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ON RECURRENCE FOR SELF-SIMILAR ADDITIVE PROCESSES

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1. Introduction

In our paper a stochastic process is called an additive process if it is a stochastically continuous process with independent increments and has right-continuous sample functions with left limits a.s. Self-similar additive processes constitute an important class of additive processes which are not assumed to be time-homogeneous. But they have not been studied except in some papers, e.g. [2], [3], [4], and [5]. We investigated their transience and recurrence in [3] and [5]. The dichotomy of recurrence and transience for this class of processes is known (see [3]). But a criterion for recurrence and transience has not been found. As an important example, there is a strictly stable process on $\boldsymbol{R}^d$, which is a self-similar Lévy process, that is, a self-similar time-homogeneous additive process. It is recurrent if its index α satisfies $\max\{1, d\} \le \alpha \le 2$, where its exponent is α^{-1} if we regard it as a self-similar additive process (see the definition below). So we attempted to find a new method to prove recurrence for strictly stable processes with index $\max\{1, d\} \le \alpha \le 2$ without using time-homogeneity. We succeeded in our attempt and we could find recurrence conditions showing great differences between self-similar additive processes and Lévy processes. We note that this problem cannot be solved by using the existing methods because of the difficulty caused by the fact that the expected occupation times on open sets containing 0 cannot determine recurrence (see [3]).

A self-similar additive process is defined by the following.

DEFINITION. A stochastic process $\{X_t : t \ge 0\}$ on $\boldsymbol{R}^d$, which is defined on a probability space $(\Omega, \mathscr{F}, P)$, is called a self-similar additive process, or a process of class L, with exponent $H > 0$ if it satisfies the following conditions:

(i) $\{X_{ct}\}$ and $\{c^H X_t\}$ have the same finite-dimensional distributions for every $c > 0$,

(ii) $X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \dots, X_{t_n} - X_{t_{n-1}}$ are independent for any n and any choice of $0 \le t_0 < t_1 < t_2 < \cdots < t_n$,

(iii) almost surely X_t is right-continuous in $t \ge 0$ and has left limits in $t > 0$.

Throughout this paper let $\{X_t\}$ be a self-similar additive process on $\boldsymbol{R}^d$ with

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exponent H. Note that it is stochastically continuous and $X_0 = 0$ a.s. Let $D = \{x \in \boldsymbol{R}^d : |x| \le 1\}$, $S = \{x \in \boldsymbol{R}^d : |x| = 1\}$, and $1_D(x)$ be the indicator function of D. Then $\{X_t\}$ has the following characteristic function:

$$\begin{aligned} Ee^{i\langle z, X_t\rangle} &= Ee^{i\langle t^H z, X_1\rangle} \\ &= \exp\Bigg[-t^{2H}2^{-1}\langle Az, z\rangle + it^H\langle \gamma, z\rangle \\ &\quad + \int_S \sigma(d\xi)\int_0^\infty (e^{it^H\langle z, r\xi\rangle} - 1 - it^H\langle z, r\xi\rangle 1_D(r\xi))\frac{k_\xi(r)}{r}\,dr\Bigg] \end{aligned}$$

for $z \in \boldsymbol{R}^d$, where A is a symmetric and nonnegative matrix, $\gamma \in \boldsymbol{R}^d$, σ is a probability measure on $S, k_\xi(r)$ is nonnegative, nonincreasing right continuous in r and Borel measurable in ξ, and

$$\int_S \sigma(d\xi)\int_0^\infty \min\{r, r^{-1}\}k_\xi(r)\,dr < \infty.$$

For any random variable Z we denote by P_Z the distribution of Z, and by $\hat{P}_Z(z)$ the characteristic function of P_Z. We have the following main results.

