VALIRON'S THEOREM ON PICARD'S CURVES

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Let $w=f(z)$ be meromorphic in $1\leq|z|<\infty$ with an essential singularity at $z=\infty$. For any $-\pi\leq\delta<\pi$, $-\infty<k<+\infty$, we denote by $\Lambda(k,\delta)$ the logarithmic spiral

$$z(t)=e^{i\delta}t^{1+ik}\qquad(1\leq t<\infty,\ z(1)=e^{i\delta}).$$

For any z_0 and $\varepsilon>0$, let $\Gamma(z_0,\varepsilon)$ denote the disc $|z-z_0|\leq\varepsilon|z_0|$, and let $\Delta(k,\delta,\varepsilon)$ be the part of $1\leq|z|<\infty$, which is covered by the discs $\Gamma(z(t),\varepsilon)$ $(1\leq t<\infty)$. Then Valiron proved[1]:

There exists in $-\infty<k<+\infty$ a set E of measure zero, such that for any $k\notin E$ and for any δ and ε, $f(z)$ takes any value, with two possible exceptions, infinitely often in $\Delta(k,\delta,\varepsilon)$.

Let $\sigma_n\to\infty$ (n=1, 2, ...) be a sequence of points on z-plane. If the family of functions $\{f(\sigma_n z)\}$ is normal in $0<|z|<\infty$ for any sequence $\{\sigma_n\}$, then we call f(z) a Julia's exceptional function or J-exceptional[2]. If f(z) is not J-exceptional, there exists a sequence $\{\sigma_n\}$ and a point z_0 in $0<|z|<\infty$, such that $\{f(\sigma_n z)\}$ is not normal at z_0.

We will prove:

Theorem 1. Let $w=f(z)$ be meromorphic in $1\leq|z|<\infty$ with an essential singularity at $z=\infty$. If f(z) is not J-exceptional, there exists in $-\infty<k<+\infty$ a set E of measure zero, such that,

(i) if D_1, D_2, D_3 are three simply connected closed domains on w-plane lying outside each others, then, for a certain one D among these three, $\Delta(k,\delta,\varepsilon)$ contains infinitely many simply connected islands above D for any $k\notin E$ and for any δ and ε; and

(ii) if $D_1,\ldots,D_5$ are five domains of mentioned sort, then, for a certain one D among these five, $\Delta(k,\delta,\varepsilon)$ contains infinitely many schlicht islands above D for any $k\notin E$ and for any δ and ε.

Theorem 2. Let $w=f(z)$ be meromorphic in $1\leq|z|<\infty$ with an essential singularity at $z=\infty$. Then there exists in $-\infty<k<+\infty$ a set E of measure zero, such that,

(i) if a is any point on w-plane other than certain exceptional values a_i which are at most two in number, then $\Delta(k,\delta,\varepsilon)$ contains infinitely many a-points of f(z) for any $k\notin E$ and for any δ and ε; and

(ii) if a is any point other than certain exceptional values a_i which are at most four in number, then $\Delta(k,\delta,\varepsilon)$ contains infinitely many simple a-points of f(z) for any $k\notin E$ and for any δ and ε.

The first part of Theorem 2 contains Valiron's theorem. We remark that Theorem 1 does not hold for J-exceptional functions. In fact, if there exists a sequence of islands Δ_n (n=1, 2,...) above a closed domain D, which are contained in $\Delta(k,\delta,1/n)$ respectively, it is easily seen that f(z) can not be J-exceptional.

First we will prove the following

Lemma 1. Let E be a set of positive measure in $-\infty<k<+\infty$. Then, for any δ and ε, the sum of all $\Delta(k,\delta,\varepsilon)$ for $k\in E$ covers a certain neighbourhood of $z=\infty$.

Proof. Without loss of generality we can assume $\delta=0$. By $\zeta=\log z$ we map $1\leq|z|<\infty$ on the right half of $\zeta=\xi+i\eta$-plane. Then $\Lambda(k,0)$ is mapped on a countable number of parallel half straight-lines: $\eta\equiv k\xi$ (mod. 2π), $0\leq\xi<\infty$. It suffices to prove that the sum of all the strips: $k\xi+2n\pi-\varepsilon<\eta<k\xi+2n\pi+\varepsilon$, $0<\xi<\infty$, for $k\in E$ and n=0, ±1, ±2,..., covers a certain half-plane $\xi>\xi_0(\varepsilon)$.

