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ON THE MONODROMY OF A FUNCTION GERM DEFINED ON AN ARRANGEMENT OF HYPERPLANES

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1. Introduction

Let $\mathscr{A} = \{H_1, \dots, H_k\}$ be an arrangement of hyperplanes in $\boldsymbol{C}^{n+1}$ such that $0 \in H_1 \cap \cdots \cap H_k$ and let $\mathscr{L}(\mathscr{A})$ denote the intersection poset of $\mathscr{A}$. Let $f : (H_1 \cup \cdots \cup H_k, 0) \to \boldsymbol{C}$ be a germ of a holomorphic function in the origin with the property that the restriction of f to any $X \in \mathscr{L}(\mathscr{A})$, $X \neq \boldsymbol{C}^{n+1}$, $X \neq \{0\}$, has an isolated critical point in 0. It is known that f defines a Milnor fibration (see [2]) and that the Milnor fiber of f, denoted by F, has the homotopy type of a bouquet of spheres of (real) dimension $n-1$.

Let $h : H_{n-1}(F) \to H_{n-1}(F)$ be the (algebraic) monodromy and $\Delta(t) = \det(tI - h)$ be its characteristic polynomial. For $X \in \mathscr{L}(\mathscr{A})$, $X \neq \boldsymbol{C}^{n+1}$, $X \neq \{0\}$, let $\Delta_X(t)$ denote the characteristic polynomial of the monodromy and $\mu(f|_X)$ denote the Milnor number of the restriction of f to X. If $\{0\} \in \mathscr{L}(\mathscr{A})$ we put $\Delta_{\{0\}}(t) = t - 1$ and $\mu(f|_{\{0\}}) = 1$. Let $\mu : \mathscr{L}(\mathscr{A}) \to \boldsymbol{Z}$ be the Möbius function of $\mathscr{L}(\mathscr{A})$.

In this article we shall prove the following theorem (we consider the reduced homology with integer coefficients):

THEOREM 1.1. *Under the above conditions, we have*:

$$\Delta(t) = \prod_{X \in \mathscr{L}(\mathscr{A}),\, X \neq \boldsymbol{C}^{n+1}} \Delta_X^{|\mu(X)|}(t).$$

In Section 3 we shall use Theorem 1.1 to obtain formulas for the ζ function of the monodromy, the Lefschetz number of f and the Milnor number of f, depending on the similar objects of the restriction of f to the linear spaces $X \in \mathscr{L}(\mathscr{A})$ and on the values of the Möbius function of $\mathscr{L}(\mathscr{A})$.

These results answer question raised by Professor D. Siersma to whom I would like to thank. I am also grateful to the referee for useful suggestions.

We shall remind some facts on arrangements of hyperplanes in a vector space, which we shall need in the proof of Theorem 1.1. These facts can be found in [4].

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Let $\mathscr{A} = \{H_1, \dots, H_k\}$ be an arrangement of hyperplanes in a vector space V such that $0 \in T(\mathscr{A}) = H_1 \cap \cdots \cap H_k$. Let $\mathscr{L} = \mathscr{L}(\mathscr{A})$ be the intersection poset of $\mathscr{A}$:

$$\mathscr{L}(\mathscr{A}) = \{V\} \cup \{W | \exists \{i_1, \dots, i_p\} \subseteq \{1, \dots, k\} \text{ such that } W = H_{i_1} \cap \cdots \cap H_{i_p}\}.$$

On $\mathscr{L}$ a partial order is defined by reverse inclusion: $X \le Y \Leftrightarrow Y \subseteq X$.

Definition 1.2. The Möbius function $\mu_{\mathscr{A}} : \mathscr{L} \times \mathscr{L} \to \boldsymbol{Z}$ is defined by

$$\mu(X, X) = 1, \quad \text{if } X \in \mathscr{L},$$

$$\sum_{X \le Z \le Y} \mu(X, Z) = 0, \quad \text{if } X, Y, Z \in \mathscr{L} \quad \text{and} \quad X < Y,$$

$$\mu(X, Y) = 0, \text{ otherwise.}$$

Definition 1.3. For $X \in \mathscr{L}$, we define $\mu(X) = \mu(V, X)$, $\mathscr{A}_X = \{H \in \mathscr{A} \mid X \subseteq H\}$ and $r(X) = \operatorname{codim} X = \dim V - \dim X$. We denote $\mu(\mathscr{A}) = \mu(T(\mathscr{A}))$.

