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# TANGENT SCROLLS IN PRIME FANO THREEFOLDS

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**Abstract**

In this paper we prove that any smooth prime Fano threefold, different from the Mukai-Umemura threefold $X'_{22}$, contains a 1-dimensional family of intersecting lines. Combined with a result in [Sch] this implies that any morphism from a smooth Fano threefold of index 2 to a smooth Fano threefold of index 1 must be constant, which gives an answer in dimension 3 to a question stated by Peternell.

## §1. Introduction

**1.1.** A smooth projective variety $X$ is called a Fano variety if the anti-canonical bundle $-K_X$ is ample. Then the index of $X$ is the largest positive integer $r = r(X)$ such that $-K_X = rH$ for some line bundle $H$ on $X$.

The smooth Fano threefold $X = X_d \subset \boldsymbol{P}^{g+1}$ $(d = \deg X)$ is called *prime* if $\rho(X) = \mathit{rank}\,\mathrm{Pic}(X) = 1$, $r(X) = 1$, and $-K_X$ is the hyperplane bundle on $X$. By the classification of Fano threefolds smooth prime Fano threefolds exist *iff* $3 \leq g \leq 12$ $(g \neq 11)$, and then $d = 2g - 2$ (see [I1]).

**1.2.** (see §4.2, §4.4 in [IP], or §1 in [I2]). Let $l$ be a line on the smooth prime Fano threefold $X$, and let $N_{l/X}$ be the normal bundle of $l \subset X$. Then

(1). *either* (a). $N_{l/X} = \mathcal{O} \oplus \mathcal{O}(-1)$; *or* (b). $N_{l/X} = \mathcal{O}(1) \oplus \mathcal{O}(-2)$.

(2). The Hilbert scheme $\mathscr{H}_X$ of lines on $X$ is non-empty, any irreducible component $\mathscr{H}_o$ of $\mathscr{H}_X$ is one-dimensional, and *either* $\mathscr{H}_o$ is *non-exotic*, i.e. $N_{l/X}$ is of type (1)(a) for the general $l \in \mathscr{H}_o$; *or* $\mathscr{H}_o$ is *exotic*, i.e. $N_{l/X}$ is of type (1)(b) for any $l \in \mathscr{H}_o$.

(3). The component $\mathscr{H}_o$ is exotic if *either* the elements $l \in \mathscr{H}_o$ sweep out the tangent scroll $R_o \subset X$ to an irreducible curve $C \subset X$; *or* $g = 3$ (i.e. $X = X_4$ is a quartic threefold), and then the lines $l \in \mathscr{H}_o$ sweep out a hyperplane section $R_o \subset X_4$ which is a cone over a plane quartic curve, centered at some point $x \in X_4$.

**1.3.** For example, the scheme $\mathscr{H}_X$ of the Fermat quartic $X = X_4 = (x_0^4 + \cdots + x_4^4 = 0)$, which is a prime Fano threefold of $g = 3$, is a union of 40 double components each of which is of type (1.2)(1)(b) (see Remark 3.5(ii) in [I1]).

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The only known example of a prime Fano threefold $X$ of $g \geq 4$ such that $\mathscr{H}_X$ has an exotic component $\mathscr{H}_o$, is the Mukai-Umemura threefold $X'_{22}$. The scheme $\mathscr{H}_{X'_{22}} = 2\mathscr{H}_o$ and the surface $R_o$ is the hyperplane section of $X'_{22}$ swept out by the tangent lines to a rational normal curve $C_{12} \subset X'_{22}$ of degree 12 (see [MU]).

**1.4.** By a theorem of Kobayashi and Ochiai the index $r = r(Y)$ of a smooth Fano $n$-fold $Y$ can't be greater than $n+1$; and the only smooth Fano $n$-folds of $r \geq n$ are $\boldsymbol{P}^n$ for which $r = n+1$ and the $n$-dimensional quadric $Q_2^n$ for which $r = n$ (see e.g. [Pe], p. 106). In particular, except $\boldsymbol{P}^3$ and $Q_2^3$, any smooth Fano 3-fold must have index $r \leq 2$.

It is shown by Remmert and Van de Ven (for $n = 2$) and later by Lazarsfeld (for any $n$) that the projective space $\boldsymbol{P}^n$ does not admit surjective morphisms to a smooth projective $n$-fold $X \neq \boldsymbol{P}^n$ (see [RV], [L]). The same is true for morphisms $f : Q_2^n \to X \neq \boldsymbol{P}^n$, $Q_2^n$ (see [PS]). In particular, $\boldsymbol{P}^3$ and $Q_2^3$ do not admit surjective moprphisms to smooth Fano threefolds $X$ of *smaller* index $r(X)$.

Let $f : Y \to X$ be a non-constant morphism between smooth Fano 3-folds of $\rho = 1$. By Kor. 1.5 in [RV], $\rho(Y) = 1$ implies that $f$ must be surjective, and by the preceding $r(Y)$ can't be $\geq 3$. Therefore $r(Y) = 2$, $r(X) = 1$. This gives rise to the following question stated originally by Peternell (see (2.12)(2) in [Pe]).

Question (Pe). Are there non-constant (hence surjective) morphisms $f : Y \to X$ from a smooth Fano 3-fold $Y$ of $\rho(Y) = 1$ and $r(Y) = 2$ to a smooth Fano 3-fold $X$ of $\rho(X) = 1$ and $r(X) = 1$?

In this paper we give the expected negative answer to (Pe).

Let $f : Y \to X$ be as above, and assume that $f$ is non-constant. Then $f$ must be surjective and finite since $\rho(Y) = 1$ (see Kor. 1.5 in [RV]). Therefore $f^* : H^3(X, \boldsymbol{C}) \to H^3(Y, \boldsymbol{C})$ will be an embedding, in particular $h^3(X) \leq h^3(Y)$ (see also [Sch]). For any Fano threefold $h^{3,0} = 0$ and $h^3 = 2h^{2,1}$ since the anti-canonical class is ample. Therefore $h^{2,1}(X) \leq h^{2,1}(Y)$. Since $h^{2,1}(Y) \leq 21$ for any Fano 3-fold $Y$ of $r = 2$ (see [I1]), then the existence of a non-constant morphism $f : X \to Y$ as in (Pe) implies that $h^{2,1}(X) \leq 21$. This gives a negative answer to (Pe) whenever $h^{2,1}(X) > 21$.

The only smooth non-prime Fano threefolds of $\rho = 1$ and $r = 1$ are the sextic double solid $X'_2$ and the double quadric $X'_4$ for which the answer to (Pe) is negative since $h^{2,1}(X'_2) = 52 > 21$ and $h^{2,1}(X'_4) = 30 > 21$. By the same argument the answer to (Pe) is negative also for the quartic threefold $X_4$ since $h^{2,1}(X_4) = 30 > 21$. Any other smooth prime Fano threefold $X = X_{2g-2} \subset \boldsymbol{P}^{g+1}$, $4 \leq g \leq 12$, $g \neq 11$ has $h^{2,1}(X) \leq 20$ (see [I1]).

In [Sch] is given a negative answer to (Pe) provided $X$ contains a conic of rank 2 (a pair of intersecting lines). The only known Fano threefold $X$ of $\rho(X) = 1$ and $r(X) = 1$ without intersecting lines is the Mukai-Umemura threefold $X = X'_{22}$, and a negative answer to (Pe) in case $X = X'_{22}$ is given by E. Amerik (see [Sch]).

Therefore, in order to give a negative answer to (Pe), it is enough to prove the following

PROPOSITION (B). *Any smooth prime Fano threefold* $X_{2g-2} \subset \boldsymbol{P}^{g+1}$, $4 \leq g \leq 12$, $g \neq 11$, *different from the Mukai-Umemura threefold* $X'_{22}$, *contains a* 1-*dimensional family of conics of rank* 2.

In Section 2 we prove Proposition (B) for $4 \leq g \leq 9$ on the base of the following technical

LEMMA (A). *A smooth prime Fano threefold* $X = X_{2g-2} \subset \boldsymbol{P}^{g+1}$, $3 \leq g \leq 9$ *can't contain the tangent scroll* $S_{2g-2}$ *to a rational normal curve* $C_g$ *of degree* $g$.

By a result of Yu. Prokhorov, the only smooth prime $X = X_{2g-2}$, $g = 10, 12$ such that the scheme of lines $\mathscr{H}_X$ has an exotic component is the Mukai-Umemura threefold $X'_{22}$ (see [Pr]). This implies Proposition (B) for $g = 10, 12$. Indeed, let $X = X_{2g-2}$ be a smooth prime Fano threefold such that the scheme of lines $\mathscr{H}_X$ on $X$ has a non-exotic component $\mathscr{H}_o$. Then, by Lemma 3.7 in [I1], the general element of $\mathscr{H}_o$ will represent a line $l \subset X$ which intersects at least one other line on $X$.

This completes the proof of Proposition (B), which yields a negative answer to (Pe).

In Section 3 we prove Lemma (A) for any particular value of $g$, $3 \leq g \leq 9$.

For the prime Fano threefolds $X_{2g-2} \subset \boldsymbol{P}^{g+1}$ $(g = 3, 4, 5, 6, 8)$ we prove Lemma (A) by using the Mukai's representation (see (3.1)) of the smooth $X_{2g-2}$, $3 \leq g \leq 10$ as a complete intersection in a homogeneous or almost-homogeneous variety $\Sigma(g)$. More concretely we see that if the threefold $X \subset \Sigma(g)$ $(g = 3, 4, 5, 6, 8)$ is a complete intersection in $\Sigma(g)$ of the same type as the smooth prime $X_{2g-2}$, and if $X$ contains the tangent scroll $S_{2g-2}$ to the rational normal curve $C_g$ of degree $g$, then $X$ must be singular—see (3.4), (3.5), (3.6), (3.13), (3.15), (3.19).

For $g = 7$ we use the properties of the projection from a special line $l \subset X_{2g-2}$, $g \geq 7$ to reduce the proof of Lemma (A) for $g = 7$ to the already proved Lemma (A) for $g = 5$—see (3.20)–(3.24). To prove Lemma (A) in case $g = 9$ we can use the same approach as for $g = 7$. But a more elegant proof, based on the description of the double projection from a line, had been suggested by the Referee—see (3.25).

## §2. Lemma (A) ⇒ Proposition (B) for $4 \leq g \leq 9$.

**2.0**. Assume that the smooth prime Fano threefold $X = X_{2g-2}$ $(4 \leq g \leq 9)$ does not contain a 1-dimensional family of conics of rank 2.

LEMMA 2.1. *Under the assumption* (2.0):

(i). *The Hilbert scheme* $\mathscr{H}_X$ *of lines on* $X$ *has a unique irreducible component* $\mathscr{H}_o$;

(ii). *$\mathcal{H}_o = \mathcal{H}_X$ is exotic, and the lines $l \in \mathcal{H}_o$ sweep out a tangent scroll $R_o \in |\mathcal{O}_X(d)|$, $d \geq 2$.*

*Proof of Lemma* 2.1.

(i). Let $\mathcal{H}_o$ and $\mathcal{H}_\infty$ be two different irreducible components of $\mathcal{H}_X$, and let $R_o$ and $R_\infty$ be the surfaces swept out by the lines $l \in \mathcal{H}_o$ and $l \in \mathcal{H}_\infty$. Since $\mathrm{Pic}(X) = \boldsymbol{Z}.H$, where $H$ is the hyperplane section, any effective divisor on $X$ must be ample. In particular the general line $l \in \mathcal{H}_o$ intersects the surface $R_\infty \subset X$. If moreover $l \subset R_\infty$ for the general $l \in \mathcal{H}_o$, then $R_o = R_\infty$ and this surface contains two 1-dimensional families of lines. Therefore $R_o = R_\infty$ is a quadric surface on $X$, which contradicts $\mathrm{Pic}(X) = \boldsymbol{Z}.H$. Therefore the general $l \in \mathcal{H}_o$ intersects $R_\infty$ and does not lie in $R_\infty$; and since $R_\infty$ is swept out by lines then there exists a line $l' \subset R_\infty$ which intersects $l$. Since $l \in \mathcal{H}_o$ is general this produces a 1-dimensional family of intersecting lines $l + l'$ ($=$ conics of rank 2 on $X$)—contradiction.

