) \int_0^y \exp(-(\sigma-\sigma_0))\, dx \\
&< K(\sigma_0) \exp(-(\sigma-\sigma_0)y) \left\{ 1 + \frac{|s-\sigma_0|}{\sigma_0 - \sigma} \right\} \qquad (\sigma - \sigma_0 < 0) \\
(3.3) \quad &< K(\sigma_0) \exp(-(\sigma-\sigma_0)y) \left\{ 1 + \frac{d(\sigma_0, \mathcal{D})}{\sigma_0 - \eta} \right\},
\end{aligned}$$

where $d(\sigma_0, \mathcal{D}) = \max_{s \in \mathcal{D}} |s - \sigma_0|$, so that, for $s \in \mathcal{D}$ and $\sigma \leqq \eta < \sigma_0$ $(\eta > 0)$,

$$\begin{aligned}
&\frac{1}{y} \log |S_y(s)| \\
&< -\sigma + \sigma_0 + \frac{1}{y}\, C_1(\sigma_0, \eta, \mathcal{D}),
\end{aligned}$$

where $C_1(\sigma_0, \eta, \mathcal{D}) = \log\left\{ K(\sigma_0) \left(1 + \frac{d(\sigma_0, \mathcal{D})}{\sigma_0 - \eta}\right) \right\}$,

q. e. d.

(4) <u>PROOF OF THEOREM I.</u> If $s_0 = it_0$ were not an accumulation point of zeros of $S_y(s)$, $[x_\nu] \leqq y \leqq x_\nu$ $(\nu = 1, 2, \cdots)$, there would exist a sufficiently small circle $|s - s_0| \leqq \delta$ in which

$$\frac{1}{y} \log S_y(s)$$

$$([x_\nu] \leqq y \leqq x_\nu, \quad \nu = 1, 2 \quad \cdots)$$

would be regular. By Lemma 1, $\frac{1}{y} \log |S_y(s)|$ is bounded and since $S_y(s)$ converges uniformly in any finite domain contained in $\sigma > 0$ ([3] p.54), we have uniformly

$$\lim_{y \to \infty} \frac{1}{y} \log |S_y(s)| = 0 \quad \text{in } \mathcal{D},$$

where $\mathcal{D}$ is contained in $|s - s_0| < \delta$, $\sigma > 0$. Hence, by a well-known Vitali's theorem, we have

$$(4.1) \quad \lim_{y \to \infty} \frac{1}{y} \log |S_y(s_1)| = 0 \qquad s_1 = -\frac{\delta}{2} + it$$

$$\left(\begin{array}{c} [x_\nu] \leqq y \leqq x_\nu \\ \nu = 1, 2 \ \cdots \end{array} \right)$$

Putting $R_{y, y'}(s) = S_{y'}(s) - S_y(s)$, $y' > y$, and by integration by parts,

$$\begin{aligned}
&\int_{[x_\nu]}^{x_\nu} d\alpha(x) \\
&= R_{[x_\nu], x_\nu}(s_1) \exp(s_1 x_\nu) - s_1 \int_{[x_\nu]}^{x_\nu} R_{[x_\nu], x}(s_1) \exp(s_1 x)\, dx \\
&= \left\{ S_{x_\nu}(s_1) - S_{[x_\nu]}(s_1) \right\} \exp(s_1 x_\nu) \\
&\qquad - s_1 \int_{[x_\nu]}^{x_\nu} \left\{ S_x(s_1) - S_{[x_\nu]}(s_1) \right\} \exp(s_1 x)\, dx
\end{aligned}$$

Setting

$$\max_{[x_\nu] \leqq y \leqq x_\nu} |S_y(s_1)| = |S_{y_\nu}(s_1)|, \quad ([x_\nu] \leqq y_\nu \leqq x_\nu),$$

$$\begin{aligned}
&\left| \int_{[x_\nu]}^{x_\nu} d\alpha(x) \right| \\
&\leqq 2 |S_{y_\nu}(s_1)| + \frac{s_1}{|\kappa(s_1)|}\, 2 |S_{y_\nu}(s_1)| \left[\exp(\kappa(s_1) x) \right]_{[x_\nu]}^{x_\nu} \\
&< 2 |S_{y_\nu}(s_1)| \frac{|s_1|}{|\kappa(s_1)|} \exp(\kappa(s_1)[x_\nu])
\end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{1}{x_\nu} \log \left| \int_{[x_\nu]}^{x_\nu} d\alpha(x) \right| \\
&< \frac{1}{x_\nu} \log K(s_1) + \kappa(s_1) \frac{[x_\nu]}{x_\nu} + \frac{y_\nu}{x_\nu}\, \frac{1}{y_\nu} \log |S_{y_\nu}(s_1)|
\end{aligned}$$

where $K(s_1) = 2 \frac{|s_1|}{|\kappa(s_1)|}$. By (4.1)

$$0 = \lim_{\nu \to \infty} \frac{1}{x_\nu} \log \left| \int_{[x_\nu]}^{x_\nu} d\alpha(x) \right| \leqq -\frac{\delta}{2} < 0$$

which is impossible. Thus, every point on $\sigma = 0$ is an accumulation point of zeros of $S_y(s)$, $[x_\nu] \leqq y \leqq x_\nu$ $(\nu = 1, 2, \cdots)$, q.e.d.