Remark 1.4. It is well-known, see for instance [4], that $\sum_{X \in \mathscr{L}} \mu(X) = 0$ and that for $X \in \mathscr{L}$, we have $\mu(X) = (-1)^{r(X)} |\mu(X)| = \mu(\mathscr{A}_X)$.

2. Proof of Theorem 1.1

We prove Theorem 1.1 by double induction on the number of hyperplanes in the arrangement, k, and the dimension of the base space, $n + 1$.

For $k = 1$ and *any* n, we have one hyperplane H in $\boldsymbol{C}^{n+1}$ so $\dim H = n$ and we work in fact with $f|_H$. We have $\Delta(t) = \Delta_H(t)$.

For $n = 1$ and *any* k we have k (complex) lines, $H_1, \dots, H_k$ in $\boldsymbol{C}^2$ and $H_1 \cap \cdots \cap H_k = \{0\}$. The Milnor fiber F of f is a finite set of points and consequently the only nonzero homology group is $H_0(F)$. We have $\mathscr{L}(\mathscr{A}) = \{\boldsymbol{C}^2, H_1, \dots, H_k, \{0\}\}$ with $\mu(\boldsymbol{C}^2) = 1$, $\mu(H_i) = -1$, $\forall i \in \{1, \dots, k\}$, $\mu(\{0\}) = k - 1$. The formula to prove is

$$\Delta(t) = (t - 1)^{k-1} \prod_{i=1}^{k} \Delta_{H_i}(t).$$

For $k = 2$: Let $F_1 = F \cap H_1$ be the Milnor fiber of the restriction $f|_{H_1}$ and let $F_2 = F \cap H_2$ be the Milnor fiber of the restriction $f|_{H_2}$. Then F_1 consists of $\mu(f|_{H_1}) + 1$ points, say $x_0, x_1, \dots, x_{\mu(f|_{H_1})}$, and F_2 consists of $\mu(f|_{H_2}) + 1$ points, say $y_0, y_1, \dots, y_{\mu(f|_{H_2})}$. Since $F_1 \cap F_2 = \emptyset$, $F = F_1 \cup F_2$ consists of $\mu(f|_{H_1}) + \mu(f|_{H_2}) + 2$ points and $\dim H_0(F) = \mu(f|_{H_1}) + \mu(f|_{H_2}) + 1$. Let us consider the Mayer-Vietoris sequence for $F = F_1 \cup F_2$:

$$0 \to H_0(F_1) \oplus H_0(F_2) \to H_0(F).$$

A basis in $H_0(F_1)$ is $\{x_0 - x_j \mid j = 1, 2, \dots, \mu(f|_{H_1})\}$.

A basis in $H_0(F_2)$ is $\{y_0 - y_i \mid i = 1, 2, \ldots, \mu(f|_{H_2})\}$.
A basis in $H_0(F)$ is

$$\{x_0 - x_j, y_0 - y_i, x_0 - y_0 \mid j = 1, 2, \ldots, \mu(f|_{H_1}) \text{ and } i = 1, 2, \ldots, \mu(f|_{H_2})\}.$$

By [2], the monodromy respects the stratification. Thus, if $h(x_0) = x_j$ and $h(y_0) = y_i$ for some $j \in \{0, 1, \ldots, \mu(f|_{H_1})\}$ and some $i \in \{0, 1, \ldots, \mu(f|_{H_2})\}$, then

$$h(x_0 - y_0) = x_j - y_i = -x_0 + x_j + y_0 - y_i + x_0 - y_0.$$

Thus, the matrix of the monodromy $h : H_0(F) \to H_0(F)$ in the above basis is

$$\begin{pmatrix} \begin{matrix}\text{Matrix of the}\\ \text{monodromy}\\ \text{of } f|_{H_1}\end{matrix} & 0 & 0 \\ 0 & \begin{matrix}\text{Matrix of the}\\ \text{monodromy}\\ \text{of } f|_{H_2}\end{matrix} & 0 \\ * & * & 1 \end{pmatrix}.$$