(ii). If $\mathcal{H}_X = \mathcal{H}_o$ is non-exotic then, by Lemma 3.7 of [I1], the general element of $\mathcal{H}_o$ will represent a line $l \subset X$ which will intersect at least one other line $m \subset X$, i.e. $X$ will contain a 1-dimensional family of intersecting lines ($=$ conics of rank 2). Therefore $\mathcal{H}_o$ is exotic.

Since $\mathcal{H}_o$ is exotic and $g \geq 4$ then $R_o$ is the tangent scroll to a curve $C \subset X$ (see (1.2)(3)), and since $\mathrm{Pic}(X) = \boldsymbol{Z}.H$ then $R_o \in |\mathcal{O}_X(d)|$ for some ineteger $d \geq 1$. If $R_o$ is a hyperplane section of $X$ (i.e. $d = 1$) then, by Lemma 6 in [Pr], $C = C_g$ must be a rational normal curve of degree $g$. However the last is impossible since then Lemma (A) will imply that $X$ is singular. Therefore $d \geq 2$. q.e.d.

We shall show that nevertheless $X$ contains a 1-dimensional family of conics $l + m$ of rank 2 where $l, m \in \mathcal{H}_o$.

*Remark.* Let $C_s$ be the (possibly empty) set of singular points of $C$. For any $x \in C - C_s$ denote by $l_x$ the tangent line to $C$ at $x$. For a point $x = x(0) \in C_s$ define a tangent line to $C$ at $x$ to be any limit $\lim_{x(t) \to x(0)} l_{x(t)}$ of tangent lines $l_{x(t)}$ to points $x(t) \in C - C_s$ (see Chapter 2 §4 in [GH]). Clearly, $C$ can have only a finite number of tangent lines to $x(0) \in C_s$ (see also Chapter 2 §1.5 in [Sh]).

**2.2.** By the initial assumption (2.0), $X$ does not contain a 1-dimensional family of conics of rank 2. In particular, $X$ does not contain a 1-dimensional family of pairs of intersecting tangent lines to $C$.

Since the surface $R_o \in |\mathcal{O}_X(d)|$, $d \geq 2$, and *Span* $X = \boldsymbol{P}^{g+1}$ then *Span* $R_o = \boldsymbol{P}^{g+1}$, $g \geq 4$ (see also Lemma 6 in [Pr]). Since $R_o$ is swept out by the tangent lines to $C$ then *Span* $C =$ *Span* $R_o = \boldsymbol{P}^{g+1}$. In particular $C$ does not lie on a plane. Since $R_o$ is the tangent scroll to the non-plane curve $C$ then $S_C$ is singular along the curve $C$.

Let $L \neq C$ be (if exists) an irreducible curve on $R_o$ such that $R_o$ is singular along $L$. If $L$ is not a tangent line to $C$, then the general point of $L$ will be an intersection point of two or more tangent lines to $C$ (see §4 in [P2]). The last is impossible since, by assumption, on $X$ can lie at most a finite number of pairs of intersecting lines.

Therefore any irreducible curve $L \neq C$ such that $L \subset Sing\, R_o$ must be a tangent line to $C$. In addition, the tangent scroll $R_o$ to $C$ still can be singular along a tangent line $L$ to $C$—for example if $L$ is a common tangent line to two or more branches of $C$ at $x$, or if $C$ has a branch with a cusp at $x$, or if $x \in C - C_s$ but $x$ is an inflexion point of $C$ and then $R_o$ has a cusp along $l_x$, etc. (see §2, §4 in [P2]).

Let $\Delta \subset R_o$ be the union of all the irreducible curves $L$ on $R_o$ such that $L \neq C$ and $R_o$ is singular along $L$. By the above argument, either $\Delta = \emptyset$ or $\Delta$ is a union of a finite number of tangent lines to $C$.

Let $\nu : R_n \to R_o$ be the normalization of $R_o$. Fix a desingularization $\tau : \tilde{R}_n \to R_n$, and let $\sigma = \tau \circ \nu : \tilde{R}_n \to R_o$. Let $E_1, \dots, E_k$ $(k \geq 0)$ be all the irreducible *contractable* curves on $\tilde{R}_n$, i.e. all the irreducible curves $E_\iota \subset \tilde{R}_n$ such that $\sigma(E_\iota) \in R_o$ is a point.

Denote by $\sim$ the linear equivalence of divisors on the smooth surface $\tilde{R}_n$, and let $E$ be a divisor on $\tilde{R}_n$. Call $E$ a *zero* divisor on $\tilde{R}_n$ if $E \sim 0$; call the non-zero divisor $E$ *contractable* if $E \sim a_1E_1 + \cdots + a_kE_k$ for some $a_1, \dots, a_k \in \boldsymbol{Z}$. Let $C' \subset \tilde{R}_n$ be the proper $\sigma$-preimage of $C$ on $\tilde{R}_n$. Since $R_o$ is the tangent scroll to the irreducible curve $C$ then the curve $C'$ is irreducible and $\sigma|_{C'} : C' \to C$ is an isomorphism over a dense open subset of $C$ (see also Lemma 2.3 below).

Let $C'_1, \dots, C'_r$ be all the irreducible curves on $\tilde{R}_n$ such that $\sigma(C'_i)$ is an irreducible component of $\Delta$. Therefore

$$K_{\tilde{R}_n} \sim \sigma^* K_{R_o} - mC' - \Sigma_{\iota=1,\dots,r} p_i C'_i + E$$

for some positive integers $m, p_1, \dots, p_r$, and a contractable (or zero) divisor $E$ on $\tilde{R}_n$.

LEMMA 2.3. *Let $X$ fulfills* (2.0). *Then the tangent scroll $R_o \subset X$ to $C$ has a cusp of type $v^2 = u^3 + \cdots$ along $C$, at a neighbourhood of the general point $x \in C \subset R_o$; in particular $m = \mathrm{mult}_C R_o = 2$* (see also §5 in [P2] and §4 in [P1]).

*Proof of Lemma* 2.3.

(1). We shall see first that $R_o$ is irreducible at any neighbourhood of the general point $x \in C$, i.e. $R_o$ has one local branch at $x$.

Assume the contrary, and let $x \in C$ be general. Let $U \subset X$ be a complex-analytical neighbourhood of $x$ such that $R_U = R_o \cap U$ is reducible. Since $R_o$ is swept out by the tangent lines to $C$, the last imlies that for the general point $y \in C_U = C \cap U$ (hence for the general $y \in C$) there exists (possibly non-unique) $z \in C$, $z \neq y$ such that $y$ lies on a tangent line to $C$ at $z$. Since the set $C_s = \{x_1, \dots, x_s\}$ of singular points of $C$ is finite (or empty), and any such $x_\iota$ has at most a finite number of tangent lines, then the general $y \in C$ doesn't lie on a tangent line to $z \in C_s$. Therefore the general $y \in C$ lies on *the* tangent line $l_z$ to $C$ at some (possibly non-unique) $z \in C - C_s$.

If moreover $l_y \neq l_z$ (where $l_y$ is the tangent line to $C$ at $y$) then all such $l_y + l_z$

will produce a 1-dimensional family of conics of rank 2 on $X$, which contradicts the initial assumption (2.0) about $X$.

If $l_y = l_z$ then this will imply that the tangent line $l_y$ to $C$ at the general $y \in C$ is tangent to $C$ at two or more points. But then the projection $\bar{C}$ of $C \subset \boldsymbol{P}^{g+1}$ from the general subspace $\boldsymbol{P}^{g-2} \subset \boldsymbol{P}^{g+1} = Span\, X$ will be a plane curve with a 1-dimensional family of lines tangent to $C$ at two or more points, which is impossible.

(2). It rests to see that the unique local branch of $R_o$ at the general $x \in C$ has a cusp of type $v^2 = u^3 + \cdots$ along $C$ at a neighbourhood of $x$.

Since $R_o \in |\mathcal{O}_X(d)|$ and $d \geq 2$, then $Span\, C = Span\, R_o = \boldsymbol{P}^{g+1}$, $g \geq 4$ (see above).

Let $x$ be a general point of $C$. In order to prove that the tangent scroll $R_o \subset X$ to $C$ has a cusp of type $v^2 = u^3 + \cdots$ at a neighbourhood of $x$ it is enough to see that the projection of $R_o$ from a general $\boldsymbol{P}^{g-3} \subset \boldsymbol{P}^{g+1}$ has a cusp at $x$. This reduces the check to the case when $R_o$ is the tangent scroll to a curve $C \subset \boldsymbol{P}^3$.

Since $x$ is a *general* point of $C \subset \boldsymbol{P}^3$ then, after a possible linear change of coordinates in $\boldsymbol{P}^3$, the curve $C$ has (at $x = (1:0:0:0)$) a local parameterization, or a *normal form* (see §2 in [P2], or Chapter 2 §4 in [GH]):

$$C_U : (x_o(z) : \cdots : x_n(z)) = (1 : z + o(z^2) : z^2 + o(z^3) : z^3 + o(z^4)), \quad |z| < 1,$$

where $o(z^k) = \Sigma_{j \geq k} a_j z^j$. Since the coefficient at $z^k$ in $x_k(z) = z^k + o(z^{k+1})$ is $1 \neq 0$ $(k = 2, 3)$ then, after a possible linear change of $(x_1, x_2, x_3)$, the local parameterization of $C$ at $x = (1:0:0:0)$ can be written as

$$\begin{aligned} C_U &: (x_o(z) : x_1(z) : x_2(z) : x_3(z)) \\ &= (1 : z + o(z^4) : z^2 + o(z^4) : z^3 + o(z^4)), \quad |z| < 1, \end{aligned}$$

i.e. $C_U$ approximates, upto $o(z^4)$, the twisted cubic $C_3 = \{(1 : z : z^2 : z^3)\}$. Therefore, at a neighbourhood of $x = (1:0:0:0)$, the unique local branch (see (1)) of the tangent scroll $R_o$ to $C$ is parameterized by

$$\begin{aligned} R_U &: (x_o(z,t) : x_1(z,t) : x_2(z,t) : x_3(z,t)) \\ &= (1 : z + t + o(z^4) + o(z^3)t : z^2 + 2zt + o(z^4) \\ &\quad + o(z^3)t : z^3 + 3z^2 t + o(z^4) + o(z^3)t). \end{aligned}$$

In affine coordinates $(x_1, x_2, x_3)$ the tangent line to $C$ at $x = (0,0,0)$ is spanned by the vector $n_x = (1,0,0)$, and the normal space $\boldsymbol{C}^2_o \subset \boldsymbol{C}^3(x_1, x_2, x_3)$ to $n_x$ at $x$ is defined by $x_1 = 0$. In order to prove that $R_U$ has a cusp along $C$ at a neighbourhood of $x$ we shall see that the curve $D_U = R_U \cap (x_1 = 0) \subset \boldsymbol{C}^2_o$ has a cusp at $x$.

