NOTE ON A MEASURE PROBLEM

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Messers. Anzai and Ito and others proved independently the following fact with help of an integral inequality.

Let within a set Ω whose total measure is 1, there be infinitely many subsets A_i, each having a measure greater than α. Then, for a given integer N and a positive number ε there are N suitable integers $i_1 < \cdots < i_N$ such that $m A_{i_1} \cap \cdots \cap A_{i_N} > \alpha^N - \varepsilon$

The object of this note is to give a more precise result by an arithmetical approach.

Theorem. Let within a set Ω whose total measure is 1, there be n sets $A_1, A_2, \cdots, A_n$ whose mean measure equals to α. Then, if n is greater than a certain integer n_0 which is determined by an integer $N (< n)$ and a positive number ε, we can choose a set of N integers $i_1 < \cdots < i_N$ suitably such that $m A_{i_1} \cap \cdots \cap A_{i_N} > \alpha^N - \varepsilon$.

Proof. Let us consider a set B_0 of points which do not belong to any one of A_i's, and let $m B_0 = b_0$. Let us consider a set B_k of points which belong to just k of A_i's, and let $m B_k = b_k$ $(k = 1, \cdots, n)$. Then the conditions can be written as follows:

$$(1) \quad \begin{cases} b_k \geqq 0 \quad (k = 0, 1, \cdots, n) \\ b_0 + b_1 + \cdots + b_n = 1 \\ b_1 + 2b_2 + \cdots + n b_n = n\alpha \end{cases}$$

And the quantity in question $a_N = \frac{1}{\binom{n}{N}} \sum_{i_1 \cdots i_N} m A_{i_1} \cap \cdots \cap A_{i_N}$ is given by

$$(2) \qquad \binom{n}{N} a_N = \binom{N}{N} b_N + \binom{N+1}{N} b_{N+1} + \cdots + \binom{n}{N} b_n .$$

And conversely when subsets B_k are given satisfying these relations (1) and (2) (where $m B_k = b_k$), we can compose sets A_i by dividing B_k arbitrarily into $\binom{n}{k}$ parts $C_{i_1 \cdots i_k}$ $(i_j \in 1, 2, \cdots, n)$ and by collecting sets $C_{i_1 \cdots i_k}$ whose suffices contain i to a set A_i $(i = 1, 2, \cdots, n)$.

Now,

$$-\alpha^N + a_N + \varepsilon = \underset{\lambda \geqq 0}{\mathrm{Min}} \left\{ (N-1)\lambda^N - \alpha N \lambda^{N-1} + a_N + \varepsilon \right\}$$

$$= \underset{\lambda \geqq 0}{\mathrm{Min}} \left\{ \sum_{k=0}^{n} b_k \left[(N-1)\lambda^N - \frac{k}{n} N \lambda^{N-1} + \frac{k}{n}\frac{k-1}{n-1} \cdots \frac{k-N+1}{n-N+1} + \varepsilon \right] \right\}.$$

In order that $-\alpha^N + a_N + \varepsilon \geqq 0$ it is sufficient that every bracket is positive or zero, as $b_k \geqq 0$. But

$$\underset{\lambda \geqq 0}{\mathrm{Min}} \left\{ (N-1)\lambda^N - \frac{k}{n} N \lambda^{N-1} + \frac{k}{n}\frac{k-1}{n-1} \cdots \frac{k-N+1}{n-N+1} + \varepsilon \right\}$$

$$= -\left(\frac{k}{n}\right)^N + \frac{k}{n}\frac{k-1}{n-1} \cdots \frac{k-N+1}{n-N+1} + \varepsilon$$

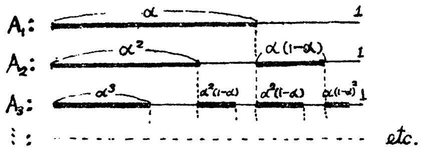
$$(k = 0, 1, \cdots, n)$$

and $\left(\frac{k}{n}\right)^N - \frac{k}{n}\frac{k-1}{n-1} \cdots \frac{k-N+1}{n-N+1} \leqq \left(\frac{k}{n}\right)^N - \left(\frac{k-N+1}{n-N+1}\right)^N$

$$\leqq \frac{N(N-1)(n-k)}{n(n-N+1)} \leqq \frac{N(N-1)}{n-N+1} \longrightarrow 0$$

when $n \to \infty$. Hence we surely get $a_N > \alpha^N - \varepsilon$ when n is greater than a certain value n_0. When we estimate the above inequality a little more precisely, we shall see that it is always enough that n_0 is not less than $(N-1)/\varepsilon$.

A typical example is an ideal mixture. In this case $m A_{i_1} \cap \cdots \cap A_{i_N}$ are all equal to α^N.



(*) Received March 7, 1949.

(1) Zenkoku Shijyo Danwakai-shi, 1948.

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