ON THE MIXED MARKOFF PROCESS.

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§1. Introduction. Let X_t be an one-dimensional simple Markoff process with a continuous parameter t. Such a process is characterized by the transition probability $P(t,y;\ t',dx)$, i.e., the conditional probability for $X_{t'} \in dx$ under the condition $X_t = y$ $(t' > t)$. According to the properties of $P(t,y;\ t+\Delta t,dx)$ in an infinitesimal time interval $(t,\ t+\Delta t)$, this process is generally divided into many cases. These cases have the transition probabilities just matching to an infinitely divisible law or its component-laws in a differential stochastic process. In fact, the case corresponding to Gaussian law is ordinarily called to be continuous, and to the law generated by the convolution of at most infinitely many Poisson laws we obtain a process which is called to be purely discontinuous. The former was discussed by A. Kolmogoroff [1], A. Khintchine [2], W. Feller [3] and J. L. Doob [4] and the later by W. Feller [5]. More generally, we get a process corresponding to an infinitely divisible laws, which contains the above two cases. We shall call it a mixed Markoff process. Recently, K. Itô [6] introduced a stochastic integral equation having this process as a solution and showed that it also satisfies a certain stochastic differential equation.

The object of this paper is to derive directly the canonical form of the mixed Markoff process in an infinitesimal time interval from some assumptions on the transition probability.

§2. Theorem. We lay down the following assumptions (1), (2) and (3).

(1) There exists a function $p(t,x,\xi)$ of $(t,x,\xi) \in \mathcal{D}$ ($\mathcal{D}$: $t_0 \leq t \leq T$, $-\infty < x < \infty$, $-\infty < \xi < x-0$, $x+0 < \xi < +\infty$) which for any fixed t and x is non-decreasing over $-\infty < \xi < x-0$ and $x+0 < \xi < +\infty$ and uniformly dominated totally variated over $\mathcal{D}$, i.e.,

(1.1) $$\int_{|\xi|>0} p(t,x,d\xi(+)x) \equiv M(t,x) \leq A_1, \quad (t_0 \leq t \leq T,\ -\infty < x < \infty)$$ [7]

and

(1.2) $$\lim_{t'\to t} \frac{1}{t'-t} \int_{|\xi|\geq\eta\geq 1} P(t,x;\ t',d\xi(+)x) = \int_{|\xi|\geq\eta\geq 1} p(t,x,d\xi(+)x),$$

$$\lim_{t'\to t} \frac{1}{t'-t} \int_{1\geq|\xi|\geq\eta>0} \xi^2 P(t,x;\ t',d\xi(+)x) = \int_{1\geq|\xi|\geq\eta>0} p(t,x,d\xi(+)x)$$

at the continuity points $\eta+x$ of $p(t,x,\eta+x)$ for fixed t and x. And further

(1.3) $$\int_{|\xi|>0} |p(t,x,d\xi(+)x) - p(t,y,d\xi(+)y)| \leq A_2|x-y|$$

where A_1 and A_2 are absolute constants.

(2) There exists a function $\sigma^2(t,x)$ of t,x $(t_0 \leq t \leq T,\ -\infty < x < \infty)$ and satisfies

(2.1) $$\lim_{\varepsilon\to 0} \lim_{t'\to t} \frac{1}{t'-t} \int_{-\varepsilon}^{\varepsilon} \xi^2 P(t,x;\ t',d\xi(+)x) = \sigma^2(t,x)$$

(2.2) $$|\sigma^2(t,x) - \sigma^2(t,y)| \leq B_1|x-y|$$

and

(2.3) $$|\sigma^2(t,x)| \leq B_2 \quad (t_0 \leq t \leq T,\ -\infty < x < \infty),$$

where B_1 and B_2 are absolute constants.