Consequently, the characteristic polynomial of h is

$$\Delta(t) = \Delta_{H_1}(t) \cdot \Delta_{H_2}(t) \cdot (t-1).$$

The induction step $k \mapsto k+1$: The induction hypothesis is: for k lines in $\boldsymbol{C}^2$ the characteristic polynomial of the monodromy is

$$\Delta(t) = \Delta_{H_1}(t) \cdots \Delta_{H_k}(t) \cdot (t-1)^{k-1}.$$

Let $H_1, \ldots, H_{k+1}$ be $k+1$ lines in $\boldsymbol{C}^2$ such that $H_1 \cap \cdots \cap H_{k+1} = \{0\}$. The sets of points representing the Milnor fibres of the restrictions $f|_{H_i}$ do not intersect, hence

$$\dim H_0(F) = \sum_{i=1}^{k+1} \mu(f|_{H_i}) + k.$$

We put $F_1 = F \cap (H_1 \cup \cdots \cup H_k)$ and $F_2 = F \cap H_{k+1}$ and we note that

$$\dim H_0(F_1) = \sum_{i=1}^{k} \mu(f|_{H_i}) + (k-1).$$

If we fix basis in $H_0(F_1)$ and $H_0(F_2)$ we can get a basis in $H_0(F)$ in the same way we did in the case $k = 2$ and, like there, we get

$$\Delta(t) = \Delta_{H_{k+1}}(t) \cdot (t-1) \cdot (t-1)^{k-1} \prod_{i=1}^{k} \Delta_{H_i}(t) = (t-1)^k \prod_{i=1}^{k+1} \Delta_{H_i}(t).$$

Let us consider now that Theorem 1.1 is true for any p hyperplanes in a $(m+1)$ dimensional vector subspace of $\boldsymbol{C}^{n+1}$ for $p \le k$ and $m \le n$ and let us prove it for $k+1$ hyperplanes in $\boldsymbol{C}^{n+1}$. So consider $H_1, \ldots, H_{k+1} \subseteq \boldsymbol{C}^{n+1}$,

$f:(H_1\cup\cdots\cup H_{k+1},0)\to \boldsymbol{C}$ as before and let F be the Milnor fiber of f. We put:

$$F_1=F\cap(H_1\cup\cdots\cup H_k)\quad\text{and}\quad F_2=F\cap H_{k+1}.$$

Thus, F_1 is the Milnor fiber of the restriction $f|_{H_1\cup\cdots\cup H_k}$ and $F_1\cap F_2$ is the Milnor fiber of the restriction $f|_{H_{k+1}\cap(H_1\cup\cdots\cup H_k)}$. Let us consider the following monodromies:

$h=$ the monodromy of f
$h_1=$ the monodromy of $f|_{H_1\cup\cdots\cup H_k}$
$h_2=$ the monodromy of $f|_{H_{k+1}}$
$h_{12}=$ the monodromy of $f|_{H_{k+1}\cap(H_1\cup\cdots\cup H_k)}$.

Because the monodromy respects the stratification, the Mayer-Vietoris sequence for $F=F_1\cup F_2$ gives us the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & H_{n-1}(F_1)\oplus H_{n-1}(F_2) \arrow[r] \arrow[d, "h_1\oplus h_2"] & H_{n-1}(F) \arrow[r] \arrow[d, "h"] & H_{n-2}(F_1\cap F_2) \arrow[r] \arrow[d, "h_{12}"] & 0 \\
0 \arrow[r] & H_{n-1}(F_1)\oplus H_{n-1}(F_2) \arrow[r] & H_{n-1}(F) \arrow[r] & H_{n-2}(F_1\cap F_2) \arrow[r] & 0.
\end{tikzcd}$$