(5) <u>THEOREM II.</u> In this section, we shall prove an extension of Szegő's theorem.

<u>THEOREM II.</u> <u>Let (2.1) be simply convergent for $\sigma > 0$. If, furthermore, the regularity-abscissa σ_r is 0, then every point on $\sigma = 0$ is the accumulation point of zeros of $S_{y_\nu}(s) = \int_0^{y_\nu} \exp(-sx)\, d\alpha(x)$ $(\nu = 1, 2, \cdots)$ with $\lim_{\nu \to \infty} \log \nu / y_\nu = 0$, and $\lim_{\nu \to \infty} y_{\nu+1} / y_\nu = 1$.</u>

If $d\alpha(x) \geqq 0$ $x \geqq X$, by H. Hamburger's theorem ([3] p.59), $s = 0$ is a singular point of (2.1). Hence, we have

<u>COROLLARY I.</u> <u>Let (2.1), with $d\alpha(x) \geqq 0$ for $x \geqq X$, be simply convergent for $\sigma > 0$. Then, every point on $\sigma = 0$ is the accumulation point of zeros of $S_{y_\nu}(s)$ with $\lim_{\nu \to \infty} \log \nu / y_\nu = 0$ and $\lim_{\nu \to \infty} y_{\nu+1} / y_\nu = 1$.</u>

Applying theorem 2 to Dirichlet series, we get

COROLLARY II. Let $F(s)=\sum_{n}^{\infty} a_n \exp(-\lambda_n s)$ $(s=\sigma+it)$ with $\varliminf (\lambda_{n+1}-\lambda_n^{1})>0$ be simply convergent for $\sigma>0$. Then, every point on $\sigma=0$ is an accumulation point of zeros of

$$S_{n_k}(s) = \sum_{n=1}^{n_k} a_n \exp(-\lambda_n s) \quad (k=1,2,\cdots)$$

with $\lim_{k\to\infty} \lambda_{n_{k+1}}/\lambda_{n_k}=1$

In fact, putting $\varliminf_{n\to\infty}(\lambda_{n+1}-\lambda_n)=g>0$, by G.Pólya's theorem ([4] p.140), every closed segment on $\sigma=0$ with length greater than $2\pi/g$ contains at least one singular point, and we have easily $\lim_{n\to\infty}\frac{1}{\lambda_n}\log n=0$ (a fortiori, $\lim_{k\to\infty}\frac{1}{\lambda_{n_k}}\log k=0$). Hence, corollary 2 follows immediately from Theorem 2.

Applying Theorem 2 to Laplace-transform, we get

COROLLARY III. Let $f(z)$ $(z=re^{i\theta})$ be an integral function of exponential type. If $F(s)=\int_0^\infty \exp(-sx)f(x)\,dx$ is simply convergent for $\sigma>0$, then every point on $\sigma=0$ is an accumulation point of zeros of $S_{y_\nu}(s)=\int_0^{y_\nu}\exp(-sx)f(x)\,dx$ $(\nu=1,2,\cdots)$ with $\lim_{\nu\to\infty}\log\nu/y_\nu=0$, and $\lim_{\nu\to\infty} y_{\nu+1}/y_\nu=1$.

In fact, by the theorem ([4] p.298, [5] p.75, [6]), there exists at least one singular point on $\sigma=0$, so that corollary follows from Theorem 2.

(6) LEMMA. For the proof of Theorem 2, we need next Lemma.

LEMMA 2. Let (2.1) be simply convergent for $\sigma>0$. Then we have

$$(6.1)\qquad \frac{1}{y_\nu}\log|R_\nu(s)| < -\sigma+\varepsilon_\nu(s), \quad s\in\mathcal{D}$$

where (i) $R_\nu(s)=S_{y_{\nu+1}}(s)-S_{y_\nu}(s)$, $\lim_{\nu\to\infty} y_{\nu+1}/y_\nu=1$,

(ii) $\mathcal{D}$ is a bounded domain,

(iii) $\varepsilon_\nu(s)\to 0$ as $\nu\to\infty$ uniformly in $\mathcal{D}$.