On $D_U = R_U \cap (x_1 = 0)$, one has: $0 = x_1 = z + t + o(z^4) + o(z^3)t$, i.e. $t = -z + o(z^4)$. Let $u = -x_2$, $v = -x_3/2$. Therefore, on $D_U \subset \boldsymbol{C}^2_o$

$$u = -(z^2 + 2zt + o(z^4) + o(z^3)t) = z^2 + o(z^4)$$
$$v = -1/2(z^3 + 3z^2t + o(z^4) + o(z^3)t) = z^3 + o(z^4),$$

i.e. $u^2 = z^4 + o(z^6)$, $uv = z^5 + o(z^6)$, $v^2 = z^6 + o(z^7)$, $u^3 = z^6 + o(z^8)$, $u^2v = z^7 + o(z^8), \dots$

Let $C|_U = (f_U(u,v) = 0)$ be the local equation of $D_U \subset \mathbf{C}_o^2(u,v)$ at $x = (0,0)$. Therefore, upto a constant non-zero factor, $f_U(u,v) = v^2 - u^3 + c_{2,1}u^2v + c_{1,2}uv^2 + \cdots$, i.e. $C|_U$ has a double cusp-singularity of type $v^2 = u^3 + \cdots$ at $x = (0,0)$ (see §5 in [P1], §4 in [P2], Chapter 5 Examples 3.9.5, 3.9.1 and Chapter 1 Exercise 5.14 in [H]).

Therefore $R = R_o$ has a pinch of type $v^2 = u^3 + \cdots$ along $C$ at a neighbourhood of the general point $x \in C$, which proves Lemma 2.3.

**2.4.** By the definition of $C'_1, \dots, C'_r$ any irreducible component of $\Delta$ can be represented (possibly non-uniquely) as the image $\sigma(C'_i)$ of some $C'_i$, $i = 1, \dots, r$. Since $\sigma|_{C'} : C' \to C$ is an isomorphism over an open dense subset of $C$ then the general point $x \in C$ has a unique $\sigma$-preimage $x'$ on $C'$, and the proper preimage $l'_x \subset \tilde{R}_n$ of the tangent line $l_x$ to $C$ at $x$ intersects $C'$ transversally at $x'$. Since, by assumption, on $X$ doesn't lie a 1-dimensional family of pairs of intersecting lines then the tangent line $l_x$ to $C$ at the general point $x \in C$ does not intersect any other tangent line to $C$.

Therefore the non-singular surface $\tilde{R}_n$ has a structure of a possibly non-minimal ruled surface with a general fiber $L' :=$ the proper $\sigma$-preimage of the general tangent line $l_x$ to $C$. In particular $K_{\tilde{R}_n}.L' = -2$, and since the curve $C'$ is a section of $\tilde{R}_n$ then $C'.L' = 1$.

By the definition of $C'_i$ the curves $\sigma(C'_i) \subset R_o$ are irreducible components of $\Delta$; and since by (2.2) the components of $\Delta$ can be only tangent lines to $C$ then $\sigma(C'_i)$ is a tangent line to $C$. Therefore any component of $\sigma^{-1}(\sigma(C'_i))$, in particular $C'_i$, will not intersect the general fiber $L'$ of $\tilde{R}_n$, i.e. $C'_i.L' = 0$.

Moreover a contractable curve $E_j$ can't intersect the general fiber of $\tilde{R}_n$ since otherwise the point $\sigma(E_j) \in R_o$ will be a common point of a 1-dimensional family of tangent lines to $C$. The last is impossible since $g \geq 4$ and the smooth $X = X_{2g-2}$ can't contain cones—see (1.2). Therefore $E_j.L' = 0$ for any $j = 1, \dots, k$; and since $E$ is a sum of such $E_j$ then $E.L' = 0$.

Since $K_X \sim -H$ and $R_o \sim dH$ on $X$ then, by adjunction, $K_{R_o} \sim (d-1)H|_{R_o}$. Since the hyperplane section $H$ intersects the general tangent line $l$ to $C$ at one point then $\sigma^*(H|_{R_o})$ is also a section of $\tilde{R}_n$, i.e. $\sigma^*(H|_{R_o}).L' = 1$. Therefore

$$\begin{aligned}
-2 = K_{\tilde{R}_o}.L' &= (\sigma^* K_{R_o} - mC' - \Sigma_{i=1,\dots,r} p_i C'_i + E).L' \\
&= (d-1)\sigma^*(H|_{R_o}).L' - mC'.L' - \Sigma_{i=1,\dots,r} p_i C'_i.L' + E.L' \\
&= (d-1) - m, \text{ i.e. } d = m - 1.
\end{aligned}$$

Since $X = X_{2g-2}$ is smooth and $g \geq 4$ then, by Lemma (A), $d > 1$. Therefore $m = d + 1 > 2$, which is impossible since $m = 2$ by Lemma 2.3.

This contradicts the initial assumption (2.0) that the smooth $X = X_{2g-2}$, $4 \leq g \leq 9$ does not contain a 1-dimensional family of conics of rank 2. q.e.d.

## §3. Proof of Lemma (A)

**3.1.** By [M1], [M2] any *smooth* prime Fano threefold $X_{2g-2} \subset \boldsymbol{P}^{g+1}$, $3 \leq g \leq 10$ is a complete intersection of hypersurfaces $F_1, F_2, \dots, F_N$ of degrees $d_1, d_2, \dots, d_N$ in a homogeneous (for $g = 6$—an almost homogeneous) space $\Sigma(g)$, and:

if $g = 3$ then $\Sigma(3) = \boldsymbol{P}^4$, $N = 1$, $d_1 = 4$;

if $g = 4$ then $\Sigma(4) = \boldsymbol{P}^5$, $N = 2$, $d_1 = 2$, $d_2 = 3$;

if $g = 5$ then $\Sigma(5) = \boldsymbol{P}^6$, $N = 3$, $d_1 = d_2 = d_3 = 2$;

if $g = 6$ then $\Sigma(6) = K.G(2,5) \subset \boldsymbol{P}^{10}$ is a cone over the grassmannian $G(2,5) \subset \boldsymbol{P}^9$, $N = 4$, $d_1 = d_2 = d_3 = 1$, $d_4 = 2$;

if $7 \leq g \leq 10$ then $X_{2g-2} = \Sigma(g) \cap \boldsymbol{P}^{g+1}$, where $\Sigma(7) \subset \boldsymbol{P}^{15}$ is the spinor 10-fold, $\Sigma(8) = G(2,6) \subset \boldsymbol{P}^{14}$, $\Sigma(9) \subset \boldsymbol{P}^{13}$ is the sympletic grassmann 6-fold, and $\Sigma(10) \subset \boldsymbol{P}^{13}$ is the $G_2$-fivefold.

**3.2.** To prove Lemma (A), it is enough to see that if $X = X_{2g-2} \subset \Sigma(g)$ is a 3-fold complete intersection as in (3.1) (assuming implicitly that such $X$ may have singularities) then $X_{2g-2}$ can't be smooth. We shall prove this separately for any value of $g$, $3 \leq g \leq 9$.

For $g = 3,4,5,6,8$ we use that $\Sigma(g)$ is either a projective space or a (cone over) grassmannian, which makes it possible to compute directly that the general such $X_{2g-2} \supset S_{2g-2}$ must have $12 - g$ singular points on the curve $C_g$.

For $g = 7,9$ we assume that $X = X_{2g-2} \subset S_{2g-2}$ is smooth, and then project $X$ from a tangent line to $C_g$ to derive a contradiction on the base of the already known Lemma (A) for $g = 5$.

**3.3.** Notation. Let $n \geq 1$, $m \geq 0$ be integers, let $\boldsymbol{P}^{n+m}(z:w) = \boldsymbol{P}^{n+m}(z_0 : \cdots : z_n : w_{n+1} : \cdots : w_{n+m})$ be the complex projective $(n+m)$-space, and let $F(z:w) = F(z_0 : \cdots : z_n : w_{n+1} : \cdots : w_{n+m})$ be a homogeneous form. Denote by $\nabla_z F = (\partial F/\partial z_0, \dots, \partial F/\partial z_n)$ the gradient vector of $F$ with respect to $(z) = (z_0 : \cdots : z_n)$.

Let $F_1(z:w), \dots, F_k(z:w)$ $(k \geq 1)$ be homogeneous forms. Denote by:

$(F_1, \dots, F_k) \subset \boldsymbol{C}[z:w] = \boldsymbol{C}[z_0 : \cdots : z_n : w_{n+1} : \cdots : w_{n+m}]$—the homogeneous ideal generated by $F_1, \dots, F_k$;

$V(F_1, \dots, F_k)$—the projective variety defined by $F_1, \dots, F_k$;

$J_z|_{(a:b)} = J_z(F_1, \dots, F_k)|_{(a:b)} = [\nabla_z F_1; \dots; \nabla_z F_k]|_{(a:b)}$—the Jacobian matrix $J_z$ of partial derivatives of $F_1, \dots, F_k$ with respect to $(z) = (z_0, \dots, z_n)$, computed at the point $(a:b) \in \boldsymbol{P}^{n+m}(z:w)$, (where $\nabla_z F_i$ are regarded as *rows* of $J_z$).

Let e.g. $m = 0$, let $X = V(F_1, \dots, F_k) \subset \boldsymbol{P}^n(z_0 : \cdots : z_n)$, and let $\dim X = d$. Then $\dim T_x X \geq d$ for any $x \in X$, where $T_x X$ is the tangent space to $X$ at $x$; and the point $x \in X$ is singular if $\dim T_x X > d$ (see e.g. Chapter 2, §1.4 in [Sh]).

Equivalently $x \in X$ is singular if $\operatorname{rank} J_z|_x < n-d$. The subset $\operatorname{Sing} X = \{x \in X : \operatorname{rank} J_z|_x < n-d\} \subset X$ of all the singular points of $X$ is a proper closed subset of the projective algebraic variety $X$, defined on $X$ by vanishing of all the $(n-d) \times (n-d)$ minors of $J_z$.

**3.4. Proof of Lemma (A) for** $g = 3$. The tangent scroll to the twisted cubic $C_3 : (x_0 : x_1 : x_2 : x_3) = \vec{s} := (s_0^3 : s_0^2 s_1 : s_0 s_1^2 : s_1^3)$ is the quartic surface $S_4 = V(f) \subset \boldsymbol{P}^3(x)$ where $f(x) = 3x_1^2x_2^2 + 6x_0x_1x_2x_3 - 4x_1^3x_3 - x_0^2x_3^2 - 4x_0^2x_2^3$. The surface $S_4$ is singular along $C_3$ since the gradient vector $\nabla_x f|_{\vec{s}} = 0$ for any $\vec{s} \in C_3$.

Let the quartic threefold $X_4 \subset \boldsymbol{P}^4(x:u)$ be such that $S_4 = X_4 \cap \boldsymbol{P}^3(x)$, and let $X_4 = V(F) \subset \boldsymbol{P}^4(x:u)$ where $F(x:u) = \Sigma_{0 \le k \le 4} f_i(x) u^{4-i}$. Therefore $f_4 \in (f)$, i.e. $f_4 = cf$ for some constant $c \in \boldsymbol{C}$.

Let $x \in X_4$. Then $x \in \mathit{Sing}\, X_4$ *iff* $\nabla_{x,u} F_x = 0$. Let $s = (s_0 : s_1) \in \boldsymbol{P}^1$. Then $(\vec{s}:0) \in \mathit{Sing}\, X_4$ *iff* $0 = \nabla_{x,u} F|_{(\vec{s}:0)} = (\nabla_x F, \partial F/\partial u)|_{(\vec{s}:0)} = (\nabla_x f|_{\vec{s}}, f_3(\vec{s})) = (0, f_3(\vec{s}))$.

Therefore either $f_3(\vec{s}) \equiv 0$ (i.e. $f_3(\vec{s}) = 0$ for any $s = (s_0 : s_1)$), and then $X_4$ is singular along $C_4$, or $f_s(\vec{s}) \not\equiv 0$, and then $(\vec{s} : 0) \in \mathit{Sing}\, X_4$ *iff* $s = (s_0 : s_1)$ is a zero of the (non-vanishing) homogeneous form $F_9(s) = f_3(\vec{s}) = f_3(s_0^3 : s_0^2 s_1 : s_0 s_1^2 : s_1^3)$ of degree 9.