(3) There exists a function $a(t,x)$ of t,x $(t_0 \leq t \leq T,\ -\infty < x < \infty)$ and satisfies

(3.1) $$\lim_{t'\to t} \frac{1}{t'-t} \int_{|\xi|\leq 1} \xi P(t,x;\ t',d\xi(+)x) = a(t,x),$$

(3.2) $$|a(t,x) - a(t,y)| \leq C_1|x-y|$$

and

(3.3) $$|a(t,x)| \leq C_2, \quad (t_0 \leq t \leq T,\ -\infty < x < \infty),$$

where C_1 and C_2 are absolute constants. Under the above assumptions (1), (2) and (3) we can conclude

$$\left\{ \int e^{iz\xi}\, P(t_0,x;\ T,\ d\xi(+)x) \right\}^{\frac{1}{T-t_0}}$$

$$\to \exp\Big[iz\,a(t_0,x) - \frac{z^2}{2}\sigma^2(t_0,x) + \int_{|\xi|>1} (e^{iz\xi}-1)\, p(t_0,x,d\xi(+)x)$$

$$+ \int_{1\geq|\xi|>0} \frac{(e^{iz\xi}-1-iz\xi)}{\xi^2}\, p(t_0,x,d\xi(+)x) \Big]$$

uniformly for any finite interval of $(-\infty < z < +\infty)$.

§ 3°. Proof of Theorem. It will be executed through the five steps.

1°. For any $t < t'$ $(t_0 \leq t < t' \leq T)$ and for any real x, we have.

$$\Big\{ \int_{-\infty}^{\infty} e^{iz\xi} P(t,x;t',d\xi(t)x) - 1 \Big\}$$

$$= \frac{1}{t'-t}\Big\{ \int_{|\xi|>1} (e^{iz\xi}-1) P(t,x;t',d\xi(t)x) \Big\}$$

$$+ \frac{1}{t'-t}\Big\{ \int_{1\geq|\xi|>\eta>0} \frac{(e^{iz\xi}-1-iz\xi)}{\xi^2} d\int_{1\geq|u|>\xi} u^2 P(t,x;t',du(t)x)$$

$$+ \frac{iz}{t'-t}\int_{|\xi|\leq 1} \xi\, P(t,x;t',d\xi(t)x)$$

$$+ \frac{1}{t'-t}\Big\{ \int_{|\xi|\leq\eta} \Big(-\frac{z^2}{2}\xi^2 + \frac{z^3}{6}\xi^3\vartheta\Big) P(t,x;t',d\xi(t)x)\Big\},$$

where $|\vartheta| \leq 1$. Now, let $t' \to t$ and let $\eta \to 0$, then we have, applying the assumptions (1.2), (2.1) and (3.1)

$$\lim_{t'\to t} \frac{1}{t'-t}\Big\{ \int_{-\infty}^{\infty} e^{iz\xi} P(t,x;t',d\xi(t)x) - 1 \Big\}$$

(4)
$$= \int_{|\xi|>1} (e^{iz\xi}-1) p(t,x,d\xi(t)x) + \int_{1\geq|\xi|>0} \frac{(e^{iz\xi}-1-iz\xi)}{\xi^2} p(t,x,d\xi(t)x)$$

$$+ iz\, a(t,x) - \frac{z^2}{2}\sigma^2(t,x).$$

Hence for any real x, y

$$\Big|\lim_{t'\to t}\frac{1}{t'-t}\Big\{ \int_{-\infty}^{\infty} e^{iz\xi} P(t,x;t',d\xi(t)x) - \int_{-\infty}^{\infty} e^{iz\xi} P(t,y;t',d\xi(t)y)\Big\}\Big|$$

$$\leq 2\int_{|\xi|>1} |p(t,x,d\xi(t)x) - p(t,y,d\xi(t)y)|$$

$$+ \frac{z^2}{2}\int_{1\geq|\xi|>0} |p(t,x,d\xi(t)x) - p(t,y,d\xi(t)y)|$$

$$+ |z|\,|a(t,x) - a(t,y)| + \frac{z^2}{2}|\sigma^2(t,x) - \sigma^2(t,y)|$$

$$\leq \Big(2+\frac{z^2}{2}\Big)\int_{|\xi|>0} |p(t,x,d\xi(t)x) - p(t,y,d\xi(t)y)|$$

