COMPLEX CONTACT MANIFOLDS AND HYPERKÄHLER GEOMETRY*

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Abstract

We investigate the properties of a certain class of hermitian metrics on complex contact manifolds with global complex contact forms. These are called complex Sasakian metrics. In the main theorem, we prove a relationship between complex Sasakian geometry and hyperkähler geometry. We then compare these complex contact manifolds to the more famous twistor space examples.

1. Introduction

In times past, there have been many interesting and useful correspondences proven between real contact and almost complex geometries. For example, Hatakeyama proved in [12] that every Sasakian manifold locally fibres over a Kähler manifold. Also, the Boothby-Wang Fibration gives a construction by which an S^1-bundle with a real contact structure is formed over a symplectic manifold [5]. Furthermore, Hatakeyama's result can be generalized to make the Boothby-Wang fibration a Riemannian submersion of associated metrics.

A great deal work has also been done to find similar correspondences dealing with complex contact and quaternionic structures on manifolds. However, a large obstacle has been finding a proper quaternionic context to work with. An almost quaternionic structure is defined easily enough; it is simply a metric g together with a three-dimensional bundle E of endomorphisms with local bases $\{A, B, C = AB\}$ of local anti-commuting almost complex structures such that g is hermitian with respect to each endomorphism.

What is the analogue for g being Kähler? If we just insist that each almost complex structure of E be parallel, then the quaternionic projective space HP^n would be excluded (see [3]). To remedy this, we define an almost quaternionic manifold (M, g, E) to be *quaternionic-Kähler*, if $\nabla E \subset T^*M \otimes E$. Under these circumstances, g is Einstein, and so we can categorize these manifolds into three categories by the sign of the scalar curvature: zero, positive, or negative.

For any of these cases, we can create the *twistor space* Z of M by locally setting

$$Z = \{xA + yB + zC : x^2 + y^2 + z^2 = 1\} \subset E.$$

Salamon and Bérard-Bergery ([22], [2]) both showed that, if the scalar curvature of g is nonzero, then Z carries both a natural complex structure and complex contact structure derived from a connection for the fibration $E \rightarrow M$. Thus, we get a correspondence between certain classes of quaternionic and complex contact manifolds.

When the scalar curvature is zero, however, this construction will not produce a complex contact manifold. In fact, E will be trivial, and we will have a global set of parallel complex structures $\{A, B, C\}$. Such an almost quaternionic manifold is called *hyperkähler*.

In this paper, we show that this class of quaternionic manifolds does correspond with a certain class of complex contact manifolds, called *strict normal complex contact manifolds*. These complex contact manifolds are very different from twistor spaces, especially since they reside in the so-called "strict complex contact" category of manifolds. This category of complex contact manifolds has for the most part been ignored, even though a certain element of it, the complex Heisenberg group, is very well known.

In the first section, we introduce and describe some properties of these types of manifolds. Then we investigate their hermitian geometry. In the fourth section, we introduce the notion of a complex Sasakian metric and prove the main theorem which relates complex Sasakian metrics with hyperkähler geometry. In the fifth section, we briefly describe some examples of complex Sasakian manifolds. In the last section, we compare some aspects of complex Sasakian manifolds with the much more famous examples of complex contact manifolds, the twistor spaces of quaternionic Kähler manifolds with non-zero scalar curvature; and we prove that, like complex Sasakian manifolds, we can think of these twistor spaces as complex contact manifolds satisfying certain curvature equations.

2. Strict complex contact manifolds

Recall that a *complex contact manifold* is a complex manifold (M, J) of complex dimension $2n + 1$ such that there is an open atlas $\mathscr{U} = \{\mathcal{O}\}$ of M for which the following statements hold:

1. On each $\mathcal{O} \in \mathscr{U}$, there is a holomorphic 1-form η such $\eta \wedge (d\eta)^n \neq 0$ everywhere on $\mathcal{O}$.

2. For $\mathcal{O}, \mathcal{O}' \in \mathscr{U}$ with respective 1-forms η, η', there is holomorphic function $f : \mathcal{O} \cap \mathcal{O}' \rightarrow C^*$ such that $\eta' = f\eta$.

If we define $\mathscr{H}^{1,0} = (\bigcup_{\mathcal{O} \in \mathscr{U}} \ker(\eta)) \cap T^{1,0}M$, then $\mathscr{H}^{1,0}$ is a well-defined, holomorphic subbundle of $T^{1,0}M$ of maximal rank and complex dimension $2n$, called

the holomorphic contact subbundle of M.   Set $\mathcal{H} = \{X \in T^R M : X - iJX \in \mathcal{H}^{1,0}\}$.
Then $\mathcal{H}$ is a subbundle of TM with maximal rank, which is holomorphic in the
sense that there are local bases of $\mathcal{H}$ $\{X_1, \ldots, X_{4n}\}$ such that $Ł_{X_j} J = 0$ for
$j = 1, \ldots, 4n$.   We will call $\mathcal{H}$ the *holomorphic contact subbundle* or simply the
contact subbundle.

Let (M, J) be a complex contact manifold with complex contact subbundle
$\mathcal{H}$.   We say that a set of C-valued 1-forms $\{\pi\}$ whose domains cover M is a
normalized contact structure, if any two 1-forms π, π' in this set with respective
domains $\mathcal{O}, \mathcal{O}'$ satisfy:

1. $\mathcal{H} = \ker(\pi)$ on $\mathcal{O}$.