In addition, for the general $f_3(x)$ all the zeros of $F_9(s) = f_3(\vec{s})$ are simple, i.e. different from each other. Therefore the general $X_4 \supset S_4$ has $9 = 12 - g(X_4)$ singular points on $C_3$. In coordinates as above, these singular points of $X_4$ are the images of the 9 zeros of $F_9(s)$ under the Veronese map $v_3 : \boldsymbol{P}^1 \to C_3 \subset X_4$, $v_3 : s = (s_0 : s_1) \mapsto (\vec{s} : 0)$.

**3.5. Proof of Lemma (A) for** $g = 4$. The tangent scroll to the rational normal quartic $C_4 : (x_0 : x_1 : \cdots : x_4) = \vec{s} := (s_0^4 : s_0^3 s_1 : \cdots : s_1^4)$ is a complete intersection $S_6 = V(q, f) \subset \boldsymbol{P}^4(x)$ where $q(x) = 3x_2^2 - 4x_1x_3 + x_0x_4$ and $f(x) = x_2^3 - 2x_0x_3^2 - 2x_1^2x_4 + 3x_0x_2x_4$. The surface $S_6$ is singular along $C_4$ since the gradients of $q$ and $f$ are linearly dependent along $C_4$; more precisely $\nabla_x f|_{\vec{s}} = s_0^2 s_1^2 \nabla_x q|_{\vec{s}}$ for any $\vec{s} \in C_4$.

Let $X_6 = V(Q, F) \subset \boldsymbol{P}^5(x:u)$ be a complete intersection of the quadric $Q(x:z) = \Sigma_{0 \le k \le 2} q_k(x) u^{2-k} = 0$ and the cubic $F(x:z) = \Sigma_{0 \le l \le 3} q_l(x) u^{3-l} = 0$, and let $S_6 = X_6 \cap \boldsymbol{P}^4(x)$. In particular, $(q_2, f_3) \subset (q, f)$ as homogeneous ideals in $\boldsymbol{C}[x] = \boldsymbol{C}[x_0 : \cdots : x_4]$.

For the fixed $X_6 = V(Q, F)$ the generators $Q, F$ of the homogeneous ideal $(Q, F)$ can be replaced by $c'Q$ and $c''F + L(x:u)Q$, for any pair of nonzero constants $c'$ and $c''$, and for any linear form $L(x:u)$. Now, $(q_2, f_3) \subset (q, f)$ yields that one can choose $Q$ and $F$ such that $q_2 = \varepsilon' q$ and $f_3 = \varepsilon'' f$, where $\varepsilon', \varepsilon''$ are either 0 or 1.

Consider the general case $\varepsilon' = \varepsilon'' = 1$; the study in the degenerate case $\varepsilon'.\varepsilon'' = 0$ is similar. The subscheme $\mathit{Sing}\, X_6 = \mathit{Sing}\, V(Q, F)$ is defined by $\mathit{rank}[\nabla_{x,u} Q; \nabla_{x,u} F] \le 1$. By the choice of $Q$ and $F$, $\nabla_{x,u} Q|_{(\vec{s}:0)} = (\nabla_x q_2|_{\vec{s}}, q_1(\vec{s}))$ and $\nabla_{x,u} F|_{(\vec{s}:0)} = (\nabla_x f_3|_{\vec{s}}, f_2(\vec{s}))$, where $1.q_2 = q$ and $1.f_3 = f$. Just as in case $g = 3$, the last together with the identity $\nabla_x f|_{\vec{s}} - s_0^2 s_1^2 \nabla_x q|_{\vec{s}} \equiv 0$ imply that $(\vec{s}:0) \in Sing\, X_6$ *iff* $F_8(s) := f_2(\vec{s}) - s_0^2 s_1^2.q_1(\vec{s}) = 0$, where $\vec{s} = (s_0^4 : s_0^3 s_1 : \cdots : s_1^4)$.

The Veronese map $v_4 : \boldsymbol{P}^1 \to C_4 \subset X_6$, $v_4 : s = (s_0 : s_1) \mapsto (\vec{s} : 0)$ states an isomorphism between $\boldsymbol{P}^1$ and $C_4$. Therefore either $F_8(s) \equiv 0$, and then $X_6$ is singular along $C_4$, or $F_8(s) \not\equiv 0$, and then the singular points of $X_6$ on $C_4$ are the $v_4$-images of the zeros of the homogeneous form $F_8(s)$ of degree 8. As in case $g = 3$, for the general $f_2(x)$, $q_1(x)$ the form $F_8(s)$ has only simple zeros. Therefore the general $X_6 \supset S_6$ has $8 = 12 - g(X_6)$ singular points on $C_4$.

**3.6. Proof of Lemma (A) for** $g = 5$. The tangent scroll to the rational normal quintic $C_5 : x_i = s_0^{5-i} s_1^i$, $0 \le i \le 5$ is a complete intersection $S_8 = V(q', q'', q''') \subset \boldsymbol{P}^5(x)$ where $q'(x) = 4x_1x_3 - 3x_2^2 - x_0x_4$, $q''(x) = 3x_1x_4 - 2x_2x_3 - x_0x_5$ and $q'''(x) = x_1x_5 - 4x_2x_4 + 3x_3^2$. The surface $S_8$ is singular along $C_4$ since the gradients of $q', q''$ and $q'''$ are linearly dependent along $C_5$. More precisely

$$(*) \qquad s_1^2 \nabla_x q'|_{\vec{s}} - s_0 s_1 \nabla_x q''|_{\vec{s}} - s_0^2 \nabla_x q'''|_{\vec{s}} = 0 \quad \text{for any } \vec{s} \in C_5.$$

Let $X_8 = V(Q', Q'', Q''') \subset \boldsymbol{P}^6(x:u)$ be a complete intersection of the quadrics $Q^\iota(x:z) = \Sigma_{0 \le k \le 2}\, q_k^\iota(x) u^{2-k}$ $(\iota = ', '', ''')$, and such that $S_8 = X_8 \cap \boldsymbol{P}^5(x)$. In particular $(q_1', q_1'', q_1''') \subset (q', q'', q''')$ as homogeneous ideals in $\boldsymbol{C}[x] = \boldsymbol{C}[x_0 : \cdots : x_4]$. Therefore $q_1', q_1''$ and $q_1'''$ are linear combinations of $q', q''$ and $q'''$. Since the ideal $(Q', Q'', Q'')$ of $X_8$ is generated also by any $GL(3)$-transform of the triple $(Q', Q'', Q''')$, we may assume that $q_2' = \varepsilon' q'$, $q_2'' = \varepsilon'' q''$ and $q_2''' = \varepsilon''' q'''$, where $\varepsilon', \varepsilon''$ and $\varepsilon'''$ are 0 or 1. The subscheme $(Sing\, X_8)|_{C_5} \subset \boldsymbol{P}^1$ is defined by

$$(**) \qquad \begin{aligned} 2 &\ge \operatorname{rank}[\nabla_{x,u} Q'; \nabla_{x,u} Q''; \nabla_{x,u} Q''']|_{(\vec{s}:0)} \\ &= \operatorname{rank}[(\nabla_x \varepsilon' q'|_{\vec{s}}, q_1'(\vec{s})); (\nabla_x \varepsilon'' q''|_{\vec{s}}, q_1''(\vec{s})); (\nabla_x \varepsilon''' q'''|_{\vec{s}}, q_1'''(\vec{s}))]. \end{aligned}$$

Let $F_7(s) = \varepsilon' s_1^2 q_1'(\vec{s}) - \varepsilon'' s_0 s_1 q_1''(\vec{s}) - \varepsilon''' s_0^2 q_1'''(\vec{s})$.

The Veronese map $v_5 : \boldsymbol{P}^1 \to C_5 \subset X_8$, $v_5 : s = (s_0 : s_1) \mapsto (\vec{s} : 0)$ states an isomorphism between $\boldsymbol{P}^1$ and $C_5$. Just as in (3.5), $(*)$ and $(**)$ imply that either $F_7(s) \equiv 0$, and then $X_8$ is singular along $C_5$, or $F_7(s) \not\equiv 0$, and then the singular points of $X_6$ on $C_5$ are the $v_5$-images of the zeros of the homogeneous form $F_7(s)$ of degree 7. Moreover, for the general linear forms $q_1'(x), q_1''(x), q_1'''(x)$ the form $F_7(s)$ has only simple zeros. Therefore the general $X_8 \supset S_8$ has $7 = 12 - g(X_8)$ singular points on $C_5$.

## Lemma (A) for $g = 6, 8$.

LEMMA 3.7. *Let $n \ge 3$, and let $G = G(1:n) = G(2, n+1) \subset \boldsymbol{P}^{n(n+1)/2-1}$ be the grassmannian of lines in $\boldsymbol{P}^n = \boldsymbol{P}(\boldsymbol{C}^{n+1})$. Let $C \subset G \subset \boldsymbol{P}^{n(n+1)/2-1}$ be a smooth irreducible curve such that* $\dim Span(C) \ge 3$. *Let the surface* $\mathrm{Gr}(C) \subset \boldsymbol{P}^n$ *be the union of lines $l \subset \boldsymbol{P}^n$ such that $l \in C \subset G = G(1:n)$, and let $S_C \subset Span(C) \subset \boldsymbol{P}^{n(n+1)/2-1}$ be the surface swept out by the tangent lines to $C$ (or the tangent scroll to $C$—see above).*

*Then the tangent scroll $S_C$ to $C$ lies on $G = G(1:n)$ iff*
*either all the lines $l \in C$ have a common point, i.e.* Gr$(C)$ *is a cone,*
*or there exists an irreducible curve $Z \subset \boldsymbol{P}^n$ such that all the lines $l \in C$ are tangent lines to $Z$, i.e.* Gr$(C)$ *is the tangent scroll to $Z$.*

*Proof.* For $n = 3$ this result can be found in [AS]. For $n > 3$ one can apply induction, using the fact that projection from a point in $\boldsymbol{P}^n$ induces a projection from $G(2, n+1)$ onto $G(2, n)$. q.e.d.

LEMMA 3.8. *Let $C_g \subset G(2, g/2+2) = G(1 : \boldsymbol{P}^{g/2+1})$ $(g = 6, 8)$ be a rational normal curve such that the tangent scroll $S_{2g-2}$ to $C_g$ is contained in $G(2, g/2+2)$. Then:*

(i). *If $g = 6$ then the lines $l \subset \boldsymbol{P}^4$, $l \in C_6$ sweep out the tangent scroll to a rational normal curve $C_4 \subset \boldsymbol{P}^4$.*

(ii). *If $g = 8$, and if there exists a 3-fold linear section $X_{14} = G(2,6) \cap \boldsymbol{P}^9$ such that $X_{14} \supset S_{14}$, then the lines $\{l \subset \boldsymbol{P}^5 : l \in C_8\}$ sweep out the tangent scroll to a rational normal curve $C_5 \subset \boldsymbol{P}^5$.*

*Proof.* Let $g = 6, 8$, let the lines $l \in C_g$ sweep out a cone $\mathrm{Gr}(C_g) \subset \boldsymbol{P}^{g/2+1}$— see Lemma 3.7, and let $x$ be the vertex of $\mathrm{Gr}(C_g)$. Then $C_g$ is contained in the Schubert $g/2$-space $\boldsymbol{P}^{g/2}_x = \sigma_{g/2,0}(x) = \{l \subset \boldsymbol{P}^{g/2+1} : x \in l\}$. Since $C_g$ is projectively normal it must span a $g$-space. Therefore $g \leq g/2+1$ which contradicts $g = 6, 8$. Therefore, by Lemma 3.7, the lines $l \in C_g$ must sweep out the tangent scroll to a rational curve $C_d \subset \boldsymbol{P}^{g/2+1}$.