THEOREM 1.1. *Let $d = 1$. The process $\{X_t\}$ is recurrent if it satisfies one of the following:*

(i) $A \neq 0$,

(ii) *$k(r) - \varphi(1/r)$ is nonnegative and nonincreasing on $(0, \varepsilon)$ for some $\varepsilon > 0$, where $k(r) = k_{+1}(r) + k_{-1}(r)$ and $\varphi(r)$ is a nonnegative strictly increasing convex function on $(1/\varepsilon, \infty)$.*

THEOREM 1.2. *Let $d = 2$. If the rank of the matrix A is 2, then the process $\{X_t\}$ is recurrent.*

Remark. When $d = 2$, there is a transient self-similar additive process such that the matrix A has rank 1. For example, $\{X_t\}$ is transient if X_1 is full and has the following characteristic function: Suppose that

$$\int_S \sigma(d\xi)k_\xi(0+) < \infty.$$

$$\hat{P}_{X_1}(z) = \exp\left[-2^{-1}\langle Az, z\rangle + i\langle \gamma_0, z\rangle + \int_S \sigma(d\xi)\int_0^\infty (e^{i\langle z, r\xi\rangle} - 1)\frac{k_\xi(r)}{r}\,dr\right],$$

where $\gamma_0 = \begin{pmatrix} \gamma_{01} \\ \gamma_{02} \end{pmatrix}$, $c_1, c_2 \in \boldsymbol{R}$, and $A = \begin{pmatrix} c_1\gamma_{01} & c_2\gamma_{01} \\ c_1\gamma_{02} & c_2\gamma_{02} \end{pmatrix}$. In fact $\{\Pi R X_t\}$ is transient according to Theorem 1.1 in [5], where the 1×2-matrix Π is equal to $(1\ 0)$ and R is a rotation matrix such that the first coordinate of $R\gamma_0$ is 0.

Our theorems show differences between a self-similar additive process and a Lévy process, that is, when the two processes have the same distribution at time

1, there is a case where one of them is recurrent and the other is transient. For example, let $\{B_t\}$ be a d-dimensional Brownian motion and let $\gamma \neq 0$. If $d \leq 2$, then $\{B_t + \sqrt{t}\gamma\}$, which is a self-similar additive process with exponent 2^{-1}, is recurrent by the above theorems. On the other hand we know that the Lévy process $\{B_t + t\gamma\}$ is transient (see [1]).

2. Proofs

Let

$$f(x) = \prod_{j=1}^{d} \max\{1 - |x_j|, 0\}.$$

Then

$$\int_{\boldsymbol{R}^d} f(x)\, dx = 1,$$

and the Fourier transform of f is

$$\hat{f}(z) = \int_{\boldsymbol{R}^d} e^{i\langle z, x\rangle} f(x)\, dx = \prod_{j=1}^{d} \left(\frac{\sin 2^{-1} z_j}{2^{-1} z_j}\right)^2.$$

We note that

$$f(x) = \frac{1}{(2\pi)^d} \int_{\boldsymbol{R}^d} e^{-i\langle z, x\rangle} \hat{f}(z)\, dz.$$

Define

$$a_n = \sum_{k=1}^{n} k^{-Hd}, \quad W_n = a_n^{-1} \sum_{k=1}^{n} f(X_k).$$

We will show two facts:

(2.1) $$\lim_{n\to\infty} EW_n > 0,$$

(2.2) $$\sup_n E[W_n^2] < \infty,$$

for some H with $0 < Hd \leq 1$. Then the recurrence of $\{X_t\}$ is shown in the following way. As (2.2) implies the uniform integrability of $\{W_n\}$, we have

$$E\left[\limsup_{n\to\infty} W_n\right] \geq \lim_{n\to\infty} EW_n > 0.$$

Hence we have

$$P\left(\lim_{n\to\infty} \sum_{k=1}^{n} f(X_k) = \infty\right) > 0$$

and $\{X_t\}$ is not transient. Therefore it is recurrent by Theorem 3.2 in [3].

The proof of (2.1) and (2.2) will be complete in Lemma 2.3.

LEMMA 2.1. *Let* $0 < \beta \le 1$.
(i) *If* $\gamma > \beta > 0$, *then we have* $\lim_{n\to\infty} \sum_{j=1}^{n} j^{-\gamma} (\sum_{j=1}^{n} j^{-\beta})^{-1} = 0$.
(ii) *We have* $\sup_{n\ge 2} \sum_{1\le j_1<j_2\le n} j_1^{-\beta}(j_2 - j_1)^{-\beta}(\sum_{j=1}^{n} j^{-\beta})^{-2} < \infty$.

Proof is omitted.