2. On $\mathcal{O} \cap \mathcal{O}'$, $\pi' = f\pi$ for some $f : \mathcal{O} \cap \mathcal{O}' \to S^1$.

Note that for any π in this set, $\pi = u - iv$ where u and v are real 1-forms such
that $v = u \circ J$.   By taking a hermitian metric on the complex line bundle $L =
\{\omega \in \Lambda^{1,0} M : \omega(\mathcal{H}) = 0\}$, it is fairly easy to construct a normalized contact
structure on M.   This very construction shows that such structures are far from
unique.   However, they are handy objects to use because of the next theorem
(see [8] for proof).

THEOREM 2.1.   *Given a normalized contact structure $\{\pi\}$ on a complex contact
manifold M, there is a unique, two-dimensional, J-invariant subbundle $\mathcal{V}$ of TM
such that*

1. *$TM \cong \mathcal{H} \oplus \mathcal{V}$.*

2. *For any element $\pi = u - iv$ of the normalized contact structure, there is a
local basis of $\mathcal{V}$, $\{U, JU\}$ defined by:*

   (a) $u(U) = 1$, $v(U) = 0$, $u(JU) = 0$, $v(JU) = -1$,

   (b) $du(U, X) = 0$, $dv(JU, X) = 0$ *for any* $X \in \mathcal{H}$.

Since there is a splitting $TM \cong \mathcal{H} \oplus \mathcal{V}$, we have two natural projections
$TM \to \mathcal{H}$ and $TM \to \mathcal{V}$.   We will denote these respective projections by the
same name as their corresponding subbundles, $\mathcal{H}$ and $\mathcal{V}$.

We say M is a *strict complex contact manifold*, if there is a global holo-
morphic 1-form η such that $\mathcal{H} = \ker(\eta)$.   With such a manifold, we let $\tilde{U}$ be
the unique holomorphic vector field given by the equations $\eta(\tilde{U}) = 1$, and
$\iota(\tilde{U}) d\eta = 0$.   Also, the singleton set $\{\eta\}$ is a normalized contact structure on M.
The resulting vertical subbundle $\mathcal{V}$ is given by $\mathcal{V} = \mathrm{span}_R \{U, JU\}$ where U and
JU are given by $\tilde{U} = (1/2)(U - iJU)$.   Since $\tilde{U}$ is holomorphic, both U and
JU are infinitesimal automorphisms of J, i.e., $Ł_U J = Ł_{JU} J = 0$.   In particular,
$[U, JU] = 0$ so that $\mathcal{V}$ is a foliation on M.

Note that, under these circumstances, $du(U, JU) = -(1/2)u([U, JU]) = 0$.
Also, $dv(U, JU) = 0$.   This departs sharply from the general case of complex
contact manifolds (e.g., see [17]).

3. Hermitian geometry on strict complex contact manifolds

We now examine some aspects of the hermitian geometry of strict complex contact manifolds. To this end, we will restrict ourselves to hermitian metrics which exploit the complex contact structure on the given manifold. From this point on, we will assume that (M, J, η) is a compact strict complex contact manifold with global complex contact structure η. We will use the same notation as the previous section. In particular, $\tilde{U} = (1/2)(U - iJU)$ is the holomorphic vertical vector field defined as before by η.

A hermitian metric g on (M, J, η) is called *associated* to η, if it satisfies the following statements:

1. The real endomorphisms G, H defined by

$$G(X) = -\text{skew}(X \mapsto \nabla_X U),$$

$$H(X) = \text{skew}(X \mapsto \nabla_X(JU))$$

satisfy $G^2 = H^2 = -id_{\mathscr{H}}, H = GJ = -JG, G\mathscr{V} = 0$.

2. $\eta(X) = g(X, U) + ig(X, JU)$.

Thus, the covariant derivatives ∇U and $\nabla(JU)$ of an associated metric induce a quaternionic structure $\{G, H, J' = J \circ \mathscr{H}\}$ on $\mathscr{H}$. Associated metrics were shown to exist in [17]. In fact, the family of associated metrics forms an infinite-dimensional manifold (see [8]). Note that an associated metric on a *general* complex contact manifold is simply defined by saying that the *local* endomorphisms G and H given above by a normalized contact structure $\{\pi\}$ form a local quaternionic structure on $\mathscr{H}$.

If we set $\eta = u - iv$, then $du(X, Y) = g(X, GY)$ and $dv(X, Y) = g(X, HY)$. Also, $u(X) = g(X, U)$ and $v(X) = -g(X, JU)$.

Since $\mathscr{V}$ is a foliation, we may locally fibre out the vertical leaves of M, $\pi_{\mathcal{O}} : \mathcal{O} \to \tilde{\mathcal{O}} = \mathcal{O}/\mathscr{V}$. We say that an associated metric g is *projectable*, if, for each such fibration, there is a metric $\tilde{g}$ on $\tilde{\mathcal{O}}$ such that $g_{\mathscr{H}} = (\pi_{\mathcal{O}})^* \tilde{g}$. It is known ([15]) that this is equivalent to the condition $(\pounds_W(g \circ \mathscr{H}))_{\mathscr{H}} = 0$ for any vertical vector field W.

We define real endomorphisms h_U, h_{JU} by $h_U(X) = \text{sym}(X \mapsto \nabla_X U)$ and $h_{JU}(X) = \text{sym}(X \mapsto \nabla_X JU)$. Then $\nabla_X U = -GX + h_U(X)$, and $\nabla_X(JU) = HX + h_{JU}X$. It can be shown that $G \circ h_U = -h_U \circ G$ and $H \circ h_{JU} = -h_{JU} \circ H$, however the proof is far too complicated for this paper. See [8] for the full proof.