Let $g=6$, and let $C_d \subset \boldsymbol{P}^3 \subset \boldsymbol{P}^4$. Then $C_6 \subset G(1:\boldsymbol{P}^3) = \sigma_{11}(\boldsymbol{P}^3) \subset G(1:\boldsymbol{P}^4)$. Therefore $6 = \dim(\mathit{Span}\, C_6) \leq \dim(\mathit{Span}\, G(1:\boldsymbol{P}^3)) = 5$—contradiction. Therefore $d \geq 4$ since $C_d$ must span $\boldsymbol{P}^4$, and now it is easy to see that the rational normal curve $C_6$ is the curve of tangent lines to $C_d$ *iff* $d = 4$.

Let $g = 8$, and let $C_d \subset \boldsymbol{P}^4 \subset \boldsymbol{P}^5$. Then $C_8 \subset G(1:\boldsymbol{P}^4) = \sigma_{11}(\boldsymbol{P}^4) \subset G(1:\boldsymbol{P}^5)$. Let $\boldsymbol{P}^9_o = \mathit{Span}\, G(1:\boldsymbol{P}^4)$. Then $\boldsymbol{P}^8 = \mathit{Span}\, C_8 \subset \boldsymbol{P}^9_o$. By condition $C_8 \subset X_{14} = G(1:\boldsymbol{P}^5) \cap \boldsymbol{P}^9$. Therefore $\boldsymbol{P}^8 \subset \boldsymbol{P}^9 \cap \boldsymbol{P}^9_o$ and $X_{14} \supset Z_o := G(1:\boldsymbol{P}^4) \cap \boldsymbol{P}^8$. Since $\boldsymbol{P}^8 \subset \boldsymbol{P}^9_o = \mathit{Span}\, G(1:\boldsymbol{P}^4)$, $Z_o$ is at least a hyperplane section of the 6-dimensional grassmannian $G(1:\boldsymbol{P}^4)$. This contradicts $X_{14} \supset Z_o$ and $\dim X_{14} = 3$. Therefore $d \geq 5$ since $\mathit{Span}\, C_d = \boldsymbol{P}^5$, and now it is easy to see that the rational normal curve $C_8$ is the curve of tangent lines to $C_d$ *iff* $d = 5$. q.e.d.

## Proof of Lemma (A) for $g = 6$.

**3.9.** By (3.1) any smooth prime $X_{10}$ is a complete intersection of three hyperplanes and a quadric in the cone $K.G(2,5) \subset \boldsymbol{P}^{10}$. Let $o$ be the vertex of the cone $K.G(2,5) \subset \boldsymbol{P}^{10}$, and let $X \subset K.G(2,5)$, be as in (3.1). There are two kinds of such threefolds $X_{10}$ (see [I1], [Gu]):

(i). $g = 6$—first kind: $o \notin \mathit{Span}(X)$, and then the projection $p_o$ from $o$ sends $X$ isomorphically to $X_{10} = G(2,5) \cap \boldsymbol{P}^7 \cap Q$, where $G(2,5) \subset \boldsymbol{P}^9$ by the Plücker embedding, $\boldsymbol{P}^7 \subset \boldsymbol{P}^9$ and $Q$ is a quadric.

(ii). $g = 6$—second kind: $o \in Span(X)$, and then $\pi = p_o|_X : X = X'_{10} \to Y_5$ is a double covering of a threefold $Y_5 = G(2,5) \cap \boldsymbol{P}^6$. In particular, if $X'_{10}$ is smooth then the intersection $Y_5 = G(2,5) \cap \boldsymbol{P}^6$ is smooth.

### $g = 6$ (first kind)

**3.10.** Let $X_{10} = G(2,5) \cap \boldsymbol{P}^7 \cap Q$ be a (possibly singular) complete intersection as in (3.9)(i), and assume that $X_{10}$ contains the tangent scroll $S_{10}$ to a rational normal curve $C_6$ of degree 6. By Lemma 3.8(i) the points of $C_6$ are the Plücker coordinates $x_{ij}(s)$ of the tangent lines to a rational normal quartic $C_4 = x_i = s^i$, $0 \le i \le 4$, i.e.

$$(x_{ij}(s)) = \begin{pmatrix} 0 & 1 & 2s & 3s^2 & 4s^3 \\ \cdots & 0 & s^2 & 2s^3 & 3s^4 \\ \cdots & \cdots & 0 & s^4 & 2s^5 \\ \cdots & \cdots & \cdots & 0 & s^6 \\ \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

Therefore the subspace $\boldsymbol{P}^6 = Span(C_6) \subset \boldsymbol{P}^9$ is defined by $H_0 = H_1 = H_2 = 0$ where $H_0 = x_{03} - 3x_{12}$, $H_1 = x_{04} - 2x_{13}$ and $H_2 = x_{14} - 3x_{23}$.

**3.11.** Introduce in $\boldsymbol{P}^6$ the coordinates $(v) = (v_0 : \cdots : v_6) = (x_{01} : x_{02} : x_{12} : x_{13} : x_{23} : x_{24} : x_{34})$, and let $I_{S_{10}} \subset \boldsymbol{C}[v] := \boldsymbol{C}[v_0, \dots, v_6]$ be the homogeneous ideal of the tangent scroll $S_{10} \subset \boldsymbol{P}^6$ to $C_6$. Let $Pf_k = x_{ab}x_{cd} - x_{ac}x_{bd} + x_{ad}x_{bc}$, $0 \le a < b < c < d \le 4 : a, b, c, d \ne k$ $(k \in \{0,1,2,3,4\})$ be the 5 Plücker quadrics in the coordinates $x_{jk}$. In coordinates $(v)$ of $\boldsymbol{P}^6 = \boldsymbol{P}^6(v)$ the restrictions $q_k$ of $Pf_k$ to $\boldsymbol{P}^6$ are

$$q_0(v) = v_2v_6 - v_3v_5 + 3v_4^2, \quad q_1(v) = v_1v_6 - 3v_2v_5 + 2v_3v_4,$$
$$q_2(v) = v_0v_6 - 9v_2v_4 + 2v_3^2, \quad q_3(v) = v_0v_5 - 3v_1v_4 + 2v_2v_3,$$
$$q_4(v) = v_0v_4 - v_1v_3 + 3v_2^2.$$

In the open subset $U_0 = \{v_0 = 1\} \subset \boldsymbol{P}^6(v)$ the curve $C_6$ is parameterized by $C_6 = \{(v) = \vec{s} := (1, 2s, s^2, 2s^3, s^4, 2s^5, s^6)\}$.

Now, it is easy to see that the quadric $q(v) = 5v_2v_4 - 2v_1v_5 + 3v_0v_6$ vanishes at the points of the tangent scroll $S_{10}$ to $C_6$, and the homogeneous ideal $I_{S_{10}} = (q_0, \dots, q_4, q) \subset \boldsymbol{C}[v]$.

Let $J_v = J_v(q_0, \dots, q_4, q)$ be the Jacobian matrix of $(q_0, \dots, q_4, q)$ with respect to $v = (v_1, \dots, v_6)$. The surface $S_{10} \subset V_5$ is singular along $C_6$ since $\operatorname{rank} J_v|_{\vec{s}} < 4 = \operatorname{codim}_{\boldsymbol{P}^6} S_{10}$ for any $\vec{s} \in C_6$. For the special choice $(v)$ of the coordinates this can be verified directly:

$$s^{-2}\nabla_v q_0|_{\vec{s}} = (0, s^4, -2s^3, 6s^2, -2s, 1), \quad s^{-1}\nabla_v q_1|_{\vec{s}} = (s^5, -6s^4, 2s^3, 4s^2, -3s, 2),$$
$$\nabla_v q_2|_{\vec{s}} = (0, -9s^4, 8s^3, -9s^2, 0, 1), \quad s\nabla_v q_3|_{\vec{s}} = (-3s^5, 4s^4, 2s^3, -6s^2, s, 0),$$
$$s^2\nabla_v q_4|_{\vec{s}} = (-2s^3, 6s^4, -2s^3, s^2, 0, 0), \quad \text{and} \quad \nabla_v q|_{\vec{s}} = (-4s^5, 5s^4, 0, 5s^2, -4s, 3).$$

$(*)$. Therefore $rank\, J_v|_{\vec{s}} = 4$ for any $\vec{s} \in C_6$, and the linear 4-space of linear equations between the gradients $\nabla_v q_0|_{\vec{s}}, \dots, \nabla_v q_4|_{\vec{s}}$ and $\nabla_v q|_{\vec{s}}$ is spanned on the Pfaff-equation $s^{-2}\nabla_v q_0|_{\vec{s}} + s^{-1}\nabla_v q_1|_{\vec{s}} + \nabla_v q_2|_{\vec{s}} + s\nabla_v q_3|_{\vec{s}} + s^2\nabla_v q_4|_{\vec{s}} = 0$ and the 3-space of equations $\nabla_v q|_{\vec{s}} = a_0.s^{-2}\nabla_v q_0|_{\vec{s}} + a_1.s^{-1}\nabla_v q_1|_{\vec{s}} + a_2.\nabla_v q_2|_{\vec{s}} + a_3.s\nabla_v q_3|_{\vec{s}} + a_4.s^2\nabla_v q_4|_{\vec{s}} = 0$ where:

$(**)$. $a_0 + 4a_3 + 3a_4 = 8$, $a_1 - 3a_3 - 2a_4 = -4$, $a_2 + 2a_3 + a_4 = 3$.

**3.12.** In the dual space $\check{\boldsymbol{P}}^9$, let $\check{G} = \check{G}(2,5) \subset \check{\boldsymbol{P}}^9$ be the grassmannian of hyperplane equations represented by the skew-symmetric $5 \times 5$ matrices of rank 2. It is easy to see that the plane $\Pi = \langle H_0, H_1, H_2 \rangle \subset \check{\boldsymbol{P}}^9$ of hyperplane equations of $\boldsymbol{P}^6 = Span\, C_6$ does not intersect $\check{G}$.

Let $\Lambda \subset \Pi$ be any line in $\Pi$. In $\boldsymbol{P}^9$, the line $\Lambda$ defines, by duality, the subspace $\boldsymbol{P}^7(\Lambda) \supset \boldsymbol{P}^6 = Span\, C_6$. It is easy to see that the fourfold $W(\Lambda) = G \cap \boldsymbol{P}^7(\Lambda)$, where $G = G(2,5)$, is smooth. In fact, $W(\Lambda)$ will be smooth *iff* the line $\Lambda$ does not intersect $\check{G}$. The last is true since $\Lambda \subset \Pi$ and $\Pi \cap \check{G} = \emptyset$. Therefore any $X_{10} \supset S_{10}$ is a quadratic section of the smooth 4-fold $W = W(\Lambda)$.

**3.13.** Let $X_{10} = G(2,5) \cap \boldsymbol{P}^7 \cap Q \supset S_{10}$, where $Q$ is a quadric. We shall show that the singularities of $X_{10}$ on $C_6$ are the zeros of a homogeneous form of degree 6 on $C_6 \cong \boldsymbol{P}^1$. To simplify the notation, we shall show this for one special choice of the line $\Lambda \subset \Pi$ (see (3.12)); the check for any other $\Lambda \subset \Pi$ is similar.

Let $\Lambda = \{H_2 = 0\} \subset \Pi$. Then, in coordinates $(v)$ and $u = x_{14} - 3x_{23}$ in $\boldsymbol{P}^7(\Lambda)$, the subspace $\boldsymbol{P}^6(x) = (u = 0)$. Therefore any quadric $Q \subset \boldsymbol{P}^7(\Lambda)$, such that $Q \cap X_{10} = S_{10}$, can be written in the form $Q = Q(v,u) = cu^2 + L(v)u + q(v)$ where $c \in \boldsymbol{C}$ and $L$ is a linear form of $(v)$.

Let $Q_k(v,u)$ $(k = 0, 1, \dots, 4)$ be the restriction of the Pfaff quadric $Pf_k$ on $\boldsymbol{P}^7(v,u)$, and let $J_{v,u} = [\nabla_{v,u}Q_0; \dots; \nabla_{v,u}Q_4; \nabla_{v,u}Q]$ be the Jacobian matrix of $(Q_0, \dots, Q_4, Q)$.

