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EHRESMANN CONNECTIONS FOR A FOLIATED MANIFOLD AND CERTAIN KINDS OF RECTANGLES WITHOUT TERMINAL VERTEX

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Abstract

We define the notion of a non-extendable rectangle without terminal vertex for a foliated manifold $(M, \mathfrak{F})$ with a complementary distribution D and classify them into non-singular ones and singular ones. It is easy to show that D is an Ehresmann connection in the sense of R. A. Blumenthal and J. J. Hebda if and only if there is no non-extendable rectangle without terminal vertex. One of our purposes is to investigate the existence of singular non-extendable rectangle without terminal vertex. Another purpose is to obtain a new sufficient condition for the orthogonal complementary distribution of a foliation on a Riemannian manifold to be an Ehresmann connection by investigating a property of singular non-extendable rectangles without terminal vertex.

Introduction

Throughout this paper, unless otherwise mentioned, we assume that all objects are smooth (i.e., of class C^∞) and all manifolds are connected ones without boundary. For a foliated manifold $(M, \mathfrak{F})$ with a complementary distribution D, R. A. Blumenthal and J. J. Hebda considered a piecewise smooth map $\delta : [0,1] \times [0,1] \to M$ such that, for every fixed s_0, the curve $\delta_{\cdot s_0} := \delta(\cdot, s_0)$ is a horizontal curve, and, for every fixed t_0, the curve $\delta_{t_0} := \delta(t_0, \cdot)$ is a vertical curve, where a horizontal curve is a piecewise smooth map from $[0,1]$ to M whose velocity vector field lies in D and a vertical curve is a piecewise smooth map from $[0,1]$ to a leaf of $\mathfrak{F}$. They called such a piecewise smooth map δ a *rectangle*. If, for every vertical curve α and every horizontal curve β with $\alpha(0) = \beta(0)$, there is the rectangle δ with $\delta_{0\cdot} = \alpha$ and $\delta_{\cdot 0} = \beta$, then they called D an *Ehresmann connection for* $\mathfrak{F}$ (see [2]). They proved the following so-called global stability theorem and decomposition theorem (see [2]):

(i) *If $\mathfrak{F}$ admits an Ehresmann connection, then the universal coverings of leaves of $\mathfrak{F}$ are diffeomorphic to one another.*

(ii) *If D is an integrable Ehresmann connection for $\mathfrak{F}$, then for each $p \in M$, there is a covering map π of the product manifold $\hat{L}_p^V \times \hat{L}_p^H$ onto M satisfying $\pi_*(T\hat{L}_p^V) = F$ and $\pi_*(T\hat{L}_p^H) = D$, where $\hat{L}_p^V$ (resp. $\hat{L}_p^H$) is the universal covering of*

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a leaf of $\mathfrak{F}$ *through* p (*resp. that of the maximal integral manifold of* D *through* p), π_* *is the differential of* π, $T\hat{L}_p^V$ (*resp.* $T\hat{L}_p^H$) *is the tangent bundle of the foliation* $\hat{L}_p^V \times \{\cdot\}$ (*resp.* $\{\cdot\} \times \hat{L}_p^H$) *on* $\hat{L}_p^V \times \hat{L}_p^H$ *and* F *is the tangent bundle of* $\mathfrak{F}$.

Thus we can apply the study of an Ehresmann connection to those of the global stability of a foliation and the decomposition of a manifold into a product manifold (furthermore, the geometric decomposition of a manifold with a geometric structure). Therefore, it is very interesting to investigate what kind of foliation admits an Ehresmann connection.