These endormorphisms have a nice geometric interpretation. It is easily seen that for any vertical vector field $W = aU + bJU$, $X, Y \in \mathscr{H}$, $(\pounds_W g)(X, Y) = ag(h_U(X), Y) + bg(h_{JU}(X), Y)$. So, we see that g is projectable if and only if $h_U = h_{JU} = 0$. This shows that W is Killing. Thus, a projectable associated metric is the complex analogue of a K-contact metric on a real contact manifold. To this effect and for future reference, we mention one more projectability equivalence.

PROPOSITION 3.1. *An associated metric g on a strict complex contact manifold is projectable if and only if the sectional curvature of any non-vertical plane section containing an element of $\mathcal{V}$ is equal to* 1.

We now prove a few identities of associated metrics.

LEMMA 3.2. $\nabla_U G = \nabla_{JU} H = 0$.

Proof. Since $du(X, Y) = g(X, GY)$, we know $0 = g((\nabla_X G)Y, Z) + g((\nabla_Y G)Z, X) + g((\nabla_Z G)X, Y)$. Substituting '$U$' for '$X$' and using the skew-symmetry of ∇G, we see that $g((\nabla_U G)Y, Z) = g((\nabla_Y G)U, Z) - g((\nabla_Z G)U, Y)$. However, $(\nabla_Y G)U = -G(\nabla_Y U) = -\mathcal{H}Y + h_U Y$. Since both $\mathcal{H}$ and h_U are symmetric operators, $\nabla_U G = 0$. The proof for $\nabla_{JU} H = 0$ is similar.

LEMMA 3.3. $\nabla_U J = -2H$, $h_U \circ J = J \circ h_U$.

Proof. Since U is an infinitesimal automorphism of J,

$$0 = [U, JX] - J[U, X]$$
$$= \nabla_U(JX) - \nabla_{JX} U - J\nabla_U X + J\nabla_X U$$
$$= (\nabla_U J)X + 2HX - h_U JX + Jh_U X.$$

However, the operator $(\nabla_U J + 2H)$ is skew-symmetric with respect to g, whereas $(-h_U J + Jh_U)$ is symmetric. Thus, $0 = \nabla_U J + 2H$, and $0 = h_U JX - Jh_U X$.

LEMMA 3.4. $g(R_{XY} U, Z) = g((\nabla_Z G)X, Y) + g((\nabla_X h_U)Y, Z) - g((\nabla_Y h_U)X, Z)$.

Proof. This follows from carrying $R_{XY} U$ out by using the definitions of G and h_U then using the fact that the 2-form $\hat{G}(X, Y) = g(X, GY)$ is closed.

4. Complex Sasakian metrics

For a nice exposition of the upcoming propositions, we need some notation. Let V be a vector space over $\mathbf{R}$. For a 1-form $\alpha : V \to \mathbf{R}$ and endomorphism $\beta : V \to V$, we let $\alpha \wedge \beta$ be the rather obvious skew-symmeric, bilinear function $V \times V \to V$ given by $(\alpha \wedge \beta)(X, Y) = (1/2)(\alpha(X)\beta(Y) - \alpha(Y)\beta(X))$.

PROPOSITION 4.1. *Suppose that g is an associated metric for a strict complex contact manifold (M, J, η). Then the following statements are equivalent.*

1. $R_{XY} U = -2(u \wedge \mathcal{H})(X, Y) + 2(v \wedge J')(X, Y) + 2g(J'X, Y)V$ *for any* X, Y.

2. $(\nabla_X G)Y = g(\mathcal{H}X, Y)U + g(J'X, Y)V - u(Y)\mathcal{H}X - v(Y)J'X - 2v(X)J'Y$ *for any X, Y.*

Proof. It is easily seen that, by Lemma (3.4), the statements are equivalent, provided that $h_U = 0$. Thus, we need only to show that for both statements $h_U = 0$.

If 1. is true, then, for $X \in \mathscr{H}$, $R_{UX}U = -X$. By Proposition (3.1), we see that g is projectable. For any associated metric g, $(\nabla_X G)U = -\mathscr{H}X - Gh_U X$ for all X. If 2. is true, then we also have $(\nabla_X G)U = -\mathscr{H}X$. Thus, $h_U = 0$. This proves the proposition.

There is an analogous proposition concerning JU and H which we now give.

PROPOSITION 4.2. *Suppose that g is an associated metric for a strict complex contact manifold (M, J, η). Then the following statements are equivalent.*

1. $R_{XY}JU = 2(v \wedge \mathscr{H})(X, Y) + 2(u \wedge J')(X, Y) + 2g(J'X, Y)U$ *for any* X, Y.

2. $(\nabla_X H)Y = g(J'X, Y)U - g(\mathscr{H}X, Y)V + u(Y)J'X - v(Y)\mathscr{H}X - 2u(X)J'Y$ *for any X, Y.*

An associated metric satisfying one statement from each of the two propositions above is called *complex Sasakian.* This definition is a special case of the so-called *normal complex contact metrics* which is due to B. Korkmaz (nee Karaman) in [20]. The original definition involves associated metrics on any complex contact manifold, not just strict ones. Thus, it is a great deal more complicated. In particular, twistor spaces over quaternionic Kähler manifolds with positive scalar curvature have normal complex contact metrics–namely the so-called Salamon-Bérard-Bergery metrics. Results about normality and twistor spaces will be forthcoming in [10].