$$+ |z|\,|a(t,x) - a(t,y)| + \frac{z^2}{2}|\sigma^2(t,x) - \sigma^2(t,y)|$$

Now by (1.3), (2.2) and (3.2)

$$\Big|\lim_{t'\to t}\Big\{ \int_{-\infty}^{\infty} e^{iz\xi} (P(t,x;t',d\xi(t)x) - P(t,y;t',d\xi(t)y))\Big\}\Big|$$

$$\leq \Big(2+\frac{z^2}{2}\Big)A_2|x-y| + |z|C_1|x-y| + \frac{z^2}{2}B_1|x-y|$$

$$\equiv D_1|x-y|,$$

where

$$D_1 \equiv \Big(2+\frac{z^2}{2}\Big)A_2 + |z|C_1 + \frac{z^2}{2}B_1.$$

Hence we get the following relation

(5)
$$\Big|\int_{-\infty}^{\infty} e^{iz\xi} P(t,x;t',d\xi(t)x) - \int_{-\infty}^{\infty} e^{iz\xi} P(t,y;t',d\xi(t)y)\Big|$$

$$\leq D_1|x-y||t-t'| + o(|t'-t|), \quad (t'\to t).$$

2. By (4), we have

(6)
$$\int_{-\infty}^{\infty} e^{iz\xi} P(t,x;t',d\xi(t)x) - 1$$

$$= (t'-t)\Big\{ \int_{|\xi|>1} (e^{iz\xi}-1) p(t,x,d\xi(t)x)$$

$$+ \int_{1\geq|\xi|>0} \frac{(e^{iz\xi}-1-iz\xi)}{\xi^2} p(t,x,d\xi(t)x)$$

$$+ iz\, a(t,x) - \frac{z^2}{2}\sigma^2(t,x)\Big\} + o(t'-t), \quad (t'\to t).$$

Hence

(7)
$$\Big|\int_{-\infty}^{\infty} e^{iz\xi} P(t,x;t',d\xi(t)x) - 1\Big|$$

$$\leq |t'-t|\Big\{\Big(2+\frac{z^2}{2}\Big)M(t,x) + |z||a(t,x)| + \frac{z^2}{2}\sigma^2(t,x)\Big\} + o(t'-t)$$

$$\leq |t'-t|\Big\{\Big(2+\frac{z^2}{2}\Big)A_1 + |z|C_2 + \frac{z^2}{2}B_2\Big\} + o(t'-t)$$

$$\equiv |t'-t|\,D_2 + o(t'-t), \quad (t'\to t),$$

where

$$D_2 \equiv \Big(2+\frac{z^2}{2}\Big)A_1 + |z|C_2 + \frac{z^2}{2}B_2.$$

3°. Next divide the time interval $[t_0, T]$ into n subintervals $t_0 < t_1 < t_2 < \dots < t_n = T$ and put $t_{k+1} - t_k = \Delta t_k$ $(k = 0,1,\dots,n-1)$. Then by Smoluchowski-Chapman s equation we have

$$P(t_0,x;t_{k+1},d\xi) = \int_{-\infty}^{\infty} P(t_0,x;t_k,dy)P(t_k,y;t_{k+1},d\xi).$$

Hence

(8) $$\int_{-\infty}^{\infty} e^{iz(\xi-x)} P(t_0,x;t_{k+1},d\xi)$$
$$= \int_{-\infty}^{\infty} e^{iz(\xi-x)} \int_{-\infty}^{\infty} P(t_0,x;t_k,dy) P(t_k,y;t_{k+1},d\xi)$$
$$= \int_{-\infty}^{\infty} e^{iz(y-x)} \left\{ \int_{-\infty}^{\infty} e^{iz(\xi-y)} P(t_k,y;t_{k+1},d\xi) \right\} P(t_0,x;t_k,dy)$$