In this paper, we consider a piecewise smooth map $\delta : [0,1]\times[0,1]\backslash\{(1,1)\} \to M$ such that, for every fixed $s_0 \in [0,1)$, the curve $\delta_{\cdot s_0}$ is a horizontal curve, for every fixed $t_0 \in [0,1)$, the curve δ_{t_0} is a vertical curve and $\delta_{\cdot 1}$ (resp. δ_1) is a horizontal (resp. vertical) curve without terminal point. We shall call such a piecewise smooth map δ a *rectangle without terminal vertex*. If there is not a rectangle $\tilde{\delta}$ satisfying $\tilde{\delta}|_{([0,1]\times[0,1]\backslash\{(1,1)\})} = \delta$, then we shall say that δ *is non-extendable*. By imitating the proof of Proposition 2.3 of [13], it is shown that D is an Ehresmann connection for $\mathfrak{F}$ if and only if there is no non-extendable rectangle without terminal vertex. Thus the study of a non-extendable rectangle without terminal vertex leads to that of an Ehresmann connection. According to Lemma 3.5 of [11], if δ is a non-extendable rectangle without terminal vertex, then $\lim_{s\to 1-0}\delta(1,s)$ does not exist. However, $\lim_{t\to 1-0}\delta(t,1)$ is possible to exist. If $\lim_{t\to 1-0}\delta(t,1)$ exists (resp. does not exist), then we shall say that δ is *singular* (resp. *non-singular*).

Remark. If δ is singular, then a continuous curve $c : [0,1] \to M$ defined by

$$c(t) := \begin{cases} \delta(t,1) & (0 \le t < 1) \\ \lim\limits_{t\to 1-0} \delta(t,1) & (t = 1) \end{cases}$$

is not of class C^1 at $t = 1$. In fact, it is shown in terms of a foliated coordinate neighbourhood about $c(1)$ that δ is extendable if c is of class C^1 at $t = 1$.

If $\operatorname{codim}\mathfrak{F} = 1$, that is, $\dim D = 1$, then D is integrable and hence all non-extendable rectangles without terminal vertex are non-singular. It is very important to investigate the existence of a singular non-extendable rectangle without terminal vertex in case of $\operatorname{codim}\mathfrak{F} \ge 2$. In this paper, we shall prove the following result related to its existence.

THEOREM 1. *For every* $r \ge 3$ *and every* $n \ge r+1$, *there is a triple* $(M, \mathfrak{F}, D)$ *of an n-dimensional manifold* M, *a foliation* $\mathfrak{F}$ *of codimension* r *on* M *and a complementary distribution* D *to* $\mathfrak{F}$ *which admits a singular non-extendable rectangle without terminal vertex.*

It is natural to ask what kind of foliation admits an Ehresmann connection on a manifold with a geometric structure. On a Riemannian manifold, such a study has been already done by some geometers as follows. Let $F^\perp$ be the orthogonal complementary distribution of a foliation $\mathfrak{F}$ on a Riemannian manifold (M, g). It is known that $F^{\perp}$ is an Ehresmann connection if one of the following conditions holds (see [2], [4], [8], [15]):

(I) (M, g) is complete and g is bundle-like for $\mathfrak{F}$,

(II) the induced Riemannian metrics on leaves of $\mathfrak{F}$ are complete and $\mathfrak{F}$ is totally geodesic,

(III) $\dim \mathfrak{F} \geq 3$, the induced conformal structures on leaves of $\mathfrak{F}$ are complete and $\mathfrak{F}$ is totally umbilic.

In [7], for each vertical curve α, we defined a function $G_\alpha^{\perp}$ on the set $\mathrm{Rec}(\alpha, \cdot)$ of all rectangles δ such that $\delta_{0\cdot} = \alpha$ and $\delta_{\cdot 0}$ is a regular curve by $G_\alpha^{\perp}(\delta) := l(\delta_{\cdot 1})/l(\delta_{\cdot 0})$ for $\delta \in \mathrm{Rec}(\alpha, \cdot)$, where $l(\cdot)$ is the length of a curve $\cdot$ with respect to g. Also, for each horizontal curve β, we defined a function G_β^T on the set $\mathrm{Rec}(\cdot, \beta)$ of all rectangles δ such that $\delta_{\cdot 0} = \beta$ and $\delta_{0\cdot}$ is a regular curve by $G_\beta^T(\delta) := l(\delta_{1\cdot})/l(\delta_{0\cdot})$ for $\delta \in \mathrm{Rec}(\cdot, \beta)$. In the paper, we proved that $F^{\perp}$ is an Ehresmann connection if one of the following conditions holds:

(IV) (M, g) is complete and $\sup_{s \in [0,1)} \sup G_{\alpha|_{[0,s]}}^{\perp} < \infty$ for every vertical curve $\alpha : [0, 1) \to M$ without terminal point,

(V) the induced Riemannian metrics on leaves of $\mathfrak{F}$ are complete and $\sup G_\beta^T < \infty$ for every horizontal curve β.