Suppose that this were false, then we could find a sequence of points $\xi_\nu+i\eta_\nu$ ($\nu=1, 2,\ldots;\ 0<\xi_1<\xi_2<\ldots<\xi_\nu\to\infty$, $-\pi\leq\eta_\nu<+\pi$), such that $k\xi_\nu$ mod. 2π does not fall in the interval $I_\nu=(\eta_\nu-\varepsilon,\eta_\nu+\varepsilon)$ for any $k\in E$. Let η^* be one of the limiting values of η_ν, then, by taking suitable subsequence, we can assume that η_ν ($\nu=1, 2,\ldots$) are contained in the interval $I^*=(\eta^*-\varepsilon/2,\ \eta^*+\varepsilon/2)$ and further $\xi_{\nu+1}-\xi_\nu\geq$ const.>0. Then, since $I^*\subset I_\nu$, $k\xi_\nu$ mod. 2π does not fall in the interval I^* for any $k\in E$ and for any ν.

On the other hand, H. Weyl proved[3]: if $\xi_{\nu+1}-\xi_\nu\geq$ const. >0, the sequence $k\xi_1$, $k\xi_2$,.... mod. 2π is uniformly dense in the interval $(0, 2\pi)$ for any k with exception of a set of measure zero. Hence E must be of measure zero, which contradicts the hypothesis.

From Lemma 1 follows

Lemma 1'. Let $z_n \to \infty$ $(n=1, 2, \ldots)$ be a sequence of points on z-plane. Then, there exists a set E of measure zero, such that $\Delta(k, \delta, \varepsilon)$ contains infinitely many points of $\{z_n\}$ for any $k \notin E$ and for any δ and ε.

Proof. Let δ and ε be fixed. By Lemma 1 we see that, for any positive integer λ, the set $E_\lambda(\delta, \varepsilon)$ of values of k, such that $\Delta(k, \delta, \varepsilon)$ contains none of $z_\lambda, z_{\lambda+1}, \ldots$, is of measure zero. We put $E(\delta, \varepsilon) = \sum_{\lambda=1}^{\infty} E_\lambda(\delta, \varepsilon)$, so that $mE(\delta, \varepsilon)=0$. Then, for any $k \notin E(\delta, \varepsilon)$, $\Delta(k, \delta, \varepsilon)$ contains infinitely many points of $\{z_n\}$.

Next, let $\{e^{i\delta_\mu}\}$ $(\mu = 1, 2, \ldots)$ be a sequence of points, which are dense on $|z| = 1$. For each pair of positive integers μ and ν, we construct the exceptional set $E(\delta_\mu, 1/\nu)$ and put

$E = \sum_{\mu,\nu}^{\infty} E(\delta_\mu, 1/\nu)$. This E satisfies the condition of the lemma, since, for any δ and ε, $\Delta(k, \delta, \varepsilon)$ contains $\Delta(k, \delta_\mu, 1/\nu)$ for suitable values of μ and ν.

Proof of Theorem 1.

(i). First, we will prove that there exists, for at least a certain one D among D_1, D_2, D_3, a sequence of points $z_n \to \infty$ $(n=1, 2, \ldots)$, such that each disc $\Gamma(z_n, 1/n)$ contains a simply connected island above D. Suppose that this were false, then there would exist a certain n_0, such that, for any point z_0 in $0<|z|<\infty$ and for any sequence of points $\sigma_\nu \to \infty$, any one of $f(\sigma_\nu z)$ has in $\Gamma(z_0, 1/n_0)$ no simply connected islands above any one of D_1, D_2, D_3. Then, by Ahlfors' theorem[4], the family $\{f(\sigma_\nu z)\}$ is normal in $\Gamma(z_0, 1/n_0)$. Since $\{\sigma_\nu\}$ and z_0 are arbitrary, it follows that f(z) is J-exceptional, which is a contradiction.