For simplicity, from now on, put

$$f(t_k,x) \equiv \int_{-\infty}^{\infty} e^{iz(\xi-x)} P(t_0,x;t_k,d\xi) = \int_{-\infty}^{\infty} e^{iz\xi} P(t_0,x;t_k,d\xi \oplus x),$$

$$g(t_{k+1},x) \equiv \int_{-\infty}^{\infty} e^{iz(\xi-x)} P(t_k,x;t_{k+1},d\xi) = \int_{-\infty}^{\infty} e^{iz\xi} P(t_k,x;t_{k+1},d\xi \oplus x).$$

Therefore from (8)

$$f(t_{k+1},x) = \int_{-\infty}^{\infty} e^{izy} g(t_{k+1},y+x) P(t_0,x;t_k,dy \oplus x).$$

Now for $T \to t_0$ by (5), (7), (1.2) and (2.1), we see

$$|f(t_{k+1},x) - f(t_k,x) g(t_{k+1},x)|$$
$$= \left| \int_{-\infty}^{\infty} e^{iz\xi} \{ g(t_{k+1},\xi+x) - g(t_{k+1},x) \} P(t_0,x;t_k,d\xi \oplus x) \right|$$
$$\le \int_{|\xi|>1} \{ |g(t_{k+1},\xi+x)-1| + |g(t_{k+1},x)-1| \} \times$$
$$\times P(t_0,x;t_k,d\xi \oplus x) + \left\{ \int_{|\xi|\le 1} |g(t_{k+1},\xi+x) - g(t_{k+1},x)|^2 \times \right.$$
$$\left. P(t_0,x;t_k,d\xi \oplus x) \right\}^{\frac{1}{2}}$$
$$\le \Delta t_k \Big[2D_2 \int_{|\xi|>1} P(t_0,x;t_k,d\xi \oplus x) + D_1 \Big\{ \int_{1\ge|\xi|>\eta>0} \xi^2 P(t_0,x;t_k,d\xi \oplus x) \Big\}^{\frac{1}{2}}$$
$$+ D_1 \Big\{ \int_{|\xi|\le\eta} \xi^2 P(t_0,x;t_k,d\xi \oplus x) \Big\}^{\frac{1}{2}} \Big] + o(\Delta t_k)$$
$$\le (t_k-t_0)\Delta t_k\, 2D_2 \int_{|\xi|>1} P(t_0,x,d\xi \oplus x) + (t_k-t_0)^{\frac{1}{2}} \Delta t_k D_1 \times$$
$$\times \Big\{ \int_{1\ge|\xi|>0} P(t_0,x,d\xi \oplus x) \Big\}^{\frac{1}{2}} + (t_k-t_0)^{\frac{1}{2}} \Delta t_k D_1 \sigma(t_0,x) + o(\Delta t_k)$$
$$\le (t_k-t_0)^{\frac{1}{2}} \Delta t_k \{ (2D_2+D_1) M(t_0,x) + D_1 \sigma(t_0,x) \} + o(\Delta t_k)$$
$$\le (t_k-t_0)^{\frac{1}{2}} \Delta t_k \{ (2D_2+D_1) A_1 + D_1 B_2 \} + o(\Delta t_k)$$
$$= (t_k-t_0)^{\frac{1}{2}} \Delta t_k D_3 + o(\Delta t_k),$$

where

$$D_3 \equiv (2D_2+D_1) A_1 + D_1 B_2$$

Thus we have

(9) $$|f(t_{k+1},x) - f(t_k,x) g(t_{k+1},x)|$$
$$\le (t_k-t_0)^{\frac{1}{2}} \Delta t_k D_3 + o(\Delta t_k),$$
$$k = 0, 1, \cdots, n-1, \quad (T \to t_0).$$