Here we note that $G_\alpha^{\perp} \equiv 1$ holds for every vertical curve α if g is bundle-like for $\mathfrak{F}$ and that $G_\beta^T \equiv 1$ holds for every horizontal curve β if $\mathfrak{F}$ is totally geodesic. Thus these results are generalizations of I and II above. In this paper, we shall furthermore improve one of these results as follows.

THEOREM 2. *If* (M, g) *is complete and* $\sup G_\alpha^{\perp} < \infty$ *for every vertical curve* α, *then* $F^{\perp}$ *is an Ehresmann connection.*

This theorem will be proved by investigating a property of singular non-extendable rectangles without terminal vertex. We shall also give examples showing that this improvement is essential (see §3). Furthermore, we shall give examples showing the topological gap between foliations admitting a Riemannian metric such that $\sup G_\alpha^{\perp} < \infty$ for every vertical curve α and foliations admitting a bundle-like metric (see §3). The following corollary is directly deduced from Theorem 2 and the sufficient condition (V) for $F^{\perp}$ to be an Ehresmann connection.

COROLLARY. *Let* $\mathfrak{F}$ *be a foliation on a Riemannian manifold* (M, g) *satisfying the above condition* (V) *or the assumption of Theorem* 2. *Then the following statements* (i) *and* (ii) *hold*:

(i) *The universal coverings of leaves of* $\mathfrak{F}$ *are diffeomorphic to one another.*

(ii) *If* $\operatorname{codim} \mathfrak{F} = 1$, *then the universal covering of* M *is diffeomorphic to* $\hat{L}_p^V \times \boldsymbol{R}$, *where* p *is an arbitrary point of* M *and* $\hat{L}_p^V$ *is the universal covering of the leaf of* $\mathfrak{F}$ *through* p.

In §1 and §2, we prove Theorems 1 and 2, respectively. In §3, we give examples of a non-extendable rectangle without terminal vertex and those of a foliated Riemannian manifold which satisfies the assumption of Theorem 2 but does not satisfy the condition (IV). Furthermore, we give examples of a foliated manifold which admits a Riemannian metric satisfying the assumption of Theorem 2 but does not admit a bundle-like metric.

1. Proof of Theorem 1

In this section, we shall prove Theorem 1 by constructing a triple $(M, \mathfrak{F}, D)$ admitting a singular non-extendable rectangle without terminal vertex. First we shall present a plan of construction of such a triple $(M, \mathfrak{F}, D)$.

Plan of Construction. Let $(x_1, \ldots, x_n)$ be the natural coordinate of an n-dimensional affine space $\boldsymbol{R}^n$ and $\mathfrak{F}$ a foliation on $\boldsymbol{R}^n$ whose leaves are fibres of the projection $\pi : \boldsymbol{R}^n \to \boldsymbol{R}^r$ defined by $\pi(x_1, \ldots, x_n) = (x_1, \ldots, x_r)$ $(r \geq 2,\, n \geq r+1)$.

(Step I) First we construct a complementary (C^∞-)distribution D_1 to $\mathfrak{F}$, a C^∞-curve $\beta = (\beta_1, \ldots, \beta_r) : [0,1) \to \boldsymbol{R}^r$ without terminal point and a C^∞-curve $\alpha : [0,1] \to \pi^{-1}(\beta(0))$ satisfying the following conditions:

(i) $\lim_{t \to 1-0} \beta(t)$ exists and a continuous curve $\bar{\beta} : [0,1] \to \boldsymbol{R}^r$ defined by

$$\bar{\beta}(t) := \begin{cases} \beta(t) & (0 \leq t < 1) \\ \lim\limits_{t \to 1-0} \beta(t) & (t = 1) \end{cases}$$

is not of class C^1 at $t = 1$.