The singularities of $X_{10}$ on $C_6$ are the points $(\vec{s} : 0) \in \boldsymbol{P}^7$ for which $rank\, J_{v,u}|_{(\vec{s}:0)} < 4 = \mathrm{codim}_{\boldsymbol{P}^7} X_{10}$.

Let $l_i(\vec{s}) = \partial Q_i/\partial u|_{\vec{s}}$, $i = 0, \dots, 4$. The rows of $J_{v,u}|_{(\vec{s},0)}$ are $\nabla_{u,v}Q_i|_{(\vec{s}:0)} = (\nabla_u q_i|_{\vec{s}}, l_i(\vec{s}))$, $i = 0, \dots, 4$ and $\nabla_{u,v}Q|_{(\vec{s}:0)} = (\nabla_u q|_{\vec{s}}, L(\vec{s}))$. For the special choice of the line $\Lambda \subset \Pi$, the linear forms $l_i = l_i(\vec{s})$ are $(l_0, l_1, l_2, l_3, l_4) = (s^4, 0, -3s^2, -2s, 0)$.

Therefore, in view of $(*)$, $X_{10}$ will have a singularity at $(\vec{s}, 0) \in C_6$ if there exist constants $a_0, \dots, a_4$ satisfying $(**)$ and such that $L(\vec{s}) = (a_0 - 3a_2 - 2a_3)s^2$ (here $\vec{s} = (1, 2s, s^2, 2s^3, s^4, 2s^5, s^6)$—see above).

By $(**)$ $a_0 - 3a_2 - 2a_3 = -1$. Therefore, in homogeneous coordinates $(s_0 : s_1)$, $s = s_1/s_0$, the variety $X_{10}$ will be singular at the point $((\vec{s}) : 0) \in C_6$ *iff* $F_6(s_0 : s_1) := L(s_0^6 : 2s_0^5 s_1 : s_0^4 s_1^2 : 2s_0^3 s_1^3 : s_0^2 s_1^4 : 2s_0 s_1^5 : s_1^6) + s_0^4 s_1^2 = 0$.

The Veronese map $v_6 : \boldsymbol{P}^1 \to C_6 \subset X_{10}$, $v_6 : (s_0 : s_1) \mapsto (\vec{s} : 0) = (s_0^6 : 2s_0^5 s_1 : s_0^4 s_1^2 : 2s_0^3 s_1^3 : s_0^2 s_1^4 : 2s_0 s_1^5 : s_1^6 : 0)$, states an isomorphism between $\boldsymbol{P}^1$ and $C_6$.

Therefore either $F_6(s_0 : s_1) \equiv 0$, and then $X_{10}$ is singular along $C_6$, or $F_6(s_0 : s_1) \not\equiv 0$, and then the singular points of $X_{10}$ on $C_6$ are the $v_6$-images of the zeros of the homogeneous form $F_6(s_0 : s_1)$ of degree 6.

The choice of an arbitrary line $\Lambda \subset \Pi$ will only change the homogeneous sextic forms defined by $l_i(\vec{s})$, $i = 0, \dots, 4$. q.e.d.

$g = 6$ **(second kind)**

**3.14.** Let $\pi : X = X'_{10} \to Y_5$, $X \subset \boldsymbol{P}^7$, be a double covering of the Del Pezzo threefold $Y_5 = G(2,5) \cap \boldsymbol{P}^6$ branched along the quadratic section $B \subset Y_5$. Below we shall identify the branch locus $B \subset Y_5$ and the ramification divisor $R \subset X'_{10}$, $R \cong B$.

**3.15.** Assume that $X$ contains the tangent scroll $S = S_{10}$ to the rational normal sextic $C = C_6$, and let $l \subset S$ be a general tangent line to $C$. Then $N_{l/X} \cong \mathcal{O}(1) \oplus \mathcal{O}(-2)$ (see (1.2)), and $\pi(l) \subset Y_5$ also is a line. If $N_{\pi(l)/Y_5} \cong \mathcal{O} \oplus \mathcal{O}$ then $d\pi : N_{l/X} \to N_{\pi(l)/Y_5}$ has one-dimensional kernel along $l$. In this case $l$ is contained in the ramification divisor $R$ of $\pi$. The last is possible only for a finite number of $l's$. Therefore, at least for the general tangent line $l \subset S$ to $C$, the line $\pi(l) \subset \pi(S)$ is a $(-1,1)$-line on $Y_5$, i.e. $N_{\pi(l)/Y_5} \cong \mathcal{O}(-1) \oplus \mathcal{O}(1)$. Therefore $\pi(S) \subset Y_5$ coincides with the surface $S_{-1,1} \subset Y_5$ swept-out by the $(-1,1)$-lines on $Y_5$, which in turn is a tangent scroll to a rational normal sextic (see [FN]). Clearly $\pi : S \to S_{-1,1}$ is an isomorphism and $S_{-1,1}$ is the tangent scroll to $\pi(C_6)$.

Assume first that $S = R$. Then $S_{-1,1} = \pi(S) = B \cong R$. Therefore $X$ is singular along $C_6$ since the branch locus $B = S_{-1,1}$ of $\pi$ is singular along $C_6$. In order to prove Lemma (A) for $g = 6$ (second kind) it rests to see the general $X'_{10} \supset S_{10}$ such that $S \neq R$ is singular. This will imply that any $X'_{10} \subset S_{10}$ must be singular—since the property $X'_{10} \subset S_{10}$ to have a singularity is a closed condition.

Since $\pi : X'_{10} \to Y_5$ is a 2-sheeted covering with a ramification divisor $R$, then $H^0(X'_{10}, O(1)) = \pi^* H^0(Y_5, O(1)) + \boldsymbol{C}.R$ where $R$ is the hyperplane equation of $R \subset X'_{10}$. Therefore, since $S \cong \pi(S)$ and $S \neq R$, then $\pi(S)$ is tangent to the branch locus $B$ of $\pi$ along a (possibly singular) canonical curve $C^6_{10} = B \cap H = \pi(S) \cap H$ for some hyperplane $H \subset \boldsymbol{P}^6$. Since $\pi(S) = S_{-1,1}$ then the general $X'_{10} \supset S = S_{10}$, $S \neq R$ comes from a branch locus $B$ totally tangent to $S_{-1,1}$ along a general hyperplane section $H \cap S_{-1,1}$. Since $H$ is general then $H$ intersects the rational normal sextic $\pi(C_6)$ at 6 points $p_1, \dots, p_6$ such that $p_i \neq p_j$ for $i \neq j$. By $p_i \in B$ and the identification $B = R$, we may cosider $p_i$ as points on $R \subset X'_{10}$. We shall see that

($*$) Lemma. $p_i \in \mathit{Sing}\, X'_{10}$, $i = 1, \dots, 6$.

*Proof of* ($*$). Let $U_i \subset Y_5$ be a sufficiently small neighborhood of the point $p_i$. Since $Y_5$ is smooth, we can identify $U_i$ with a disk in $\boldsymbol{C}^3$, and let $(u, v, w)$ be local coordinates in $U_i$ s.t. $p_i = (0,0,0)$. Let $f(u,v,w) = 0$ be the local equation of $S_{10}$ in $U_i$. Since $S_{10}$ is the tangent scroll to the smooth rational curve $C_6$, and $p_i \in C_6$, one can choose the coordinates $u, v, w$ such that $f = u^3 - v^2$ (since the scroll $S_{10}$ has a double cusp-singularity along $C_6$—see Lemma 2.3). Let $q = 0$ and $h = 0$ be the local equations of $B$ and $H$ in $U_i$. Since $B$ and $S_{10}$ are singular along $C_{10}^6 = B \cap H = S_{10} \cap H$, $h^2, f$ and $q$ are linearly dependent, i.e. $\alpha h^2 + \beta f + \gamma q = 0$ for some constants $\alpha, \beta, \gamma$. Moreover $\alpha \neq 0$ and $\gamma \neq 0$ since $\pi(S) \cong S$ and $\pi(S) \neq B$, and one may assume that $\beta \neq 0$ (otherwise $B = 2H$ and $X'_{10}$ will be singular along the surface $R_{\mathrm{red}} \cong B_{\mathrm{red}} = H$). Since $p_i \in H$, $h(0,0,0) = 0$ and $h = au + bv + cw + o(2)$, where $o(k)$ denotes a sum of terms of degree $\geq k$. Therefore the surface $B$ is singular at $p_i$ since $B \cap U_i = (q = 0)$, and the series expansion

$$q(u,v,w) = -\beta/\gamma f - \alpha/\gamma h^2 = \beta/\gamma(v^2 - u^3) - \alpha/\gamma(au + bv + cw + o(2))^2$$

has no linear term. Since $B$ is the branch locus of $\pi : X'_{10} \to Y_5$, the threefold $X'_{10}$ will be also singular at $p_i \in R \cong B$, $i = 1, \ldots, 6$. In addition, the $6 = 12 - g(X'_{10})$ singular points $p_1, \ldots, p_6$ of $X'_{10}$ lie on the rational normal curve $C_6 \subset X'_{10}$. q.e.d.

**Proof of Lemma (A) for $g = 8$.**

**3.16. The Da Palatini construction (see [Pu]).** Let $\boldsymbol{P}^5 = \boldsymbol{P}(V)$ where $V = \boldsymbol{C}^6$, and let $\hat{V} = \mathrm{Hom}_{\boldsymbol{C}}(V, \boldsymbol{C})$ be the dual space of $V$. The points $H \in \wedge^2 \hat{V} = \mathrm{Hom}_{\boldsymbol{C}}(\wedge^2 V, \boldsymbol{C})$ can be regarded as skew-symmetric linear maps $H : V \to \hat{V}$, and the hyperplanes $(H = 0) \subset \boldsymbol{P}^{14} = \boldsymbol{P}(\wedge^2 V)$ can be regarded as points of $\hat{\boldsymbol{P}}^{14} = \boldsymbol{P}(\wedge^2 \hat{V})$.

Let $Pf = \{H \in \wedge^2 \hat{V}) : rank(H) \leq 4\}/\boldsymbol{C}^*$. Then $Pf$ is the Pfaff cubic hypersurface in $\hat{\boldsymbol{P}}^{14}$ defined, in coordinates, by vanishing of the cubic Pfaffian of the skew-symmetric $(6 \times 6)$-matrix $H$. Let $U_{10} \subset \wedge^2 V$ be a 10-dimensional subspace, and let

$$\hat{U}_5 = U_{10}^{\perp} = \{H \in \wedge^2 \hat{V} = \mathrm{Hom}_{\boldsymbol{C}}(\wedge^2 V, \boldsymbol{C}) : H|_{U_{10}} = 0\}.$$

Let moreover $U_{10} \subset \wedge^2 V$ be such that $rank(H) \geq 4$ for any $H \in U_{10}^{\perp}$, and let $X_{14} = G(2,6) \cap \boldsymbol{P}(U_{10})$ and $B_3 = Pf \cap \boldsymbol{P}(\hat{U}_5)$.

The construction "Da Palatini" of G. Fano shows that any hyperplane $\boldsymbol{P}^4 \subset \boldsymbol{P}(V)$ defines a birational isomorphism $\xi : X_{14} \to B_3$ which can be described as follows (see [Pu]):

Identify the point $b \in B_3$ and (the projective equivalence class of) the skew-symmetric $6 \times 6$ matrix corresponding to $b$. Since $rank\, b = 4$ for any $b \in B_3$ the projective kernel $n_b$ of $b$ will be a line in $\boldsymbol{P}^5$. Let $W := \bigcup_{b \in B_3} \{n_b = \boldsymbol{P}(\mathrm{Ker}\, b)\} \subset \boldsymbol{P}^5$.