Because the homology groups are free $\boldsymbol{Z}$-modules,

$$H_{n-1}(F)\simeq H_{n-1}(F_1)\oplus H_{n-1}(F_2)\oplus H_{n-1}(F_1\cap F_2)$$

so there exists a basis in $H_{n-1}(F)$ with respect to which the matrix of the monodromy h consists of cells corresponding to the matrices of h_1, h_2 and h_{12} on the diagonal and zeroes above them (like in the case $k=2$ above). Consequently,

$$\Delta(t)=\Delta_{H_{k+1}}(t)\cdot\Delta_2(t)\cdot\Delta_{12}(t), \tag{1}$$

where $\Delta_2(t)$ and $\Delta_{12}(t)$ are the characteristic polynomials of h_2 and h_{12}.

The induction hypothesis applies for $f|_{H_1\cup\cdots\cup H_k}$, so we have

$$\Delta_2(t)=\prod_{X\in\mathscr{L}',X\neq \boldsymbol{C}^{n+1}}\Delta_X^{|\mu'(X)|}(t), \tag{2}$$

where $\mathscr{A}'$ is the arrangement $\{H_1,\ldots,H_k\}$ in $\boldsymbol{C}^{n+1}$ and μ' is the Möbius function of $\mathscr{L}':=\mathscr{L}(\mathscr{A}')$. Next, we can apply the induction hypothesis for $f|_{H_{k+1}\cap(H_1\cap\cdots\cap H_k)}$ because we have at most k hyperplanes, $H_1\cap H_{k+1},\ldots,$ $H_k\cap H_{k+1}$, in a n-dimensional subspace, H_{k+1}, of $\boldsymbol{C}^{n+1}$. So

$$\Delta_{12}(t)=\prod_{X\in\mathscr{L}'',X\neq \boldsymbol{C}^{n+1}}\Delta_X^{|\mu''(X)|}(t), \tag{3}$$

where $\mathscr{A}''$ is the arrangement $\{H_1\cap H_{k+1},\ldots,H_k\cap H_{k+1}\}$ in H_{k+1} and μ'' is the Möbius function of $\mathscr{L}'':=\mathscr{L}(\mathscr{A}'')$. By introducing the relations (2) and (3) in (1), we easily see that the proof of Theorem 1.1 reduces to the proof of the following:

(4) For all $X \in \mathscr{L}' \cap \mathscr{L}''$, we have: $|\mu(X)| = |\mu'(X)| + |\mu''(X)|$.

To prove (4), let $X \in \mathscr{L}' \cap \mathscr{L}''$. Then $X \subseteq H_{k+1}$ and $X = T(\mathscr{A}_X) \in \mathscr{L}(\mathscr{A}_X \setminus \{H_{k+1}\}) = \mathscr{L}(\mathscr{A}'_X)$. Thus, H_{k+1} is not a separator, in the sense of [4], Definition 2.58. And now, point (2) of Corollary 2.59 in [4] gives us that

$$|\mu(\mathscr{A}_X)| = |\mu(\mathscr{A}'_X)| + |\mu(\mathscr{A}''_X)|. \tag{5}$$

By Remark 1.4, relation (5) implies (4). Thus, Theorem 1.1 is proved.

3. Consequences of Theorem 1.1

Because the degree of the characteristic polynomial is equal to $\dim H_{n-1}(F)$, we have

PROPOSITION 3.1. *Under the conditions in Theorem* 1.1, *we have*

$$\dim H_{n-1}(F) = \sum_{X \in \mathscr{L}(\mathscr{A}),\, X \neq \boldsymbol{C}^{n+1}} |\mu(X)| \cdot \mu(f|_X).$$

Remark 3.2. For another proof of this Proposition, see [8]. In [5] it is obtained a similar result for homogeneous f. Note also that the above formula is used in [6].