Proof. Let us put $\eta=\frac{1}{2}(\sigma_0+\sigma_0^2)$, $0<\sigma_0<1$. Since $F(\sigma_0^2)$ is convergent, there exists $K_1(\sigma_0)$ such that

$$(6.2)\qquad |R_{y,y'}(\sigma_0^2)|=\left|\int_y^{y'}\exp(-\sigma_0^2x)\,d\alpha(x)\right| < K_1(\sigma_0)$$

for $y'>y\geqq 0$

By integration by parts,

$$R_{y,y'}(s)=\int_y^{y'}\exp(-sx)\,d\alpha(x)$$

$$=\int_y^{y'}\exp(-(s-\sigma_0^2)x)\exp(-\sigma_0^2x)\,d\alpha(x)$$

$$=R_{y,y'}(\sigma_0^2)\exp(-(s-\sigma_0^2)y)$$

$$+(s-\sigma_0^2)\int_y^{y'}R_{y,x}(\sigma_0^2)\exp(-(s-\sigma_0^2))\,dx$$

Hence, by (6.2)

$$\underset{s\in\mathcal{D},\ \sigma\geqq\eta}{|R_{y,y'}(s)|}$$

$$\leqq K_1(\sigma_0)\exp(-(\sigma-\sigma_0^2)y')+|(s-\sigma_0^2)|\,K_1(\sigma_0)\int_y^{y'}\exp(-(\sigma-\sigma_0^2)x)\,dx$$

$$(6.3)\qquad < \frac{|s-\sigma_0^2|}{\eta-\sigma_0}K_1(\sigma_0)\exp(-(\sigma-\sigma_0^2)y)\qquad (\sigma-\sigma_0^2>0)$$

In particular,

$$\underset{s\in\mathcal{D},\ \sigma\geqq\eta}{|R_\nu(s)|}=\left|\int_{y_\nu}^{y_{\nu+1}}\exp(-sx)\,d\alpha(x)\right|$$

$$<\frac{|s-\sigma_0^2|}{\eta-\sigma_0^2}K_1(\sigma_0)\exp(-(\sigma-\sigma_0^2)y_\nu),$$

so that

$$(6.4)\qquad \frac{1}{y_\nu}\underset{s\in\mathcal{D},\ \sigma\geqq\eta}{\log|R_\nu(s)|}$$

$$<-\sigma+\sigma_0^2+\frac{1}{y_\nu}C_1(\sigma_0,\mathcal{D}),$$

where $C_1(\sigma_0,\mathcal{D})=\log\left\{K_1(\sigma_0)\dfrac{d(\sigma_0^2,\mathcal{D})}{\eta-\sigma_0^2}\right\}$, $d(\sigma_0^2,\mathcal{D})=\max_{s\in\mathcal{D}}|s-\sigma_0^2|$

By (3.3),

$$\underset{s\in\mathcal{D},\ \sigma\leqq\eta}{|R_\nu(s)|}\leqq|S_{y_{\nu+1}}(s)|+|S_{y_\nu}(s)|$$

$$<2K(\sigma_0)\exp(-(\sigma-\sigma_0)y_{\nu+1})\left\{1+\frac{d(\sigma_0,\mathcal{D})}{\sigma_0-\eta}\right\}$$

Hence,

$$(6.5)\qquad \frac{1}{y_\nu}\underset{s\in\mathcal{D},\ \sigma\leqq\eta}{\log|R_\nu(s)|}$$

$$<(\sigma_0-\sigma)\cdot\frac{y_{\nu+1}}{y_\nu}+\frac{1}{y_\nu}C_2(\sigma_0,\mathcal{D})$$

$$=-\sigma+\sigma_0+o(1)+\frac{1}{y_\nu}\cdot C_2(\sigma_0,\mathcal{D})\quad \left(\text{by }\lim_{\nu\to\infty}\frac{y_{\nu+1}}{y_\nu}=1\right),$$

where

$$C_2(\sigma_0,\mathcal{D})=\log\left\{2K(\sigma_0)\left(1+\frac{d(\sigma_0,\mathcal{D})}{\sigma_0-\eta}\right)\right\}$$

By (6.4) and (6.5), we finally get

$$\frac{1}{y_\nu}\underset{s\in\mathcal{D}}{\log|R_\nu(s)|}$$

$$<-\sigma+\sigma_0+\sigma_0^2+o(1)+\frac{1}{y_\nu}\max\{C_1,C_2\}.$$

Since σ_0 is arbitrarily small, we can put

$$\varepsilon_\nu(s)=\sigma_0+\sigma_0^2+o(1)+\frac{1}{y_\nu}\max\{C_1,C_2\}\qquad \text{q.e.d.}$$