The fourfold $W$ can be described by an alternative way. Identify the point $l \in G(1:5)$ and the line $l \subset \boldsymbol{P}^5$, and let $W' := \bigcup \{l \subset \boldsymbol{P}^5 \mid l \in X_{14}\} \subset \boldsymbol{P}^5$. Then (see [Pu, p. 83]):

(a). *for the general* $v \in W'$ *there exists a unique* $l \in X_{14}$ *such that* $v \in l$;
(b). *for the general* $w \in W$ *there exists a unique* $b \in B$ *such that* $w \in n_b$;
(c). $W' = W$.

Let $H \subset \boldsymbol{P}^5$ be a general hyperplane. Then, by (a), (b) and (c), the maps $\phi : X_{14} \to H \cap V$, $\phi(l) = H \cap l$ and $\psi : B_3 \to H \cap V$, $\psi(b) = H \cap n_b$ are birational isomorphisms. The composition $\xi = \xi_H = \psi^{-1} \circ \phi : X_{14} \to B_3$ is a birational isomorphism, depending on the choice of the hyperplane $H \subset \boldsymbol{P}^5$ (see [Pu, p. 85]).

**3.17. The dual cubic fourfold of $S_{14}$.** By Lemma 3.8 (ii) the curve $C_8$ is the Plücker image of the tangent scroll to a rational normal quintic $C_5 : (x_0 : \cdots : x_5) = \vec{s} = (1 : s : s^2 : \cdots : s^5)$ in $\boldsymbol{P}^5(x)$. The points of the curve $C_8$ are the Plücker coordinates $x_{ij}(s)$ of the point $\vec{s} \in C_5$:

$$(x_{ij}(s)) = \begin{pmatrix} 0 & 1 & 2s & 3s^2 & 4s^3 & 5s^4 \\ \cdots & 0 & s^2 & 2s^3 & 3s^4 & 4s^5 \\ \cdots & \cdots & 0 & s^4 & 2s^5 & 3s^6 \\ \cdots & \cdots & \cdots & 0 & s^6 & 2s^7 \\ \cdots & \cdots & \cdots & \cdots & 0 & s^8 \\ \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

Therefore $S_{14} = G(2,6) \cap \boldsymbol{P}^8$ where $\boldsymbol{P}^8 = (H_0 = \cdots = H_5 = 0) \subset \boldsymbol{P}^{14}$ and:

$$H_0 = x_{03} - 3x_{12}, \quad H_1 = x_{04} - 2x_{13}, \quad H_2 = 3x_{05} - 5x_{14},$$
$$H_3 = x_{14} - 3x_{23}, \quad H_4 = x_{15} - 2x_{24}, \quad H_5 = x_{25} - 3x_{34}.$$

Let $\Pi^5 := \langle H_0, \ldots, H_5 \rangle \subset \hat{\boldsymbol{P}}^{14}$ be the projective linear span of $\{H_0, \ldots, H_5\}$, and let $\boldsymbol{B}_3 = Pf \cap \langle H_0, \ldots, H_5 \rangle$. Introduce projective coordinates $(t_0 : \cdots : t_5)$ in $\Pi^5$ such that the point $(t_0, \ldots, t_5)$ represents the vector $t_0 H_0 + \cdots + t_5 H_5$. Then the cubic 4-fold $\boldsymbol{B}_3 = Pf \cap \boldsymbol{P}^5(t_0 : \cdots : t_5)$ is defined by $\boldsymbol{B}_3 : F = 32 t_0 t_2 t_5 - t_0 t_3 t_5 - 2t_1^2 t_5 - 2t_0 t_4^2 + 3t_1 t_3 t_4 - 12 t_1 t_2 t_4 - 45 t_2^2 t_3 - 9 t_2 t_3^2 = 0$.

The 6-vector $(0, \ldots, 0)$ is the only value of $(t_0, \ldots, t_5)$ where the 15 Pfaffians $Pf_{ij}(t_0, \ldots, t_5)$ of the matrix $H(t_0, \ldots, t_5) = t_0 H_0 + \cdots + t_5 H_5$ vanish. Therefore $rank\, b = 4$ for any $b \in \boldsymbol{B}_3$.

The fourfold $\boldsymbol{B}_3 = (F = 0) \subset \Pi^5$ is singular, and it is not hard to check that $Sing\, \boldsymbol{B}_3 = (\nabla_{(t_0 : \cdots : t_5)} F = (0, \ldots, 0)) \subset \Pi^5$ is the rational normal quartic curve $C_4 = \{[r] = (1 : 2r : r^2/3 : 8r^2/3 : 2r^3 : r^4) \mid r \in \boldsymbol{C}\}$; for simplicity we let $t_0 = 1$.

**3.18.** Now we are ready to prove Lemma (A) for $g = 8$.

Let $X_{14} = G(2,6) \cap \boldsymbol{P}^9 = \boldsymbol{P}(U_{10})$ be such that $X_{14} \supset S_{14}$, and let $B_3 = Pf \cap \boldsymbol{P}(U_{10}^{\perp})$. Then $S_{14} \subset X_{14} = G(2,V) \cap \boldsymbol{P}(U_{10}) \Leftrightarrow \boldsymbol{P}^8 = Span\, S_{14} \subset \boldsymbol{P}(U_{10}) = Span\, X_{14} \Leftrightarrow \boldsymbol{P}(U_{10}^{\perp}) \subset \Pi^5 \Leftrightarrow B_3 \subset \boldsymbol{B}_3$. Since $B_3 \subset \boldsymbol{B}_3$ then $rank(H) = 4$ for any $H \in B_3$ (see (3.17)), hence the Da Palatini birationalities $\xi : X_{14} \to B_3$ (see (3.16)) are well-defined.

Assume that $X_{14}$ is smooth. Then $B_3$ must be smooth (see above or [Pu, p. 83]). But $B_3$ must be singular at any of the intersection points of the hyperplane $\boldsymbol{P}(U_{10}^{\perp}) \subset \Pi^5$ and the rational quartic curve $C_4 = Sing\,\boldsymbol{B}_3$—contradiction (see the end of (3.17)). Therefore *any* $X_{14} \supset S_{14}$ must be singular. q.e.d.

**3.19.** *Remark.* If $\boldsymbol{P}(U_{10}^{\perp}) = Span(C_4)$ then $B_3$ is singular along $C_4$, and it can be seen that then $X_{14}$ is singular along $C_8$. Let $\boldsymbol{P}(U_{10}^{\perp}) \neq Span(C_4)$. Then the hyperplane $\boldsymbol{P}(U_{10}^{\perp}) \subset \Pi^5$ intersects the rational normal quartic $C_4 = Sing\,\boldsymbol{B}_3$ in 4 possibly coincident points $b_1, b_2, b_3, b_4$. Let, for simplicity $b_i$ be different from each other. Then one can show that the general Da Palatini birationality $B_3 \leftrightarrow X_{14}$ sends $\{b_1, b_2, b_3, b_4\}$ to $4 = 12 - g(X_{14})$ singular points of $X_{14}$ which lie on $C_8$. Let $H \subset \boldsymbol{P}^5$ be a hyperplane, and let $\xi_H^{-1} : B_3 \to X_{14}$ be the Da Palatini birationality defined by $H$. We shall see that for the general $H$ the rational map $\xi_H^{-1} = \psi_H$ is regular at a neighborhood of any $b_i$, and $\xi_H^{-1}(b_i) \in C_8$. For this, by the definition of the maps $\phi$ and $\psi$, it is necessary to see that the kernel-map $\ker : B_3 \to G(1:5)$, $b \mapsto n_b$ sends the quartic $C_4$ isomorphically to $C_8$.

Let $b = (b_{ij}) \in B_3$. Then the Plücker coordinates of the line $n_b = \ker(b)$ are $(-1)^{i+j} Pf_{ij}(b)$, where $Pf_{ij}(b)$ are the 15 quadratic Pfaffians of the skew-symmetric matrix $\hat{b}$; note that $rank(b) = 4$ for any $b \in B_3 \subset \boldsymbol{B}_3$. Now, it rests only to replace $b$ by the general point $b(t) = H_0 + tH_1 + t^2/12H_2 + 2t^2/3H_3 + t^3/4H_4 + t^4/16H_5 \in C_4$, and to see that the Plücker coordinates of $b(t)$ parameterize the general point $x_{ij}(\vec{s})$ of $C_8$ (where $s = 2/t$)—see (3.17).

**Proof of Lemma (A) for $g = 7$**

**3.20.** In the proof of Lemma (A) for $g = 7$ we shall need the known by [I2] description of the projection from a line $l$ on a smooth prime Fano threefold $X_{2g-2}$ such that $N_{l/X} = \mathcal{O}(1) \oplus \mathcal{O}(-2)$.

**3.21.**

Lemma (see §1 Proposition 3 in [I2]). *Let $l$ be a line on the smooth prime Fano threefold $X = X_{2g-2} \subset \boldsymbol{P}^{g+1}$ such that $N_{l/X} = \mathcal{O}_l(-2) \oplus \mathcal{O}_l(1)$, and let $\sigma : X' \to X$ be the blow up of $l$. Let $Z' = \sigma^{-1}(l)$, let $H' \sim \sigma^*H - Z'$ be the proper preimage of the hyperplane section $H$ of $X$, and let $\pi : X \to X'' \subset \boldsymbol{P}^{g-1}$ be the projection from $l$. Then*:

(i). *If $g \geq 5$ then the composition $\phi = \pi \circ \sigma : X' \to \boldsymbol{P}^{g-1}$ is a birational morphism (given by the linear system $|H'|$), to a threefold $X'' \subset \boldsymbol{P}^{g-1}$.*

(ii). *The restriction to $Z' = \boldsymbol{P}(N_{l/V}) = \boldsymbol{F}_3$ of the linear system $|H'|$ is the complete linear system $|s' + 3f'|$, where $s'$ and $f'$ are the classes of the exceptional section and the fiber of the rational ruled surface $Z'$.*

(iii). *The restriction $\phi|_{Z'}$ of $\phi$ to $Z'$ maps $Z'$ to a cone $Z''$ over a twisted cubic curve, contracting the exceptional section $s'$ of $Z'$ to the vertex of $Z''$.*

(iv). *If $g \geq 7$ then there are only a finite number of lines $l_i \subset X$ $(i = 1, \ldots, N)$ which intersect $l$. Let $l_i' \subset X'$ of $l_i$ $(i = 1, \ldots, N)$ be the proper preimages $l_i' \subset X'$ of $l_i$ $(i = 1, \ldots, N)$. Then the morphism $\phi : X' \to X''$ is an isomorphism outside $l_1' \cup \cdots \cup l_N' \cup s'$, and $\phi$ contracts $s'$ and any of $l_i'$ to isolated double points of $X''$.*

(v). *Let $H'' = \phi(H')$ be the hyperplane section of $X''$. Then if $g \geq 7$ then $-K_{X''} \sim H''$, i.e. the variety $X'' = X''_{2(g-2)-2} \subset \boldsymbol{P}^{g-1}$ is an anticanonically embedded Fano threefold with isolated singularities as in* (iv).

**3.22.** Suppose that there exists a smooth prime $X = X_{12} \subset \boldsymbol{P}^8$ which contains the tangent scroll $S_{12}$ to the rational normal curve $C_7$. Therefore the general such $X$ is smooth, and we may suppose that $X \supset S_{12}$ is general.

Let $l \subset S_{12}$ be any of the tangent lines to $C_7$ and let $\pi : X \to X''$ be the projection from $l$. By [I2, §1] (see also (1.2)) $N_{l/X} = \mathcal{O}(1) \oplus \mathcal{O}(-2)$, therefore Lemma 3.21 (i)–(v) take place.