LEMMA 2.2. *Let* $Hd \le 1$. *If*

$$\int_{\boldsymbol{R}^d} |z|\,|\hat{P}_{X_1}(z)|\,dz < \infty,$$

then we have

$$\lim_{n\to\infty} EW_n = \frac{1}{(2\pi)^d}\int_{\boldsymbol{R}^d} \hat{P}_{X_1}(z)\,dz > 0.$$

Proof. Let $c_0 = (2\pi)^{-d}\int_{\boldsymbol{R}^d} \hat{P}_{X_1}(z)\,dz$. We have $c_0 > 0$. Indeed, the support of P_{X_1} is $\boldsymbol{R}^1$, because $\int_0^1 k(r)\,dr \ge \int_0^1 \varphi(1/r)\,dr = \infty$ and it is shown from the general theory of infinitely divisible distributions (see [1]) under the assumption of Theorem 1.1(ii), and because P_{X_1} is the convolution of a Gaussian distribution and some distribution under the other conditions. Hence, since P_{X_1} is unimodal by Yamazato's theorem [6], its density is positive on $\boldsymbol{R}^1$.

We have

$$|EW_n - c_0| \le \frac{1}{\sum_{k=1}^{n} k^{-Hd}} \sum_{k=1}^{n} k^{-Hd}|k^{Hd}Ef(X_k) - c_0| = J_n, \quad \text{(say)}.$$

Since

$$\begin{aligned} k^{Hd}Ef(X_k) &= k^{Hd}\int_{\boldsymbol{R}^d} f(x)\,dx \frac{1}{(2\pi)^d}\int_{\boldsymbol{R}^d} e^{-\imath\langle z,x\rangle}\hat{P}_{X_1}(k^H z)\,dz \\ &= \frac{1}{(2\pi)^d}\int_{\boldsymbol{R}^d} f(x)\,dx \int_{\boldsymbol{R}^d} e^{-\imath\langle z,k^{-H}x\rangle}\hat{P}_{X_1}(z)\,dz, \end{aligned}$$

we obtain that

$$\begin{aligned} J_n &\le \frac{1}{\sum_{k=1}^{n} k^{-Hd}} \sum_{k=1}^{n} \frac{1}{k^{Hd}(2\pi)^d}\int_{\boldsymbol{R}^d} f(x)\,dx \int_{\boldsymbol{R}^d} |e^{-\imath\langle z,k^{-H}x\rangle} - 1|\,|\hat{P}_{X_1}(z)|\,dz \\ &\le \frac{1}{\sum_{k=1}^{n} k^{-Hd}} \sum_{k=1}^{n} \frac{1}{k^{Hd}(2\pi)^d}\int_{\boldsymbol{R}^d} f(x)\,dx \int_{\boldsymbol{R}^d} |z|k^{-H}|x|\,|\hat{P}_{X_1}(z)|\,dz \\ &= \text{const.} \times \sum_{k=1}^{n} k^{-H(d+1)} \left(\sum_{k=1}^{n} k^{-Hd}\right)^{-1} \end{aligned}$$

Hence, from Lemma 2.1(i), we have $\lim_{n\to\infty} J_n = 0$. □

LEMMA 2.3. *If $H = 2^{-1}$ under the assumption of Theorem* 1.1(i) *or Theorem* 1.2, *or if $H = 1$ under the assumption of Theorem* 1.1(ii), *then we have* (2.1) *and* (2.2).

Proof. Under the assumption of the lemma we have $\int_{\boldsymbol{R}^d} |z|\,|\hat{P}_{X_1}(z)|\,dz < \infty$, which is proved even under the assumption of Theorem 1.1(ii) in the same way as in the proof of the inequality (2.3) shown later. Hence we obtain (2.1) from Lemma 2.2.