Going back to the strict category, the above proof and the analogous proof for Proposition (4.2) gives us this proposition.

PROPOSITION 4.3. *Every complex Sasakian metric is projectable.*

We now go to the main theorem of this paper.

THEOREM 4.4. *Let (M, J, η) be a complex contact manifold with global structure. Let g be an associated metric with horizontal endomorphisms $\{G, H, J'\}$. Then g is complex-Sasakian if and only if g is projectable and "projects" locally to a hyperkähler manifold with hypercomplex structure corresponding to $\{G, H, J'\}$.*

Proof. We know that g is projectable (i.e., $h = 0$). Also, g satisfies:

$$(\nabla_Z G)X = g(\mathscr{H}X, Z)U - g(J'X, Z)V - u(X)\mathscr{H}Z - v(X)J'Z - 2v(Z)J'X,$$

$$(\nabla_Z H)X = g(J'X, Z)U - g(\mathscr{H}X, Z)V + u(X)J'Z - v(X)\mathscr{H}Z - 2u(Z)J'X,$$

$$(\nabla_Z J)X = -2u(Z)HX + 2v(Z)GX.$$

In particular, $\mathscr{H}(\nabla_X G)Y = \mathscr{H}(\nabla_X J)Y = \mathscr{H}(\nabla_X H)Y = 0$ for any horizontal X, Y.

Locally, since $\mathscr{V}$ is a foliation, we can fibre by vertical leaves: $\pi : \mathcal{O} \to \tilde{\mathcal{O}} = \mathcal{O}/\mathscr{V}$. Since g is projectable, there is a metric $\tilde{g}$ on $\mathcal{O}/\mathscr{V}$ such that $g = \pi^*(\tilde{g}) + u \otimes u + v \otimes v$. Let ∇ and $\tilde{\nabla}$ be the respective Levi-Civita connections of g and $\tilde{g}$.

We know $U + iV$ is a holomorphic vector field on M so that $Ł_U J = Ł_V J = 0$. So J is also projectable. This means that there is an almost complex structure $\tilde{J}$ on $\tilde{\mathcal{O}}$ such that $\pi \circ J = \tilde{J} \circ \pi$. It is easily seen that $\tilde{J}$ is integrable (since J is), and so π is a holomorphic submersion of open complex manifolds.

Using the fact that $\nabla_U G = \nabla_V H = 0$, we see that $Ł_U G = Ł_V H = 0$. Also, $Ł_U H = Ł_U(GJ) = (Ł_U G)J + G(Ł_U J) = 0$. Similarly, $Ł_V G = 0$. Hence, both G and H are projectable endomorphisms, corresponding to endomorphisms $\tilde{G}$ and $\tilde{H}$ on $\mathcal{O}/\mathscr{V}$. Since $(\pi)_*$ restricted to $\mathscr{H}$ is an isomorphism onto $T\mathcal{O}/\mathscr{V}$, we see that $\tilde{G}$ and $\tilde{H}$ are almost complex structures on $\mathcal{O}/\mathscr{V}$. In particular, $\{\tilde{G}, \tilde{J}, \tilde{H}\}$ form an almost quaternionic structure on $\mathcal{O}/\mathscr{V}$.

Let $m \in \mathcal{O}$ with $\tilde{m} = \pi(m)$. Also, X and Y be basic vector fields on $\mathcal{O}$ with $\pi_*(X) = \tilde{X}$ and $\pi_*(Y) = \tilde{Y}$. Note that since X and G are projectable tensors on $\mathcal{O}$, $\pi_*(GX) = \tilde{G}(\pi_*(X)) = \tilde{G}\tilde{X}$. Similar identities exist for H and J. We denote by $\mathrm{hor}_m : T_{\tilde{m}}\tilde{\mathcal{O}} \to T_m\mathcal{O}$ the horizontal lift map on $T_{\tilde{m}}\tilde{\mathcal{O}}$. Then we have

$$(\nabla_X Y)_m = \mathrm{hor}_{\tilde{m}}(\tilde{\nabla}_{\tilde{X}} \tilde{Y}) + \frac{1}{2}\mathscr{V}([X, Y]_m).$$

In particular, we find that $(\tilde{\nabla}_{\tilde{X}} \tilde{G})\tilde{Y} = \pi_*((\nabla_X G)Y) = \pi_*(\mathscr{H}((\nabla_X G)Y)) = 0$. Similarly, we find that $\tilde{J}$ and $\tilde{H}$ are also parallel with respect to $\tilde{g}$. Thus, $\{\mathcal{O}/\mathscr{V}, \tilde{g}, \{\tilde{G}, \tilde{H}, \tilde{J}\}\}$ is a hyperkähler manifold.