(ii) For every $s \in [0,1]$, there is the D_1-lift $\tilde{\beta}_s : [0,1) \to \boldsymbol{R}^n$ of β starting from $\alpha(s)$.

(iii) For every $s \in [0,1)$, $\lim_{t \to 1-0} \sum_{i=r+1}^{n} x_i(\tilde{\beta}_s(t))^2 = \infty$.

(iv) $\lim_{t \to 1-0} \tilde{\beta}_1(t)$ exists.

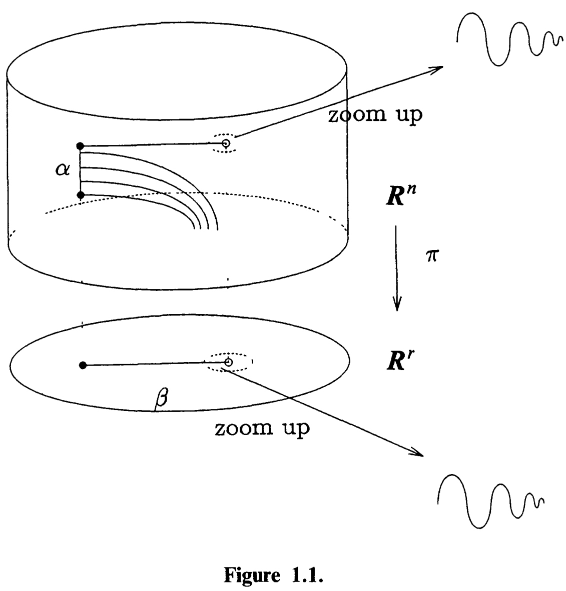


Figure 1.1.

(Step II) Next, we construct a homeomorphism ϕ of $\boldsymbol{R}^n$ which admits closed sets S_1 and S_2 of $\boldsymbol{R}^n$ satisfying the following conditions:

(v) $\phi|_{\boldsymbol{R}^n \setminus (S_1 \cup S_2)}$ is a C^∞-diffeomorphism.

(vi) $(S_1 \cup S_2) \cap (\pi^{-1}(\beta([0,1))) \cup \{\lim_{t\to 1-0} \tilde{\beta}_1(t)\}) = \emptyset$ and $S_2 \cap \pi^{-1}(\lim_{t\to 1-0} \beta(t)) \neq \emptyset$.

(vii) Let $\gamma = (\gamma_1, \dots, \gamma_{n-r}) : [0,1] \to \boldsymbol{R}^{n-r}$ be an arbitrary C^∞-curve in $\boldsymbol{R}^{n-r}$ with $\gamma(1) \in \pi'(S_2 \cap \pi^{-1}(\lim_{t\to 1-0} \beta(t)))$ and β_γ a continuous curve in $\boldsymbol{R}^n$ defined by

$$\beta_\gamma(t) := \begin{cases} (\beta_1(t), \dots, \beta_r(t), \gamma_1(t), \dots, \gamma_{n-r}(t)) & (0 \le t < 1) \\ \left(\lim\limits_{t\to 1-0} \beta_1(t), \dots, \lim\limits_{t\to 1-0} \beta_r(t), \gamma_1(1), \dots, \gamma_{n-r}(1)\right) & (t = 1), \end{cases}$$

where π' is the projection of $\boldsymbol{R}^n$ onto $\boldsymbol{R}^{n-r}$ defined by $\pi'(x_1, \dots, x_n) = (x_{r+1}, \dots, x_n)$. Then $\phi \circ \beta_\gamma$ is of class C^∞ over $[0,1]$.

(viii) Give $\boldsymbol{R}^n \setminus S_1$ a C^∞-structure $\{(\boldsymbol{R}^n \setminus S_1, \phi|_{\boldsymbol{R}^n \setminus S_1})\}$. Denote this C^∞-manifold by M. Then $\mathfrak{F}$ becomes a (C^∞-)foliation on M.

(Step III) Furthermore, we construct a complementary (C^∞-)distribution D to $\mathfrak{F}$ on M satisfying the following conditions:

(ix) $D = D_1$ on a neighbourhood of $\tilde{\beta}_1([0,1)) \cup \{\lim_{t\to 1-0} \tilde{\beta}_1(t)\}$.