4. By (9), paying attention to $g(t_1,x) = f(t_1,x)$, we see

(10) $$|f(t_n,x) - g(t_1,x) g(t_2,x) \cdots g(t_n,x)|$$
$$\le |f(t_n,x) - f(t_{n-1},x) g(t_n,x)| + |f(t_{n-1},x) g(t_n,x)$$
$$- f(t_{n-2},x) g(t_{n-1},x) g(t_n,x)| + |f(t_{n-2},x) g(t_{n-1},x) g(t_n,x)$$
$$- f(t_{n-3},x) g(t_{n-2},x) g(t_{n-1},x) g(t_n,x)| +$$
$$\cdots\cdots$$
$$+ |f(t_2,x) g(t_3,x) \cdots g(t_n,x) - g(t_1,x) g(t_2,x) \cdots g(t_n,x)|$$
$$+ \{ (t_{n-1}-t_0)^{\frac{1}{2}} \Delta t_{n-1} + (t_{n-2}-t_0)^{\frac{1}{2}} \Delta t_{n-2} + \cdots$$
$$+ (t_1-t_0)^{\frac{1}{2}} \Delta t_0 \} D_3 + o(T-t_0)$$
$$\le D_3 \int_{t_0}^{T} (t-t_0)^{\frac{1}{2}} dt + o(T-t_0)$$
$$= \frac{2}{3} D_3 (T-t_0)^{\frac{3}{2}} + o(T-t_0)$$
$$= o(T-t_0), \quad (T \to t_0).$$

5. In this step, we shall calculate the product

$$g(t_1,x) g(t_2,x) \cdots g(t_n,x)$$

By (6) and the expansion of log., we see

$$\left| \sum_{k=1}^{n} \log g(t_k,x) - \sum_{k=1}^{n} (g(t_k,x) - 1) \right|$$
$$\le \sum_{k=1}^{n} |g(t_k,x) - 1|^2$$
$$\le \sum_{k=1}^{n} \{ (\Delta t_k)^2 D_2^2 + o(\Delta t_k)^2 \}$$
$$\le \max_{1\le k\le n} \Delta t_k \{ (T-t_0) D_2^2 + o(T-t_0) \}$$
$$= o(T-t_0), \quad (T \to t_0),$$

uniformly for any finite interval of z.
Now, from (6) we have

$$\sum_{k=1}^{n}(g(t_k,x)-1)=\sum_{k=1}^{n}\Delta t_k\Big\{\int_{|\xi|>1}(e^{iz\xi}-1)p(t_k,x,d\xi(t)x)$$
$$+\int_{|z||\xi|>0}\frac{(e^{iz\xi}-1-iz\xi)}{\xi^2}p(t_k,x,d\xi(t)x)$$
$$+iza(t_k,x)-\frac{z^2}{2}\sigma^2(t_k,x)\Big\}$$
$$+o(T-t_0),\qquad (T\to t_0),$$

uniformly for any finite interval of z.
Hence

$$(11)\quad\Big|\sum_{k=1}^{n}\log g(t_k,x)-\sum_{k=1}^{n}\Delta t_k\Big\{\int_{|\xi|>1}(e^{iz\xi}-1)p(t_k,x,d\xi(t)x)$$
$$+\int_{|z||\xi|>0}\frac{(e^{iz\xi}-1-iz\xi)}{\xi^2}p(t_k,x,d\xi(t)x)$$
$$+iza(t_k,x)-\frac{z^2}{2}\sigma^2(t_k,x)\Big\}\Big|$$
$$\le o(T-t_0),\qquad (T\to t_0)$$

in the same sense as above.

6. Thus, from (10) and (11) we can conclude

$$\Big|f(t_n,x)-\exp\Big[\sum_{k=1}^{n}\Delta t_k\Big\{\int_{|\xi|>1}(e^{iz\xi}-1)p(t_k,x,d\xi(t)x)$$
$$+\int_{|z||\xi|>0}\frac{(e^{iz\xi}-1-iz\xi)}{\xi^2}p(t_k,x,d\xi(t)x)$$
$$+iza(t_k,x)-\frac{z^2}{2}\sigma^2(t_k,x)\Big\}\Big]\Big|$$
$$\le o(T-t_0),\quad (T\to t_0)$$

uniformly for any finite interval of z.
Hence let $\max_{1\le k\le n}\Delta t_k\to 0$, then paying attention to $t_n = T$, we get