Conversely, let us now suppose that we have a local Riemannian submersion $(\mathcal{O}, g) \to (\mathcal{O}/\mathscr{V}, \tilde{g})$ such that $\{G, H, J\}$ on $\mathcal{O}$ projects to $\tilde{g}$-parallel hypercomplex structure $\{\tilde{G}, \tilde{H}, \tilde{J}\}$. Let $m \in \mathcal{O}$ with $\tilde{m} = \pi(m)$, $X, Y \in \mathscr{H}$ be basic vector fields defined at m with corresponding vector fields $\tilde{X}, \tilde{Y}$ on $\mathcal{O}/\mathscr{V}$. Then

$$(\nabla_X G)Y = \nabla_X(GY) - G(\nabla_X Y)$$

$$= \mathrm{hor}_{\tilde{m}}(\tilde{\nabla}_{\tilde{X}}(\tilde{G}\tilde{Y})) + \frac{1}{2}\mathscr{V}([X, GY]) - \mathrm{hor}_{\tilde{m}}(\tilde{G}\tilde{\nabla}_{\tilde{X}}\tilde{Y})$$

$$= \frac{1}{2}u([X, GY])U + \frac{1}{2}v([X, GY])V$$

$$= -g(X, G^2 Y)U - g(X, HGY)V$$

$$= g(X, Y)U - g(X, J'Y)V.$$

Also $(\nabla_X G)U = -G(\nabla_X U) = -X$ and $(\nabla_X G)V = -J'X$. It is known already that $\nabla_U G = 0$. Finally, using the fact that $(\nabla_U J) = -2H$, we see that $\nabla_V G = -2J'$. Thus,

$$(\nabla_X G)Z = -2v(X)J'Z + g(X, \mathscr{H}Z)U - g(X, J'Y)V - u(Z)\mathscr{H}X - v(Z)J'X$$

for any $X, Y \in T\mathcal{O}$. Similarly, we get the formulae for ∇H and ∇J. Thus, g is a complex Sasakian metric on M.

5. Examples

The most famous complex Sasakian manifold is the complex Heisenberg group defined by

$$H_C = \left\{ \begin{pmatrix} 1 & z_1 & z_2 \\ 0 & 1 & z_3 \\ 0 & 0 & 1 \end{pmatrix} : z_1, z_2, z_3 \in C \right\}.$$

H_C has a global right-invariant complex contact form given by $\eta = dz_1 - z_2\, dz_3$.

Furthermore, the map $\pi : (z_1, z_2, z_3) \mapsto (z_2, z_3)$ gives us a holomorphic fibration

$$C \to H_C \to C^2.$$

Set $\mathscr{V} = \ker(\pi_*)$, the vertical foliation of this fibration. Also, η defines a connection on the resulting C-bundle, and we set $\mathscr{H} = \ker(\eta)$, the horizontal subbundle of the connection η.

If we set Γ to be the subgroup of H_C consisting of all elements of H_C with Gaussian integer entries, the 3-dimensional manifold H_C/Γ is called the *Iwasawa manifold*. This manifold is often cited as a good example of a compact complex manifold with no Kähler structure (see [11] or [7] for details).

Now, the Euclidean metric $\langle\ ,\ \rangle$ on C^2 is actually hyperkähler with a quaternionic structure $\{\tilde{J}, \tilde{G}, \tilde{H}\}$ given (as endomorphisms of R^4) by:

$$\tilde{J} = \left(\begin{array}{cc|cc} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array} \right), \quad \tilde{G} = \left(\begin{array}{cc|cc} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array} \right), \quad \tilde{H} = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array} \right).$$

Then, on C^2,

$$-dz_1 \wedge dz_2(\tilde{X}, \tilde{Y}) = \langle \tilde{X}, \tilde{G}\tilde{Y} \rangle + i\langle \tilde{X}, \tilde{H}\tilde{Y} \rangle \quad \text{for } X, Y \in T^R C^2.$$

We can then induce a metric g on H_C which projects down via π_* to $\langle\ ,\ \rangle$ on C^2, i.e., $g(X, Y) = \langle \pi_* X, \pi_* Y \rangle$ for $X, Y \in TH_C$. We do this by insisting $g(\mathscr{H}, \mathscr{V}) = 0$, setting g on $\mathscr{H}$ to be the horizontal lift of $\langle\ ,\ \rangle$ on C^2, and finally defining the real and complex components of the global vector field $\partial/\partial z_1 \in TH_C$ to be unit, orthogonal vectors. The metric g then is a hermitian metric on H_C.

Similarly, we induce endomorphisms G and H on H_C which satisfy

$$\tilde{G} \circ \pi_* = \pi_* \circ G, \quad \tilde{H} \circ \pi_* = \pi_* \circ H,$$

$$G_{|\mathscr{V}} = 0, \quad H_{|\mathscr{V}} = 0.$$

Then, it is easily seen that the structure (g, J, G, H) is associated with respect to the contact form $\eta = dz_1 - z_2\, dz_3$ on H_C, i.e.,

$$d\eta(X, Y) = g(X, GY) + ig(X, HY).$$

By our main theorem, we know then that this associated metric is complex Sasakian.

To get a compact example, we set Γ' to be the additive subgroup of $\boldsymbol{C}^2$ consisting of all elements with Gaussian integer entries. Then we get the following commutative diagram of hermitian manifolds:

$$
\begin{array}{ccc}
H_C & \longrightarrow & H_C/\Gamma \\
\downarrow & & \downarrow \\
\boldsymbol{C}^2 & \longrightarrow & \boldsymbol{C}^2/\Gamma'
\end{array}
$$

The metrics and forms of the left-hand side fibration are all left-invariant and so project over to the right-hand side fibration. Thus, the resulting metrics on H_C/Γ and $\boldsymbol{C}^2/\Gamma'$ are complex Sasakian and hyperkähler, respectively.

It is shown in [9] that this example can be generalized to every possible dimension in a way which we'll now briefly describe. Recall that a *period matrix* of $\boldsymbol{C}^N$ is an element $W \in M_{N\times 2N}(\boldsymbol{C})$ consisting of n column vectors which are linearly independent over $\boldsymbol{R}$. Then $W\boldsymbol{Z}^{2N}$ is an additive subgroup of $\boldsymbol{C}^N$, and the complex manifold $\boldsymbol{C}^N/W\boldsymbol{Z}^{2N}$ is an N-dimensional complex torus.