(x) For every $s \in [0,1)$, there is the D-lift $\beta_s^L : [0,1) \to M$ of β starting from $\alpha(s)$.

(xi) For every $s \in [0,1)$, $\lim_{t\to 1-0} \beta_s^L(t)$ exists and $\lim_{t\to 1-0} \pi'(\beta_s^L(t))$ belongs to $\pi'(S_2 \cap \pi^{-1}(\lim_{t\to 1-0} \beta(t)))$.

(xii) Let $\widehat{\pi' \circ \beta_s^L} : [0,1] \to \boldsymbol{R}^{n-r} (s \in [0,1))$ be a continuous curve in $\boldsymbol{R}^{n-r}$ defined by

$$(\widehat{\pi' \circ \beta_s^L})(t) := \begin{cases} (\pi' \circ \beta_s^L)(t) & (0 \le t < 1) \\ \lim\limits_{t\to 1-0} (\pi' \circ \beta_s^L)(t) & (t = 1). \end{cases}$$

Then $\widehat{\pi' \circ \beta_s^L}$ becomes a C^∞-curve for every $s \in [0,1)$.

Then we define a map $\delta : ([0,1] \times [0,1] \setminus \{(1,1)\}) \to M$ by

$$\delta(t,s) := \begin{cases} \beta_s^L(t) & (0 \le t < 1,\ 0 \le s < 1) \\ \lim\limits_{t\to 1-0} \beta_s^L(t) & (t = 1,\ 0 \le s < 1) \\ \tilde{\beta}_1(t) & (0 \le t < 1,\ s = 1). \end{cases}$$

It follows from the definition of δ that $\delta_{\cdot s} (s \in [0,1))$ are given by

$$\delta_{\cdot s}(t) = \begin{cases} \beta_s^L(t) & (0 \le t < 1) \\ \lim\limits_{t\to 1-0} \beta_s^L(t) & (t = 1), \end{cases}$$

which is a C^∞-curve in M by the conditions (vii), (xi) and (xii). Hence

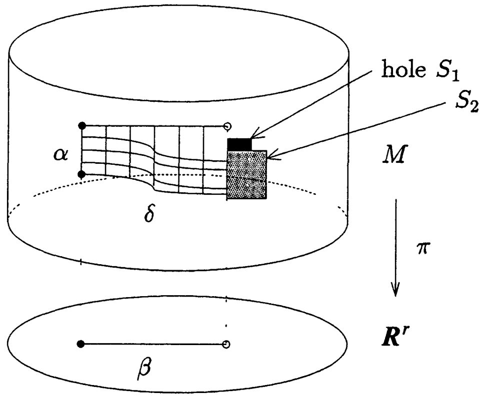


Figure 1.2.

δ_s ($s \in [0,1)$) are horizontal curves (with respect to D). Also, it follows from the conditions (v), (vi) and (ix) that $\tilde{\beta}_1 (= \delta_{\cdot 1})$ is a horizontal curve (with respect to D) without terminal point. These facts imply that δ is a rectangle without terminal vertex on $(M, \mathfrak{F}, D)$. By the condition (iv), $\lim_{t \to 1-0} \delta(t,1)$ exists. Moreover, by the conditions (i), (v) and (vi), a continuous curve $\bar{\delta}_{\cdot 1} : [0,1] \to M$ defined by

$$\bar{\delta}_{\cdot 1}(t) = \begin{cases} \delta_{\cdot 1}(t) & (0 \le t < 1) \\ \lim_{t \to 1-0} \delta(t,1) & (t = 1) \end{cases}$$

is not of class C^1 at $t = 1$. This fact implies that δ is non-extendable and singular. Thus this triple $(M, \mathfrak{F}, D)$ admits a singular non-extendable rectangle δ without terminal vertex.