Let $j_2 > j_1$. Using the Fourier inverse transformation of f, we have

$$
\begin{aligned}
Ef(X_{j_1})f(X_{j_2}) &= E[f(X_{j_1})E[f(X_{j_2} - X_{j_1} + x)]_{x=X_{j_1}}] \\
&= \frac{1}{(2\pi)^{2d}} \int_{\boldsymbol{R}^d} \hat{f}(z)\hat{P}_{X_{j_2}-X_{j_1}}(z)\,dz \int_{\boldsymbol{R}^d} \hat{f}(y)\hat{P}_{X_{j_1}}(y+z)\,dy \\
&\le \frac{1}{j_1^{Hd}(2\pi)^{2d}} \int_{\boldsymbol{R}^d} |\hat{P}_{X_{j_2}-X_{j_1}}(z)|\,dz \int_{\boldsymbol{R}^d} |\hat{P}_{X_1}(y)|\,dy.
\end{aligned}
$$

Here we have used self-similarity.

Therefore, in order to show (2.2), it suffices to prove boundedness of

$$
I_n = \sum_{j_1=1}^{n} Ef^2(X_{j_1}) \left(\sum_{j_1=1}^{n} j_1^{-Hd} \right)^{-2}
$$

and

$$
J_n = \sum_{1 \le j_1 < j_2 \le n} j_1^{-Hd} \int_{\boldsymbol{R}^d} |\hat{P}_{X_{j_2}-X_{j_1}}(z)|\,dz \left(\sum_{j_1=1}^{n} j_1^{-Hd} \right)^{-2}.
$$

Since $0 \le f \le 1$, we have $\limsup_{n\to\infty} I_n = 0$ in view of Lemma 2.2. Hence $\sup_n I_n < \infty$.

Now we shall show boundedness of J_n. First we shall consider the case that the rank of matrix A is d. Let $H = 2^{-1}$. We have

$$
\begin{aligned}
J_n &\le \sum_{1 \le j_1 < j_2 \le n} j_1^{-2^{-1}d} \int_{\boldsymbol{R}^d} e^{-2^{-1}(j_2-j_1)\langle Az, z\rangle}\,dz \left(\sum_{j_1=1}^{n} j_1^{-2^{-1}d} \right)^{-2} \\
&= \sum_{1 \le j_1 < j_2 \le n} \frac{1}{j_1^{2^{-1}d}(j_2 - j_1)^{2^{-1}d}} \int_{\boldsymbol{R}^d} e^{-2^{-1}\langle Ay, y\rangle}\,dy \left(\sum_{j_1=1}^{n} j_1^{-2^{-1}d} \right)^{-2}.
\end{aligned}
$$

Hence, from Lemma 2.1(ii), we have $\sup_n J_n < \infty$.

Next we shall consider the case of Theorem 1.1(ii). It suffices only to consider the case that $A = 0$. Let $H = 1$. Now we have

$$
\sum_{1 \le j_1 < j_2 \le n} \frac{1}{j_1} \int_{\boldsymbol{R}^1} |\hat{P}_{X_{j_2}-X_{j_1}}(z)|\,dz \le \sum_{1 \le j_1 < j_2 \le n} \frac{1}{j_1^2} \int_{\boldsymbol{R}^1} |\hat{P}_{X_{j_2/j_1}-X_1}(z)|\,dz.
$$

We divide the last integral into two parts according as $|z| \leq 1$ or $|z| > 1$. Suppose that $|z| \leq 1$. We have

$$|\hat{P}_{X_1}(z)|^{-1} = \exp\left[\int_0^\infty (1 - \cos(zr))\frac{k(r)}{r}\,dr\right] \leq C_1.$$

Here the constant C_1 does not depend on z. Since $\varphi(r) \geq c_1 r - c_2$ on $(1/\varepsilon_1, \infty)$ for some constants $c_1 > 0$, $c_2 \in \boldsymbol{R}^1$, and $0 < \varepsilon_1 < \varepsilon$, then we have $k(r) > c_3/r$ on $(0, \varepsilon_2)$ for some constants $c_3 > 0$ and $0 < \varepsilon_2 < \varepsilon_1$. Hence we obtain that

$$\begin{aligned} |\hat{P}_{X_{j_2/j_1}}(z)| &= \exp\left[\int_0^\infty (\cos(zr) - 1)\frac{k((j_1/j_2)r)}{r}\,dr\right] \\ &\leq \exp\left[\int_0^{\varepsilon_2(j_2/j_1)} (\cos(zr) - 1)\frac{c_3}{r^2}\frac{j_2}{j_1}\,dr\right] \qquad (2.3) \\ &= \exp\left[c_3\frac{j_2|z|}{j_1}\int_0^{(j_2/j_1)\varepsilon_2|z|} (\cos u - 1)\frac{du}{u^2}\right]. \end{aligned}$$