We set

$$
\tilde{J}_2 = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.
$$

We call a period matrix W of $\boldsymbol{C}^{2n}$ (i.e., $N = 2n$) a *hyperkähler Riemann matrix*, if there is a Hermitian matrix $G \in M_{2n\times 2n}(\boldsymbol{C})$ such that

1. $G > 0$,

2. $\tilde{J}_2 G = \bar{G}\tilde{J}_2$,

3. ${}^t W {}^t \tilde{J}_2 G W \in M_{4n\times 4n}(\boldsymbol{Z} + i\boldsymbol{Z})$.

This definition is actually the quaternionic analogue of a Kähler Riemann matrix; see [24] for details. We now give a hermitian version of Theorem 7.1 from [9]. This theorem gives us examples of complex Sasakian manifolds in every possible dimension. The author is actually unaware of any other examples of compact, complex Sasakian manifolds of any dimension.

THEOREM 5.1. *Let* $\{\langle\ ,\ \rangle, \{J_1, J_2, J_3\}\}$ *be the standard hypercomplex structure on* $\boldsymbol{C}^{2n} \cong \boldsymbol{H}^n$, $\underline{w} = \{w_1, \ldots, w_{4n}\}$ *be a set of linearly independent periods in* $\boldsymbol{C}^{2n}$ *with period matrix* W *and* $\Gamma = \langle w_1, \ldots, w_{4n} \rangle$. *Let* $X = \boldsymbol{C}^{2n}/\Gamma$ *with hermitian projection* $p : (\boldsymbol{C}^{2n}, \langle\ ,\ \rangle) \to (X, h)$, *and let* ω' *be the unique holomorphic 2-form on* X *such that* $\omega = p^*(\omega')$. *Then there exists a hermitian fibration* $\pi : (M, g) \to (X, h)$ *with vertical fibers* $\cong S^1 \times S^1$ *and such that* (M, g) *is complex Sasakian, if and only if* W *is a hyperkähler Riemann matrix.*

6. Twistor spaces and curvature equations

It should not be mistaken that every real Sasakian manifold can somehow be "complexified" to create a complex Sasakian example. For instance, $\boldsymbol{R}P^{2n+1}$ has

a Sasakian structure (as described in [4]), but it's "proper" complexification is CP^{2n+1}, which has no global complex contact structure, since $c_1(CP^{2n+1}) > 0$. In fact, CP^{2n+1} is the twistor space of the quaternionic-Kähler manifold HP^{2n+1}.

It is worthwhile to compare these two very contrasting types of complex contact spaces: The twistor spaces of the quaternionic-Kähler manifold with non-zero scalar curvature and complex Sasakian manifolds. Both are specific cases of normal complex contact manifolds as defined in [20].

Suppose $\pi : M \to \tilde{M}$ is twistorial fibration with $(\tilde{M}, \tilde{g})$ quaternionic-Kähler with nonzero scalar curvature τ. Then M never has a strict complex contact structure since $c_1(M)$ is necessarily nondegenerate (actually positive, if $\tau > 0$). Also, $\tilde{g}$ lifts to an Einstein associated metric g on M (it's also Kähler, if again τ is positive). Finally, the map π is never holomorphic, since $\tilde{M}$ has no global complex structure.

In complete contrast, suppose that (M, η, g) is a complex Sasakian manifold. Then, by definition, M has global complex contact form. Also, g is never Einstein, since, in particular, the sectional curvature of the vertical subbundle is flat, whereas the sectional curvature of a vertical vector and a horizontal vector is nonzero. Also, if M is compact, then we know that it has no Kähler metrics with respect to its complex structure, since it has a global holomorphic non-closed 1-form.

We end this paper with a discussion of how we can view the twistor space examples of complex contact manifolds with $c_1 > 0$ as those satisfying a certain associated metric curvature condition. Before we can get to this result, though, we need to describe the situation for general complex contact manifolds, which can vary in structure much more than those complex contact manifolds with a global contact form.

Let (M, J) be a $2n + 1$-dimensional complex manifold with complex contact structure given by a $2n$-dimensional holomorphic subbundle $\mathscr{H}$. Let $\{\pi = u - iv\}$ be a normalized contact structure on M with resulting splitting $TM \cong \mathscr{H} \oplus \mathscr{V}$ as given by Theorem 2.1. Let U and V be the vertical vector fields corresponding to u and v, respectively.

Recall that an associated metric g is a hermitian metric on M, which satisfies the following local conditions:

1. The local real endomorphisms G and H defined by:

$$GX = -\text{skew}(X \mapsto \mathscr{H}\nabla_X U)$$

$$HX = \text{skew}(X \mapsto \mathscr{H}\nabla_X JU)$$

satisfy $G^2 = H^2 = -id_{\mathscr{H}}$, $H = G \circ J = -J \circ G$.

2. $u(X) = g(U, X)$, $v(X) = g(V, X)$ for any vector X.

In [8], it is proven that an associated metric g satisfies the following conditions:

1. $\mathscr{V}$ is a totally geodesic subbundle of TM.

2. If we set $\hat{G}(X, Y) = g(X, GY)$ and $\hat{H}(X, Y) = g(X, HY)$, then

$$\hat{G} = du - \sigma \wedge v,$$

$$\hat{H} = dv + \sigma \wedge u,$$

where $\sigma(x) = g(\nabla_X U, V)$ is a 1-form representing a connection on $\mathcal{V}$ as a complex line bundle over M. This connection is called the *Ishihara-Konishi Connection* with respect to $\{\pi\}$.