By Lemma 3.21 (v) the threefold $X'' = X''_8 \subset \boldsymbol{P}^6$ is an anticanonically embedded Fano threefold of genus 5. Let $S'' \subset X''$ be the proper image of $S_{12}$, and let $C'' \subset S'' \subset X''$ be the proper image of $C_7$. It is easy to see that $C'' = C''_5$ is a rational normal quintic, and $S'' = S''_8 \subset X''$ is the tangent scroll to $C''$.

In order to use the proof of Lemma (A) for $g = 5$ we have to see whether $X''_8 \subset \boldsymbol{P}^6$ is, in fact, a complete intersection of three quadrics. If $X''$ were nonsingular then the classification of the smooth Fano threefolds will imply that $X''$ will be a complete intersection of three quadrics. But $X''$ is singular—see Lemma 3.21 (iv),(v).

However, especially in this case, $X'' = X''_8 \subset \boldsymbol{P}^6$ is *still* a complete intersection of three quadrics (see Theorem 6.1 (vii) in [I1]).

Denote by $Sing(X) \cap C$ the set of singular points of $X = X_{12}$ on $C$, and let $Sing(X'') \cap C''$ be the set of singular points of $X''$ on $C''$.

**3.23.** By the proof of Lemma (A) for $g = 5$, the elements of $Sing(X'') \cap C$ are in a $(1:1)$ correspondence with the different zeros of a homogeneous form $F_7(s_o : s_1)$ of degree 7 (see (3.6)). Clearly, the vertex $o$ of $Z''$ lies on $C''$. Moreover, by (3.21)(iv), $o$ is a double singularity of $X''$. Since $l \cap Sing\, X = \emptyset$, and since the tangent lines $l' \neq l$ to $C = C_7$ do not intersect $l$, (3.21)(iv) yield that, set-theoretically: $Sing(X) \cap C \cong Sing(X'') \cap C'' - \{o\}$.

Let $F_7(s_0 : s_1) = 0$ be as above. Since $\dim Sing\, X'' = 0$, the form $F_7$ does not vanish on $C_7$; and we can assume that $o = (1:0)$ and $F_7(0:1) \neq 0$. Therefore if $s = s_1/s_0$ and $f_7(s) = F(1:s)$ then $\deg f_7(s) = 7$. By the previous the elements of $Sing(X) \cap C$ correspond to the different zeros of the polynomial $s^{-m}.f_7(s) = 0$, where $m = \mathrm{mult}_o f_7(s)$.

By the local definition of $m$, the integer $m = m(o) = \mathrm{mult}_o f_7(s)$ does not depend on the genus $g \geq 7$ of $X_{2g-2}$ as well on the choice of the general tangent line $l$ to $C_g$. It can be seen that $m = 2$, but for the proof it is enough to know that $m \leq 2$.

(∗) Lemma. $m \leq 2$.

*Proof of* (∗). By construction $X'' \supset S'' \cup Z''$ where $S'' = S''_8$ is the tangent scroll to the rational normal quintic $C'' = C'_5$ such that $o \in C'', Z''$ is a cone over a twisted cubic, and $o$ is the vertex of $Z$. Moreover $Z''$ is triple tangent to $S''$ at the tangent line $F$ to $C''$ at $o$. Indeed $S''$ is a hyperplane section of $Z''$ which passes through the vertex $o$ of $Z$. Therefore $S''.Z = f_1 + f_2 + f_3$ is a sum of 3 rulings of $Z''$. Since $f_i$ are rulings of $Z$, $o \in f_i$ for $i = 1, 2, 3$. Therefore any $f_i$ is a line on the tangent scroll $S''$ to $C''$ which passes through $o \in C''$. Therefore $f_i = F$ must be a tangent line to $C''$ at $o$, i.e. $S''.Z'' = 3F$.

By Theorem 9.9 in [I3], the general complete intersection $X_8 \subset \boldsymbol{P}^6$ of three quadrics, containing a cone $Z_3$ over a twisted cubic, is a projection of $X_{12}$ from a line $l$ such that $N_{l/X_{12}} = \mathcal{O}(1) \oplus \mathcal{O}(-2)$. The inverse of the projection $\pi_l$ is defined by the linear system $|H + Z_3|$, where $H$ is the hyperplane section of $X_8$.

Let $X_8 \supset Z_3 \cup S_8$ be as above. Then $X_{12}$ will contain a tangent scroll $S_{12}$ to a rational normal curve $C_7$, and $l$ will be a tangent line to $C_7$. Therefore any $X_8 \supset Z_3 \cup S_8$ will be a deformation of a projection of $X_{12} \subset S_{12}$ from a tangent line to $C_7$.

It rests to see that $m(X_8) = \text{mult}_o f_7 \leq 2$ for $f_7$ corresponding, as above, to some particular such $X_8$.

*Example.* Let $\boldsymbol{P}^5(x) = \boldsymbol{P}^5(x_0 : \cdots : x_5)$, and let $q_0 = -x_0x_4 + 4x_1x_3 - 3x_2^2$, $q_1 = -x_0x_5 + 3x_1x_4 - 2x_2x_3$, $q_2 = -x_1x_5 + 4x_2x_4 - 3x_3^2$. Then $S_8 = (q_0 = q_1 = q_2 = 0) \subset \boldsymbol{P}^5(x)$ will be the tangent scroll to the rational normal quintic $C_5 : x_i = s_0^{5-i}s_1^i$ $(0 \leq i \leq 5)$.

Let $X_8 = (Q_0 = Q_1 = Q_2 = 0) \subset \boldsymbol{P}^6(x : u) = \boldsymbol{P}^6(x_0 : x_1 : x_2 : x_3 : x_4 : x_5 :\ u)$, where

$$Q_0 = q_0 + L_o(x_4 : x_5)u,$$

$$Q_1 = q_1 + (12x_1 + L_1(x_4 : x + 5))u,$$

$$Q_2 = q_2 + (27/2x_2 + L_2(x_4 : x_5))u,$$

$L_0, L_1$ and $L_2$ being linear forms of $(x_4 : x_5)$. Evidently $X_8 \cap (u = 0) = S_8$.

Let $\boldsymbol{P}^4 = \boldsymbol{P}^4(x_0 : x_1 : x_2 : x_3 : u) \subset \boldsymbol{P}^6$, and let $Z_3 = X_8 \cap \boldsymbol{P}^4$. Then $Z_3 = (P_0 = P_1 = P_2 = 0) \subset \boldsymbol{P}^4$, where $P_0 = x_1x_3/3 - x_2^2/4$, $P_1 = x_1u - x_2x_3/6$, $P_2 = x_2u - x_3^2/9$.

Therefore $Z_3$ is a cone with center $o = (1 : 0 : \cdots : 0) \in C_5$ over the twisted cubic curve $C_3 = Z_3 \cap (x_0 = 0)$, $C_3 : (x_1 : x_2 : x_3 : u) = (t_0^3 : 2t_0^2t_1 : 3t_0t_1^2 : t_1^3)$. Let $s = s_1/s_0$, and we may suppose that the point $(0 : \cdots : 0 : 1) \in C_5$ is not a singular point of $X_8$. Then, by (3.6), the equation of $(\mathit{Sing}\, X_8)|_{C_5}$ is

$$\begin{aligned} f_7(s) &= s^2\partial Q_0/\partial u(1 : s : \cdots : s^5) - s\partial Q_1/\partial u(1 : s : \cdots : s^5) + \partial Q_2/\partial u(1 : s : \cdots : s^5) \\ &= s^2L_0(s^4, s^5) - s(12s + L_1(s^4, s^5)) + (27/2s^2 + L_2(s^4, s^4)) = 3/2s^2 + o(s^3), \end{aligned}$$

where $o(s^3)$ is a sum of terms of degree $\geq 3$. Therefore $m(X_8) = \text{mult}_o f_7(s) = 2$.

q.e.d.

**3.24.** Let $X = X_{12} \supset S_{12}$ be general. Since $m \leq 2$ then $\deg s^{-m}f_7(s) \geq 7 - 2 = 5 = 12 - g(X_{12}) > 0$. In particular $g(s) := s^{-m}f_7(s)$ is not a constant. Since $g(0) \neq 0$, and since the elements of $\mathit{Sing}(X) \cap C$ are in a $(1 : 1)$ correspondence with the different zeros of $g(s) = s^{-m}f_7(s)$ (see above), then $X_{12}$ must be singular, which contradicts the initial assumption. This proves Lemma (A) for $g = 7$.

**Proof of Lemma (A) for $g = 9$.**

**3.25.** Let $X_{16} \subset \boldsymbol{P}^{10}$ contains the tangent scroll $S = S_{16}$ to the rational normal curve $C = C_9$, and suppose that nevertheless $X_{16}$ is smooth.

Let $L \subset X_{16}$ be a tangent line to $C$, and consider the double projection $\pi = \pi_{2L}$ of $X$ from the line $L$, i.e. $\pi$ is the rational map on $X$ defined by the non-complete linear system $|\mathcal{O}_X(1-2L)|$. Since $X = X_{16}$ is assumed to be smooth then, by §2 in [I2]:

($*$). *$\pi = \pi_{2L}$ sends $X$ birationally to $\boldsymbol{P}^3$. Moreover, on $\boldsymbol{P}^3$ there exists a smooth irreducible curve $C = C_7^3$ of genus* 3 *and degree* 7, *which lies on a unique cubic surface $S_3 \subset \boldsymbol{P}^3$, and such that the inverse to $\pi$ birational map $\phi : \boldsymbol{P}^3 \to X$ is given by the non-complete linear system $|\mathcal{O}_{\boldsymbol{P}^3}(7-2C)|$.*

By ($*$), the proper image $\pi(H)$ of any hyperplane section $H \subset X$ is an irreducible component of an effective divisor $S_7 \in |\mathcal{O}_{\boldsymbol{P}^3}(7-2C)|$. If moreover $H$ contains the line $L$ but $H \notin |\mathcal{O}_X(1-2L)|$ (for example if $H = S_{16}$) then $\pi(H) \subset \boldsymbol{P}^3$ will be a quartic surface containing the curve $C = C_7^3$ (see the proof of the Main Theorem in §2 of [I2]), and in this case $S_7 = \pi(H) + S_3 \in |\mathcal{O}_{\boldsymbol{P}^3}(7-2C)|$.

Therefore $S_4 := \pi(S_{16})$ is a quartic surface in $\boldsymbol{P}^3$ containing the curve $C = C_7^3$. Moreover, the double projection $\pi$ sends the general tangent line $L'$ to $C_9$ to a tangent line $\pi(L')$ to the proper image $\pi(C_9)$; and since $C_9 \cong \boldsymbol{P}^1$ then $\pi(C_9)$ is rational. Therefore the quartic surface $S_4 = \pi(S_{16})$ is the tangent scroll to the rational curve $\pi(C_9) \subset \boldsymbol{P}^3$. The last is only possible if $\pi(C_9) = C_3$ is a twisted cubic and $S_4$ is the tangent scroll to $C_3$, and we shall see that this is impossible.

The surface $S_4 \subset \boldsymbol{P}^3$ is the tangent scroll to the twisted cubic $C_3$. Then, by Lemma 1.6 and p. 498 in [MU], the normalization of $S_4$ is the quadric $\boldsymbol{P}^1 \times \boldsymbol{P}^1$, and the map $\nu : \boldsymbol{P}^1 \times \boldsymbol{P}^1 \to S_4$ is given by a linear system of bidegree $(1,2)$.

Let $\Gamma \subset \boldsymbol{P}^1 \times \boldsymbol{P}^1$ be the proper transform of $C_7^3$, and let $(a,b)$ be the bidegree of $\Gamma$. Therefore $7 = \deg(C_7^3) = 2a + b$, and $3 = g(C_7^3) = g(\Gamma) = (a-1)(b-1)$. Obviously, these two equations for the integers $a$ and $b$ have no integral solutions—contradiction.

Therefore $X_{16} \supset S_{16}$ can't be smooth, which proves Lemma (A) in case $g = 9$.

This completes the proof of Lemma (A).

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