Remark 3.3. In [7] and [8] we proved a formula to compute the algebraic codimension (when finite) of a function germ f defined on an arrangement $\mathscr{A} = \{\{x_1 = 0\}, \ldots, \{x_p = 0\}\}$ of coordinate hyperplanes in $\boldsymbol{C}^{n+1}$, when we know the Milnor numbers of its restrictions to the spaces $X \in \mathscr{L}(\mathscr{A})$. It turns out that in this case the algebraic codimension is equal to $\dim H_{n-1}(F)$. We do not know if such a property holds for any arrangement $\mathscr{A}$.

For a function germ f defined on a central arrangement of hyperplanes and having an isolated singularity in 0, as considered above, the ζ function of the monodromy is

$$\zeta_f(t) = (1-t)[t^{\nu}\Delta(t^{-1})]^{(-1)^{n-1}}$$

where Δ is the characteristic polynomial of the monodromy and ν is the degree of Δ. Using the formulas in Theorem 1.1 and Proposition 3.1, we obtain:

$$\zeta_f(t) = (1-t)\left[\prod_{X \in \mathscr{L}(\mathscr{A}),\, X \neq \boldsymbol{C}^{n+1}} (t^{\mu(f|_X)}\Delta_X(t^{-1}))^{|\mu(X)|}\right]^{(-1)^{n-1}}.$$

But, for any $X \in \mathscr{L}(\mathscr{A})$, $X \neq \{0\}$, $X \neq \boldsymbol{C}^{n+1}$, we have

$$\zeta_{f|_X} = (1-t)[t^{\mu(f|_X)} \cdot \Delta_X(t^{-1})]^{(-1)^{\dim X-1}}$$

so

$$t^{\mu(f|_X)} \cdot \Delta_X(t^{-1}) = [(1-t)^{-1}\zeta_{f|_X}(t)]^{(-1)^{\dim X-1}}.$$

If $\{0\} \in \mathscr{L}(\mathscr{A})$ we put $\zeta_{f|_{\{0\}}}(t) = 1$. Consequently, by Remark 1.4 we have:

$$\begin{aligned}
\zeta_f(t) &= (1-t)\left[\prod_{X \in \mathscr{L}(\mathscr{A}), X \neq \boldsymbol{C}^{n+1}} ((1-t)^{-1}\zeta_{f|_X}(t))^{(-1)^{\dim X-1}\cdot|\mu(X)|}\right]^{(-1)^{n-1}} \\
&= (1-t) \prod_{X \in \mathscr{L}(\mathscr{A}), X \neq \boldsymbol{C}^{n+1}} \left[((1-t)^{-1}\zeta_{f|_X}(t))^{|\mu(X)|}\right]^{(-1)^{\dim X-1-n+1}} \\
&= (1-t) \prod_{X \in \mathscr{L}(\mathscr{A}), X \neq \boldsymbol{C}^{n+1}} [(1-t)^{-1}\zeta_{f|_X}]^{|\mu(X)|(-1)^{r(X)-1}} \\
&= (1-t) \prod_{X \in \mathscr{L}(\mathscr{A}), X \neq \boldsymbol{C}^{n+1}} [(1-t)\zeta_{f|_X}^{-1}(t)]^{|\mu(X)|(-1)^{r(X)}} \\
&= (1-t) \prod_{X \in \mathscr{L}(\mathscr{A}), X \neq \boldsymbol{C}^{n+1}} (1-t)^{\mu(X)}\zeta_{f|_X}^{-\mu(X)}(t) \\
&= \prod_{X \in \mathscr{L}(\mathscr{A}), X \neq \boldsymbol{C}^{n+1}} \zeta_{f|_X}^{-\mu(X)}(t).
\end{aligned}$$

Thus, we proved

PROPOSITION 3.4. *Under the hypotheses of Theorem* 1.1, *the ζ function of the monodromy of f is*

$$\zeta_f(t) = \prod_{X \in \mathscr{L}(\mathscr{A}), X \neq \boldsymbol{C}^{n+1}} \zeta_{f|_X}^{-\mu(X)}(t).$$

