A REMARK ON RECURRENT CURVATURE SPACES[*]

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H.Ruse has studied on Laplace's differential equation in Riemann spaces, and recently arrived at the idea of recurrent curvature spaces. Direct cause of this problem was originated in the study of the case where

$$\Delta S = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\alpha}\left(\sqrt{g}\, g^{\alpha\beta} \frac{\partial S}{\partial x^\beta}\right)$$

are the function of S only, where S means the geodesic distance from a fixed point (x_0^α) to any variable point (x^α). The author of the present note can not see the original papers, and has known only some of results in Mathematical Reviews. As far as the present author knows, these studies are mainly connected to lower dimensional spaces, and it seems that there is no systematic study of the case of general dimensions.

In this paper, in order to keep away from the formerly obtained results, we characterize this space by means of geometric consideration which differ from Ruse's original stand point, and then give some answers for Professor A.Lichnerowicz's conjecture.[1]

1. At a point P of Riemann space V_n, we consider the two dimensional vector space $[\xi_1^\lambda\, \xi_2^\mu]$. Let $\xi^{\lambda\mu}$ be its current Plücker coordinates. Let us consider a geometric object (figure):

$$(1.1) \qquad R_{\lambda\mu\nu\omega}\, \xi^{\lambda\mu} \xi^{\nu\omega} = 0$$

at every point of V_n, where

$$R^\lambda{}_{\mu\nu\omega} = \frac{\partial \left\{ {\lambda \atop \mu\nu} \right\}}{\partial x^\omega} - \frac{\partial \left\{ {\lambda \atop \mu\omega} \right\}}{\partial x^\nu} + \left\{ {\alpha \atop \mu\nu} \right\}\left\{ {\lambda \atop \alpha\omega} \right\} - \left\{ {\alpha \atop \mu\omega} \right\}\left\{ {\lambda \atop \alpha\nu} \right\},$$

$$R_{\lambda\mu\nu\omega} = g_{\alpha\lambda} R^\alpha{}_{\mu\nu\omega} .$$

Following Ruse, we call the figure (1.1) Riemann complex. We assume that this complex does not degenerate, that is

$$\left| R_{(\lambda\mu)(\nu\omega)} \right| \neq 0 \quad ,$$

where $|R_{(\lambda\mu)(\nu\omega)}|$ denotes the determinant whose row and column are $(\lambda\mu)$ and $(\nu\omega)$ respectively.

We consider the space whose group of holonomy fixes the Riemann complex at every point. The definition of $\xi^{\lambda\mu}$ gives

$$\xi^{\mu_1\mu_2} = \begin{vmatrix} \xi_1^{\mu_1} & \xi_1^{\mu_2} \\ \xi_2^{\mu_1} & \xi_2^{\mu_2} \end{vmatrix} \quad ,$$

therefore, by parallelism, we obtain

$$(1.2) \quad d\xi^{\mu_1\mu_2} = -\left\{ {\mu_1 \atop \alpha\beta} \right\} \xi^{\alpha\mu_2} dx^\beta - \left\{ {\mu_2 \atop \alpha\beta} \right\} \xi^{\mu_1\alpha} dx^\beta .$$

Now we shall investigate the condition

$$(1.3) \quad d\left(R_{\lambda\mu\nu\omega}\, \xi^{\lambda\mu}\xi^{\nu\omega}\right) = \theta\, R_{\lambda\mu\nu\omega}\, \xi^{\lambda\mu}\xi^{\nu\omega} ,$$

where θ is an arbitrary scalar factor. By means of (1.2), (1.3) implies

$$(1.4) \quad R_{\lambda_1\lambda_2\mu_1\mu_2;\omega} = R_{\lambda_1\lambda_2\mu_1\mu_2}\, \theta_\omega$$

where $\theta = \theta_\omega dx^\omega$ and symbol " ; " denotes the covariant differentiation. (1.4) is a necessary and sufficient condition in order that the parallelism of this space fix the Riemann complex.

Ruse has defined the recurrent curvature space K_n as the space which satisfies the curvature condition (1.4), and looked upon it as a generalization of Cartan's symmetric space.

2. Now from (1.4), we obtain

$$(2.1) \quad R_{\lambda\mu;\alpha} = R_{\lambda\mu}\, \theta_\alpha \quad ; \quad R_{;\alpha} = R\, \theta_\alpha .$$

If $R_{\lambda\mu} = \phi\, g_{\lambda\mu}$, $\phi \neq 0$, i.e. our space is an Einstein space, then $\theta_\alpha = 0$ and K_n becomes Cartan's symmetric space. Hereafter, we must assume that our space is not an Einstein space. If

$$(2.2) \quad \begin{aligned} & R_{\lambda\mu} \neq \phi\, g_{\lambda\mu} , \\ & R_{\lambda\mu;\omega} = R_{\lambda\mu}\, \theta_\omega , \end{aligned}$$

then characteristic space of $R_{\lambda\mu}$ with respect to fundamental tensor $g_{\lambda\mu}$ become the parallel vector space respectively, and hence K_n shall be reducible (locally)[2]. Now we assume that K_n has been decomposed into two irreducible subspaces V_r and V_{n-r}. In convenience, we assume that the indices take the values

$$a, b, c, \cdots\cdots, k = 1, 2, \cdots\cdots, r \quad ;$$
$$p, q, r, \cdots\cdots, u = r+1, \cdots\cdots, n \quad .$$