$$f(T,x)=\exp\Big[\int_{t_0}^{T}dt\Big\{\int_{|\xi|>1}(e^{iz\xi}-1)p(t,x,d\xi(t)x)$$
$$+\int_{|z||\xi|>0}\frac{(e^{iz\xi}-1-iz\xi)}{\xi^2}p(t,x,d\xi(t)x)+iza(t,x)$$
$$-\frac{z^2}{2}\sigma^2(t,x)\Big\}\Big]+o(T-t_0),$$
$$(T\to t_0)$$

in the same sense as above, or

$$\{f(T,x)\}^{\frac{1}{T-t_0}}=\exp\Big[\frac{1}{T-t_0}\int_{t_0}^{T}dt\,\times$$
$$\times\Big\{\int_{|\xi|>1}(e^{iz\xi}-1)p(t,x,d\xi(t)x)$$
$$+\int_{|z||\xi|>0}\frac{(e^{iz\xi}-1-iz\xi)}{\xi^2}p(t,x,d\xi(t)x)+iza(t,x)$$
$$-\frac{z^2}{2}\sigma^2(t,x)\Big\}\Big]+o(T-t_0),\quad (T\to t_0)$$

in the same sense as above. Hence

$$\lim_{T\to t_0}\{f(T,x)\}^{\frac{1}{T-t_0}}=\exp\Big\{\int_{|\xi|>1}(e^{iz\xi}-1)p(t_0,x,d\xi(t)x)$$
$$+\int_{|z||\xi|>0}\frac{(e^{iz\xi}-1-iz\xi)}{\xi^2}p(t_0,x,d\xi(t)x)$$
$$+iza(t_0,x)-\frac{z^2}{2}\sigma^2(t_0,x)\Big\}$$

in the same sense as above, which shows the required relation.

4. <u>Special cases.</u> If in Theorem

$$\int_{|\xi|>0}p(t,x,d\xi(t)x)\equiv 0,\quad (t_0\le t\le T,\ -\infty<x<\infty)$$

we have the continuous case. That is,

<u>Theorem</u> <u>If for any</u> $\eta>0$ <u>and for any fixed</u> t <u>and</u> x

$$\lim_{t'\to t}\frac{1}{t'-t}\int_{|\xi|\ge\eta\ge 1}P(t,x;t',d\xi(t)x)=0,$$
$$\lim_{t'\to t}\frac{1}{t'-t}\int_{|z||\xi|\ge\eta>0}\xi^2P(t,x;t',d\xi(t)x)=0.$$

<u>And further, if the condition (2) and (3) in theorem of § 2 are satisfied, we can conclude</u>

$$\lim_{T\to t_0}\{f(T,x)\}^{\frac{1}{T-t_0}}=\exp\Big[iza(t_0,x)-\frac{z^2}{2}\sigma^2(t_0,x)\Big]$$

<u>uniformly for any finite interval of</u> z.

Next, if in Theorem of § 2,

$$\sigma^2(t,x)\equiv 0,\qquad (t_0\le t\le T,\ -\infty<x<\infty),$$

we have the <u>purely discontinuous case</u>. That is,

Theorem. *If in addition to the condition (1) and (3) in theorem the following condition*

$$\lim_{\varepsilon\to 0}\lim_{t'\to t}\frac{1}{t'-t}\int_{|\xi|\le\varepsilon}\xi^2 P(t,x;t',d\xi(+)x)=0$$

is satisfied, then we have

$$\lim_{T\to t_0}\left\{f(T,x)\right\}^{\frac{1}{T-t_0}}$$

$$=\exp\left\{iza(t_0,x)+\int_{|\xi|>1}(e^{iz\xi}-1)p(t_0,x,d\xi(+)x)\right.$$

$$\left.+\int_{1\ge|\xi|>0}\frac{(e^{iz\xi}-1-iz\xi)}{\xi^2}p(t_0,x,d\xi(+)x)\right\}$$

uniformly for any finite interval of z.

(*) Received June 27, 1949.

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7) $E(+)X$ denotes the set $\{e+x\}$ of points $e+x$ $(e\in E)$.

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