3. $\nabla_X U = -GX + h_U X - \sigma(X)JU$, $\nabla_X V = -HX + h_V X + \sigma(X)U$, for any $X \in TM$, where, for any $W \in \mathcal{V}$, h_W is the real endomorphism given by

$$h_W(X) = \text{sym}(X \mapsto \mathscr{H}\nabla_X W).$$

4. $\nabla_U G = \sigma(U)H$, $\nabla_{JU} H = \sigma(JU)G$.

Using this terminology and machinery, we will prove the following theorem.

THEOREM 6.1. *Let M be a complex contact manifold with contact subbundle $\mathscr{H}$ and normalized contact structure $\{\pi = u - iv\}$. Let $TM = \mathscr{H} \oplus \mathscr{V}$ be the corresponding splitting with local vertical vector fields $\{U, V\}$ corresponding to the real forms u and v. Suppose g is an associated metric on M. Then (M, g) is isometric to a twistor space for a quaternionic-Kähler manifold with positive scalar curvature if and only if*

$$R_{XY} U = u(Y)X - u(X)Y + v(Y)JX - v(X)JY + 2g(JX, Y)V,$$

$$R_{XY} V = -v(Y)X + v(X)Y - u(Y)JX + u(X)JY - 2g(JX, Y)U.$$

The proof for this theorem will actually be nothing more than pulling together various previously-known facts regarding complex contact manifolds. So, before we continue with the proof, we will now describe some of these results.

In [16], Ishihara and Konishi generated their notion of normality of complex contact manifolds. They defined local tensors

$$S(X, Y) = [G, G](X, Y) + 2\hat{G}(X, Y)U - 2\hat{H}(X, Y)V + 2(v(Y)HX - v(X)HY)$$
$$+ \sigma(GY)HX - \sigma(GX)HY + \sigma(X)GHY - \sigma(Y)GHX,$$

$$T(X, Y) = [H, H](X, Y) - 2\hat{G}(X, Y)U + 2\hat{H}(X, Y)V + 2(u(Y)GX - u(X)GY)$$
$$+ \sigma(HX)GY - \sigma(HY)GX + \sigma(X)GHY - \sigma(Y)GHX.$$

Then they called an associated metric g *normal* if $S = T = 0$. (In this paper, we call such a metric IK-*normal*).

This concept was meant to be a complex analogue of normal real contact manifolds, and in some sense that is true. The most well-known complex contact manifold CP^{2n+1} is indeed IK-normal, and it is very much the complex analogue of the equally as famous Sasakian manifold RP^{2n+1}. However, this definition of normality also leaves out the complex Heisenberg group, whose real analogue (the real Heisenberg group) is also a well-known normal real contact manifold. It is for this reason that Korkmaz in [20] expanded the definition of

normality by calling an associated metric *normal,* if it only satisfied the conditions:

$$S|_{\mathscr{H} \wedge \mathscr{H}} = S|_{\mathscr{H} \wedge \mathscr{V}} = T|_{\mathscr{H} \wedge \mathscr{H}} = T|_{\mathscr{H} \wedge \mathscr{V}} = 0.$$

Returning to IK-normality, we have the following proposition from [16].

PROPOSITION 6.2. *An associated metric g is* IK-*normal if and only if, for local* G, H, U, V, *as described above,*

$$(\nabla_X G) Y = \sigma(X) HY - u(Y) X - v(Y) JX + g(X, Y) U + g(JX, Y) V,$$

$$(\nabla_X H) Y = -\sigma(X) GY + u(X) JY - v(Y) X + g(X, JY) U + g(X, Y) V.$$

This next proposition reworks the above conditions into those dealing with the curvature.

PROPOSITION 6.3. *An associated metric g is* IK-*normal if and only if*

$$R_{XY} U = u(Y) X - u(X) Y + v(Y) JX - v(X) JY + 2g(JX, Y) V,$$

$$R_{XY} V = -v(Y) X + v(X) Y - u(Y) JX + u(X) JY - 2g(JX, Y) U.$$

Proof. It is already proven in the first part of section 4 of [16] that any IK-normal metric satisfies these curvature conditions. Conversely, we suppose that g satisfies the curvature conditions. We will show that the equation of G is true; the proof of the second will be completely analogous.

First, Proposition 3.1 tells us that g is projectable. Thus, we have $\nabla_X U = -GX - \sigma(X) JU$ for any vector X.

Direct calculation then tells us that

$$R_{XY} U = -(\nabla_X G) Y + (\nabla_Y G) X + 2d\sigma(X, Y) V - \sigma(Y) HX + \sigma(X) HY.$$

Since $\mathscr{V}$ is totally geodesic, this equation can be specialized as such:

$$\mathscr{H} R_{JUY} U = -\mathscr{H}(\nabla_{JU} G) Y + \sigma(JU) HY + (J \circ \mathscr{H}) Y.$$

Also, it is easily seen that $\mathscr{H}(\nabla_{JU} G) = (\nabla_{JU} G)$, again by the fact that $\mathscr{V}$ is totally geodesic. Comparing this result to the given curvature condition, we have that $(\nabla_{JU} G) Y = \sigma(JU) HY$ for any $Y \in TM$.