Let K_n be reducible, then we have

$$(2.3)\quad \begin{aligned} R^a{}_{bcd} &= R^a{}_{bcd}(x^a) \; ; \quad R^p{}_{qrs} = R^p{}_{qrs}(x^p), \\ R_{bc} &= R_{bc}(x^a) \; ; \quad R_{qr} = R_{qr}(x^p), \end{aligned}$$

but all the other similar quantities vanish. On the other hand, we obtain

$$(2.4)\quad \begin{aligned} 0 &= R_{ab;p} = R_{ab}\,\theta_p \; , \\ 0 &= R_{pq;a} = R_{pq}\,\theta_a \; . \end{aligned}$$

If $R_{\lambda\mu} \neq 0$, then (2.4) implies

$$\theta_p = 0 \; , \quad \theta_a = 0 \; ,$$

that is, K_n becomes a symmetric space, in the sense of Cartan.

Theorem. If $R_{\lambda\mu} \neq 0$, then K_n is a symmetric space of Cartan.

3. Now we deduce an important relation on curvature quantities in K_n.

We have the Bianchi's identities:

$$R^\lambda{}_{\mu\nu\omega;\alpha} + R^\lambda{}_{\mu\alpha\nu;\omega} + R^\lambda{}_{\mu\omega\alpha;\nu} = 0 \; .$$

Contracting λ and α, they become

$$R^\alpha{}_{\mu\nu\omega;\alpha} + R^\alpha{}_{\mu\alpha\nu;\omega} + R^\alpha{}_{\mu\omega\alpha;\nu} = 0,$$

that is

$$R^\alpha{}_{\mu\nu\omega;\alpha} = R_{\mu\nu;\omega} - R_{\mu\omega;\nu}$$

And hence, we have in K_n

$$(3.1)\quad R^\alpha{}_{\mu\nu\omega}\,\theta_\alpha = R_{\mu\nu}\,\theta_\omega - R_{\mu\omega}\,\theta_\nu,$$

$$(3.2)\quad 2R^\alpha{}_\omega\,\theta_\alpha = R\,\theta_\omega \; ,$$

In this circumstance, we have

$$\begin{aligned} &R_{\lambda\mu\nu\omega;\alpha} R^{\lambda\mu\nu\omega;\alpha} \\ &= (R_{\lambda\mu\nu\alpha;\omega} - R_{\lambda\mu\omega\alpha;\nu}) R^{\lambda\mu\nu\omega;\alpha} \\ &= (R_{\lambda\mu\nu\alpha}\theta_\omega - R_{\lambda\mu\omega\alpha}\theta_\nu) R^{\lambda\mu\nu\omega}\theta^\alpha \\ &= 2 R_{\lambda\mu\nu\alpha} R^{\lambda\mu\nu\omega}\theta^\alpha\theta_\omega \\ &= 2(R_{\nu\mu}\theta_\lambda - R_{\nu\lambda}\theta_\mu)(R^{\nu\mu}\theta^\lambda - R^{\nu\lambda}\theta^\mu) \\ &= 4(R_{\nu\mu}R^{\nu\mu}\theta_\lambda\theta^\lambda - R_{\nu\lambda}\theta^\lambda R^{\nu\mu}\theta_\mu) \\ &= 4R_{\nu\mu}R^{\nu\mu}\theta_\lambda\theta^\lambda - R^2\theta_\lambda\theta^\lambda \; . \end{aligned}$$

$$\therefore (R_{\lambda\mu\nu\omega}R^{\lambda\mu\nu\omega} - 4R_{\mu\nu}R^{\mu\nu} + R^2)\,\theta_\alpha\theta^\alpha = 0 \; .$$

Therefore

i) $\quad R_{\lambda\mu\nu\omega}R^{\lambda\mu\nu\omega} = 4R_{\mu\nu}R^{\mu\nu} - R^2$

or

ii) $\quad \theta_\alpha\theta^\alpha = 0 \; .$

If $g_{\lambda\mu}$ is positive definite, ii) implies that K_n is a symmetric space, hence we restrict ourselves to the case i). In the case i), if $R_{\lambda\mu} \equiv 0$, then $R_{\lambda\mu\nu\omega} \equiv 0$ i.e. K_n is flat.

Thus we can conclude that the K_n's may remain in its proper sense when and only when they have indefinite fundamental tensor and Ricci curvature $R_{\lambda\mu}$ being zero.

Now, it is seen from our result that any K_n which has positive definite fundamental tensor is a symmetric space what has been conjectured by Lichnerowicz.

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(1) Recently Prof. A. Lichnerowicz has corresponded to the author his kind advice and some conjectures. For his generous and stimulating councel the author eagerly extends his warm thanks.

(2) See T.Y.Thomas: The decomposition of Riemann space in the large, Monatsh. Math. Phys. 47. (1939). The present author discussed the second order symmetric tensor whose characteristic spaces are parallel vector spaces (in Japanese), The Study of Holonomy Groups, No.13, 1949, Dec.

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