Let $E(D_1, D_2, D_3)$ be the exceptional set of Lemma 1' for the above sequence $\{z_n\}$. Then since, for any δ, ε and $k \notin E(D_1, D_2, D_3)$, $\Delta(k, \delta, \varepsilon/2)$ contains infinitely many ones of z_n, $\Delta(k, \delta, \varepsilon)$ contains infinitely many discs $\Gamma(z_n, 1/n)$ and consequently infinitely many simply connected islands above D.

Next, we construct the exceptional set $E(D_1, D_2, D_3)$ for every configuration D_1, D_2, D_3, where D_i is a polygon on w-plane whose vertices are rational points. The set of all these configurations is enumerable, so that, if we put $E = \sum E(D_1, D_2, D_3)$, $mE=0$. Since any closed domain on w-plane can be enclosed and approximated by polygons with rational vertices as good as we please, we see easily that the set E satisfies the condition of the first part of Theorem 1.

The second part can be proved similarly.

For the proof of Theorem 2, we use

Lemma 2. (Valiron[5]). If f(z) is J-exceptional, then there exists a sequence of points $\sigma_n \to \infty$, such that $f(\sigma_n z)$ converges, uniformly in the wider sense in $0<|z|<\infty$, to a non-constant function F(z) meromorphic in $0<|z|<\infty$.

Proof. By Ahlfors' theorem[4], we can find on w-plane a disc D: $|w-a| \le \rho$ such that there exist on z-plane infinitely many simply connected islands Δ_n above D. Let σ_n be an a-point of f(z) in Δ_n. Since Δ_n is simply connected, any one of $f(\sigma_n z)$ takes a value lying on $|w-a| = \rho$ at a point on $|z| = 1$ and takes the value a at z=1. Hence the limiting function of any convergent subsequence of $\{f(\sigma_n z)\}$ can not be a constant.

Proof of Theorem 2.

For f(z), which is not J-exceptional, Theorem 2 is contained in Theorem 1, so that we have only to prove the theorem for J-exceptional functions.

Let $f(\sigma_n z)$ be the sequence of Lemma 2, and $\{z^{(\nu)}\}$ $(\nu = 1, 2, \ldots)$ be a sequence of points, which are dense in $0<|z|<\infty$. First we fix a value of ν. Then, for any ε, $f(\sigma_n z)$ converges to F(z) uniformly in $\Gamma(z^{(\nu)}, \varepsilon/2)$, so that, for sufficiently large n, f(z) takes in $\Gamma(\sigma_n z^{(\nu)}, \varepsilon/2)$ any value, which is taken by F(z) in $\Gamma(z^{(\nu)}, \varepsilon/4)$, with the same or less multiplicity as F(z).

Let E_ν be the exceptional set of Lemma 1' for the sequence $\{\sigma_n z^{(\nu)}\}$ $(n=1, 2, \ldots)$, and we put $E = \sum_{\nu=1}^{\infty} E_\nu$, so that $mE=0$. Then, for any $k \notin E$ and for any δ, ε and ν, $\Delta(k, \delta, \varepsilon/2)$ contains infinitely many points of $\{\sigma_n z^{(\nu)}\}$ $(n=1, 2, \ldots)$, so that $\Delta(k, \delta, \varepsilon)$ contains infinitely many ones of $\Gamma(\sigma_n z^{(\nu)}, \varepsilon/2)$ $(n=1, 2, \ldots)$ for any ν. Since the sum of $\Gamma(z^{(\nu)}, \varepsilon/4)$ for $\nu=1, 2, \ldots$ covers the whole $0<|z|<\infty$, we see that f(z) takes any value, which is taken by F(z) in $0<|z|<\infty$, infinitely often in $\Delta(k, \delta, \varepsilon)$ with the same or less multiplicity as F(z).

On the other hand, since $\Phi(\zeta)=F(e^\zeta)$ $(\zeta = \log z)$ is meromorphic on the whole finite ζ-plane and $\zeta=\infty$ is its essential singularity, $\Phi(\zeta)$ takes, by Nevanlinna's theorem[6], any value except at most two and takes any value simply except at most four. This holds also for F(z), since the mapping $z=e^\zeta$ is locally schlicht.

Thus Theorem 2 is proved.

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