For $X \in \mathscr{L}(\mathscr{A})$, let us denote by Λ_X the Lefschetz number of the monodromy of $f|_X$. If $\{0\} \in \mathscr{L}(\mathscr{A})$ we put $\Lambda_{\{0\}} = 0$. The Weil inversion formula and Proposition 3.4 imply

PROPOSITION 3.5. *Under the conditions in Theorem* 1.1, *the Lefschetz number of the monodromy is*

$$\text{(6)} \qquad \Lambda(h) = 1 + \sum_{X \in \mathscr{L}(\mathscr{A}), X \neq \boldsymbol{C}^{n+1}} (1-\Lambda_X)\mu(X) = -\sum_{X \in \mathscr{L}(\mathscr{A}), X \neq \boldsymbol{C}^{n+1}} \Lambda_X \cdot \mu(X).$$

Let $(X, 0)$ be the germ of a smooth analytic space, $m_{X,0}$ the maximal ideal of the local ring $\mathcal{O}_{X,0}$ and f a function germ defined on X. In [1] N. A'Campo proves that for the monodromy h of f, we have $\Lambda(h) = 0$ if $f \in m_{X,0}^2$ and $\Lambda(h) = 1$ if $f \in m_{X,0} \backslash m_{X,0}^2$. Because the subspaces $X \in \mathscr{L}(\mathscr{A})$ are smooth, the formula (6) implies that to compute $\Lambda(h)$ we need only the Lefschetz numbers of those restrictions of f to $X \in \mathscr{L}(\mathscr{A})$ for which $f|_X \in m_{X,0}$. Proposition 3.5 and A'Campo's result imply the following

Corollary 3.6. *Let $(X, 0)$ be the germ in 0 of a central hyperplane arrangement in $\boldsymbol{C}^{n+1}$, let $m_{X,0}$ be the maximal ideal of the local ring $\mathcal{O}_{X,0}$ and let f be a germ of function defined on $(X, 0)$. If $f \in m_{X,0}^2$ then $\Lambda(h_f) = 0$.*

Example 3.7. Let $\mathscr{A}$ be the arrangement of all coordinate hyperplanes in $\boldsymbol{C}^{n+1}$. Its defining ideal is $I = (x_1 \cdots x_{n+1})$ and the maximal ideal $\bar{m}$, of the local ring of this germ of analytic space is the image of the ideal $m = (x_1, \ldots, x_{n+1})$ in $\mathcal{O}_{n+1}/I$. Let $f \in \bar{m} \backslash \bar{m}^2$ be a function germ defined on $\mathscr{A}$. Then, by [7], Proposition 4.1, we have: Either f is $\mathscr{R}$-equivalent to $\overline{x_1 + \cdots + x_{n+1}}$, in which case

$$\Lambda(h_f) = 1 + \mu(\{0\}) = 1 + (-1)^{n+1},$$

or f is $\mathscr{R}$-equivalent to $x_1 + \cdots + x_k + h(x_{k+1}, \ldots, x_{n+1})$ for some $k \in \{1, \ldots, n\}$ and h with $j^1 h = 0$. In this situation we identify $\{(x_1, \ldots, x_{n+1}) \in \boldsymbol{C}^{n+1}) \mid x_1 = \cdots = x_k = 0\}$ with $\boldsymbol{C}^{n-k+1}$. The elements $X \in \mathscr{L}(\mathscr{A})$ which intervene in the computation of the Lefschetz number of the monodromy of f are in fact the elements of the intersection poset of the arrangement of all coordinate hyperplanes in $\boldsymbol{C}^{n-k+1}$. Let us denote the set of these elements by $\mathscr{L}_k$. For $X \in \mathscr{L}_k$, the value of $\mu(X)$ is equal to the value of the Möbius function of the arrangement of all coordinate hyperplanes in $\boldsymbol{C}^{n-k+1}$. Using [4], Proposition 2.44, we get

$$\Lambda(h_f) = 1 + \sum_{X \in \mathscr{L}_k} \mu(X) = 1.$$

Thus, in this example the Lefschetz number of the monodromy of f can take the values 0, 1 or 2.

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