However, we also have

$$\nabla_{JU} G = \nabla_{JU}(JH)$$

$$= (\nabla_{JU} J) H + J(\nabla_{JU} H)$$

$$= (\nabla_{JU} J) H + \sigma(JU) H.$$

Thus, $\nabla_U J = 0$.

Using the Bianchi identity and a comparable expression for $\mathscr{H} R_{UY} JU$ as above, we get that, for $X, Y \in \mathscr{H}$,

$$g(R_{UV} X, Y) = 2g((\nabla_U J) GX, Y) + 2g(JX, Y),$$

i.e., $g(R_{UV}X, Y) = 2g(JX, Y)$ whenever X and Y are horizontal. From the assumed curvature condition, we also have $g(R_{UV}V, U) = 2$, i.e., $g(R_{UV}X, Y) = 2g(JX, Y)$ whenever X and Y are vertical. Thus, combining these results and using the total geodesic property of $\mathscr{V}$, we get $g(R_{UV}X, Y) = 2g(JX, Y)$ for any vectors X and Y.

Recalling the formula $2d\sigma(X, Y) = X\sigma(Y) - Y\sigma(X) - \sigma([X, Y])$, we also can directly calculate $d\sigma$ to get:

$$2d\sigma(X, Y) = g(R_{UV}X, Y) + 2g(JX, Y),$$

i.e., $d\sigma(X, Y) = 2g(JX, Y)$ for any vectors X and Y.

Finally, since $\hat{G} = du - \sigma \wedge v$, we have that $d\hat{G} = -d\sigma \wedge v + \sigma \wedge \hat{H}$, i.e., in terms of the associated metric g,

$$g(Y, (\nabla_X G)Z) + g(Z, (\nabla_Y G)X) + g(X, (\nabla_Z G)Y)$$
$$= -d\sigma(X, Y)v(Z) - d\sigma(Z, X)v(Y) - d\sigma(Y, Z)v(X)$$
$$+ \sigma(X)g(Y, HZ) + \sigma(Z)g(X, HY) + \sigma(Y)g(Z, HX)$$

Combining this equation with those for $d\sigma$ and $R_{XY}U$ gives the result.

Ishihara and Konishi in the same paper also prove the following important theorem.

THEOREM 6.4 (Ishihara-Konishi). *If g is an* IK-*normal associated metric on* M, *then* g *is Kähler-Einstein and* $c_1(M) > 0$.

More recently, LeBrun in [21] proved the following theorem.

THEOREM 6.5 (LeBrun). *Let* M *be a complex contact manifold with* $c_1(M) > 0$. *Then* M *is the twistor space for a quaternionic-Kähler manifold with positive scalar curvature if and only if* M *has a Kähler-Einstein metric.*

Conversely, in the ground-breaking paper [19] of 1975, M. Konishi proved the following theorem.

THEOREM 6.6. *Let* $\tilde{M}$ *be a quaternionic-Kähler manifold of dimension* $4m$ *and positive scalar curvature. Then there exists a canonically associated* RP^3-*bundle* P *over* $\tilde{M}$ *admitting a Sasakian 3-structure, in such a way that* $P \to \tilde{M}$ *is a Riemannian submersion.*

We call this bundle the *Konishi bundle* of the quaternionic-Kähler manifold $\tilde{M}$. Recall that we call a Riemannian manifold (P, H) a *3-Sasakian manifold*, if the holonomy group of the metric cone on P, defined by $(C(P), \bar{h}) = (R^+ \times P, dr + r^2h)$, is a hyperkähler manifold. In particular, it is known that this condition is equivalent to there being three unit, Killing vector fields on P ξ_1, ξ_2, ξ_3 such that:

1. For each $j = 1, 2, 3$, the endomorphism ϕ_j defined by $\phi_j(X) = \nabla_X \xi_j$ satisfies:

$$(\nabla_X \phi_j) Y = h(\xi_j, Y) X - h(X, Y) x i_j \qquad \text{for all } X, Y \in TP,$$

i.e., each ξ_j defines a Sasakian structure on (P, h),

2. $h(\xi_j, \xi_k) = \delta_{jk}$.

3. $[\xi_j, \xi_k] = 2\varepsilon_{jkl} \xi_l$.

See either [19] or [6] for details about 3-Sasakian manifolds. It is well-known that at least one of the characteristic vector fields, say ξ_1, of the Konishi bundle P over $\tilde{M}$, a quaternionic-Kähler manifold with positive scalar curvature, is regular, i.e., $M = P/\xi_1$ is a manifold. In fact, M is the twistor space over $\tilde{M}$. Additionally, in [14], Ishihara and Konishi proved the following theorem.

THEOREM 6.7. *Let* $(P, h, \xi_1, \xi_2, \xi_3)$ *be a 3-Sasakian manifold. If* ξ_1 *is regular, then the fibration* $M = P/\xi_1 \to P$ *is a Riemannian submersion, and* M *has an* IK-*normal complex contact structure.*

Proof of Theorem 5.1. This proof now is just a matter of combining all of these results. Clearly, Proposition 5.3 and Theorem 5.7 tell us that every twistor space of a quaternionic-Kähler manifold with positive scalar curvature has an associated metric satisfying the desired curvature condition.

Conversely, if the given complex contact manifold has an associated metric satisfying the curvature condition, then this metric is IK-normal by Proposition 5.3. So, by Theorem 5.4, the metric is Kähler-Einstein, and the first Chern class of the manifold is positive. Finally, by LeBrun's result, Theorem 5.5, the complex contact manifold is the twistor space of a quaternionic-Kähler manifold with positive scalar curvature. This proves the theorem.

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