Since $|\hat{P}_{X_{j_2/j_1} - X_1}(z)| = |\hat{P}_{X_{j_2/j_1}}(z)|/|\hat{P}_{X_1}(z)|$ and $|z|\int_0^{\varepsilon_2|z|}(1 - \cos u)u^{-2}\,du \sim K|z|$ as $|z| \to \infty$ with some positive constant K, we obtain that

$$\begin{aligned} &\sum_{1\leq j_1<j_2\leq n} \frac{1}{j_1^2}\int_{|z|\leq 1} |\hat{P}_{X_{j_2/j_1} - X_1}(z)|\,dz \\ &\leq \sum_{1\leq j_1<j_2\leq n} \frac{C_1}{j_1 j_2}\int_{|z|\leq j_2/j_1} \exp\left[c_3|z|\int_0^{\varepsilon_2|z|} (\cos u - 1)\frac{du}{u^2}\right] dz \\ &\leq \text{const.} \times \sum_{1\leq j_1<j_2\leq n} \frac{1}{j_1 j_2}. \end{aligned}$$

Next Suppose that $|z| > 1$. Then, since $\varphi(j_2/(j_1 r)) - \varphi(1/r) \geq c_4(j_2/j_1 - 1)r^{-1}$ on $(0, \varepsilon_3)$ for some constants $c_4 > 0$ and $0 < \varepsilon_3 < \varepsilon$, letting $\delta = \min\{\varepsilon_3, 1\}$, we have

$$\begin{aligned} |\hat{P}_{X_{j_2/j_1} - X_1}(z)| &= \exp\left[\int_0^\infty (\cos(zr) - 1)\frac{k((j_1/j_2)r) - k(r)}{r}\,dr\right] \\ &\leq \exp\left[c_4\int_0^{\delta/|z|} (\cos(zr) - 1)\left(\frac{j_2}{j_1} - 1\right)\frac{1}{r^2}\,dr\right] \\ &\leq \exp\left[-K_1|z|^2\frac{j_2 - j_1}{j_1}\int_0^{\delta/|z|} dr\right] \\ &= \exp\left[-K_2|z|\frac{j_2 - j_1}{j_1}\right] \end{aligned}$$

with some positive constants K_1 and K_2. Hence we have

$$\sum_{1 \le j_1 < j_2 \le n} \frac{1}{j_1^2} \int_{|z| \ge 1} |\hat{P}_{X_{j_2/j_1} - X_1}(z)|\, dz$$
$$\le \sum_{1 \le j_1 < j_2 \le n} \frac{1}{j_1^2} \int_{|z| \ge 1} \exp\left[-K_2 |z| \frac{j_2 - j_1}{j_1}\right] dz$$
$$\le \sum_{1 \le j_1 < j_2 \le n} \frac{1}{j_1(j_2 - j_1)} \int_{\boldsymbol{R}^1} e^{-K_2|z|}\, dz.$$

Hence, from Lemma 2.1(ii), we have $\sup_n J_n < \infty$. This completes the proof of Lemma 2.3. □

Example 2.4. Let $\{X_t\}$ be strictly stable with index α satisfying $1 \le \alpha < 2$. Letting $\varphi(r) = cr^\alpha$ for some $c > 0$, we obtain that $\{X_t\}$ is recurrent from Theorem 1.1(ii).

The following example is pointed out by K. Sato.

Example 2.5. Let $\varphi(r) = cr^\alpha (\log r)^\beta$, where three constants c, α, and β satisfy one of the following conditions:

(i) $c > 0, 1 < \alpha < 2$, and $\beta \in \boldsymbol{R}^1$,

(ii) $c > 0$, $\alpha = 1$, and $\beta \ge 0$,

(iii) $c > 0$, $\alpha = 2$, and $\beta < -1$.

Then, if $k(r) = \varphi(1/r)$ on $(0, \varepsilon)$ for small enough ε, then $k(r)$ satisfies the condition of Theorem 1.1(ii).

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