(7) PROOF OF THEOREM II. If $s_0=it_0$ were not an accumulation point of zeros of $S_{y_\nu}(s)$ $(\nu=1,2,\cdots)$, by the entirely similar argument as in the proof of Theorem 1, we should have

$$\lim_{\nu\to\infty}|S_{y_\nu}(s)|^{1/y_\nu}=1$$

uniformly in $|s-s_0|\leqq\delta$,

whence, for any given $\varepsilon\ (>0)$,

$$(7.1)\qquad |S_{y_\nu}(s)|<(1+\varepsilon)^{y_\nu}$$

for $|s-s_0|\leqq\delta$, $\nu>N(\varepsilon)$

Since $|R_\nu(s)| \leqq |S_{y_\nu}(s)| + |S_{y_{\nu+1}}(s)|$, by (7.1),

$$\frac{1}{y_\nu} \log |R_\nu(s)|$$
$$< \frac{1}{y_\nu} \log 2 + {y_{\nu+1}}/{y_\nu} \log (1+\varepsilon)$$

for $|s - s_0| \leqq \delta$, $\nu > \nu(\varepsilon)$. By $\lim_{\nu\to\infty} {y_{\nu+1}}/{y_\nu} = 1$, we can put

$$(7.2) \qquad \frac{1}{y_\nu} \log |R_\nu(s)| < \varepsilon_\nu(s)$$
$$\text{for} \quad |s - s_0| \leqq \delta,$$

where $\varepsilon_\nu(s) \to 0$ as $\nu \to \infty$ uniformly in $|s - s_0| \leqq \delta$.

Let us define a closed analytic curve C such that C contains a singular point s_1 on $\sigma = 0$, and the arc $\widehat{\alpha\alpha'}$ contained in $|s - s_0| \leqq \delta$ lies strictly to the left of the imaginary axis. Now we define a harmonic function $h(s)$ such that

(i) on $\widehat{\alpha\alpha'}$, $h(s)$ continuous, and $0 < h(s) < -\delta \ (>0)$,
(ii) on the complementary of $\widehat{\alpha\alpha'}$, $h(s) = -\sigma$.

Then, we have evidently $h(s) < -\sigma$ in C. In particular,

$$(7.3) \qquad h(s_1) < -\Re(s_1) = 0$$

By the definition of $h(s)$, Lemma 1 and (7.2), we obtain, in C,

$$\frac{1}{y_\nu} \log |R_\nu(s)| < h(s) + \varepsilon_\nu(s)$$

Therefore, by (7.3),

$$\frac{1}{y_\nu} \log |R_\nu(s_1)|$$
$$< h(s_1) + \varepsilon_\nu(s_1) < 0 + \varepsilon_\nu(s_1),$$

so that, in a neighbourhood of s_1, we can put

$$\frac{1}{y_\nu} \log |R_\nu(s)| < \log \rho \quad (0 < \rho < 1), \quad \text{i.e}$$

$$(7.4) \qquad |R_\nu(s)| < \exp\left(-y_\nu \log(\tfrac{1}{\rho})\right)$$

Since, by $\lim_{\nu\to\infty} \log \nu / y_\nu = 0$ and G.Valiron's theorem ([4] p.4 [7]), $\sum_1^\infty \exp(-y_\nu s)$ is convergent for $\Re(s) > 0$, in a neighbourhood of s_1, by (7.4),

$$\sum_{\nu=1}^\infty |R_\nu(s)|$$
$$= \sum_{\nu=1}^\infty |S_{y_{\nu+1}}(s) - S_{y_\nu}(s)|$$
$$< \sum_1^\infty \exp\left(-y_\nu \log(\tfrac{1}{\rho})\right) < +\infty$$

Hence, $s = s_1$, is a regular point, which is impossible.

If there exists no singular point on $\sigma = 0$, and the sequence of singular points $\{s_n\} \equiv \{\sigma_n + it_n\}$ $(n = 1, 2, \cdots)$ approach $\sigma = 0$ asymptotically, then taking it$_n$ for sufficiently large n as s_1 in the above proof, we can establish our theorem in this case. This completes our proof.

(*) Received July 28, 1951.

[1] E.Landau: Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie. Berlin, 1929.

[2] T.Ugaheri: On the abscissa of convergence of Laplace-Stieltjes integral. Annals of the Institute of Statistical Mathematics. Vol.II. No.1. 1950;
C.Tanaka: Note on Laplace-transforms. (I) On the determination of three convergence-abscisses.

[3] D.V.Widder: The Laplace-transform. Princeton. 1946.

[4] V.Bernstein: Leçons sur les progrès récents de la théorie des séries de Dirichlet. Paris. 1933.

[5] G.Doetsch: Theorie und Anwendung der Laplace-Transformation. 1937.

[6] C.Tanaka: Note on Laplace-transforms. (III) On some class of Laplace-transform. (II)

[7] G.Valiron: Sur l'abscisse de convergence des series de Dirichlet. Bull.Soc.Math.de France, t.52. 1924.

Mathematical Institute,
Waseda University, Tokyo.