Proof of Theorem 1. Following to the above plan of construction, we shall concretely construct a triple $(M, \mathfrak{F}, D)$ which admits a singular non-extendable rectangle without terminal vertex in case of $r \ge 3$. Let $\mathfrak{F}$, $(x_1, \ldots, x_n)$, π and π' be as above. First we define a complementary (C^∞-)distribution D_1 to $\mathfrak{F}$ on $\boldsymbol{R}^n$, a C^∞-curve $\beta = (\beta_1, \ldots, \beta_r) : [0,1) \to \boldsymbol{R}^r$ without terminal point and a C^∞-curve $\alpha : [0,1] \to \pi^{-1}(\beta(0))$ by

$$D_1 := \operatorname{Span}\left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} + x_3 x_n \frac{\partial}{\partial x_n}, \frac{\partial}{\partial x_3} - x_2 x_n \frac{\partial}{\partial x_n}, \frac{\partial}{\partial x_4}, \ldots, \frac{\partial}{\partial x_r} \right\},$$

$$\beta(t) := \left(t-1, (t-1)\sin\frac{1}{(t-1)^2}, (t-1)\cos\frac{1}{(t-1)^2}, 0, \ldots, 0 \right) \quad (0 \le t < 1),$$

$$\alpha(s) := (-1, -\sin 1, -\cos 1, 0, \ldots, 0, s-1),$$

respectively. Clearly β satisfies the condition (i). The D_1-lift $\tilde{\beta}_s$ of β starting from $\alpha(s)$ is given by

$$\tilde{\beta}_s(t) = \left(t-1, (t-1)\sin\frac{1}{(t-1)^2}, (t-1)\cos\frac{1}{(t-1)^2}, 0, \dots, 0, \frac{s-1}{(1-t)^2}\right),$$

where $s \in [0,1]$. Hence $\tilde{\beta}_s(s \in [0,1))$ is defined over $[0,1)$. Furthermore, we have

$$\lim_{t\to 1-0} \sum_{i=r+1}^{n} x_i(\tilde{\beta}_s(t))^2 = \lim_{t\to 1-0} \frac{(s-1)^2}{(1-t)^4} = \infty \quad (s \in [0,1)),$$

$$\lim_{t\to 1-0} \tilde{\beta}_1(t) = (0,\dots,0).$$

Thus the conditions (ii)–(iv) hold.

Next we define a homeomorphism $\phi = (y_1,\dots,y_n)$ of $\boldsymbol{R}^n$ by

$$\phi(x_1,\dots,x_n) := (x_1, \mu(2x_1+x_2, \lambda(x_n)), \mu(2x_1+x_3, \lambda(x_n)), x_4, \dots, x_n),$$

where μ is a C^∞-function over $\boldsymbol{R}^2$ defined by

$$\mu(z,w) := \begin{cases} z \cdot e^{-(w^2/|z|)} & (z \neq 0) \\ 0 & (z = 0) \end{cases}$$

and λ is a C^∞-function over $\boldsymbol{R}$ with $\lambda^{-1}(0) = [-(1/2), \infty)$, $\lambda^{-1}(1) = (-\infty, -1]$ and $0 \leq \lambda \leq 1$. Now we shall show that ϕ admits closed sets S_i $(i = 1,2)$ of $\boldsymbol{R}^n$ satisfying the above conditions (v)–(viii). Define closed sets S_i $(i = 1,2)$ of $\boldsymbol{R}^n$ by

$$S_1 := \left\{(x_1,\dots,x_n) \mid (2x_1+x_2)(2x_1+x_3) = 0 \text{ and } -1 \leq x_n \leq -\frac{1}{2}\right\}$$

and

$$S_2 := \{(x_1,\dots,x_n) \mid (2x_1+x_2)(2x_1+x_3) = 0 \text{ and } x_n \leq -1\},$$

respectively. Clearly S_1 and S_2 satisfy the conditions (v) and (vi). Take an arbitrary C^∞-curve $\gamma = (\gamma_1,\dots,\gamma_{n-r}) : [0,1] \to \boldsymbol{R}^{n-r}$ with $\gamma(1) \in \pi'(S_2 \cap \pi^{-1}(\lim_{t\to 1-0}\beta(t)))$. Let β_γ be a continuous curve defined as in (vii). We must show that $\phi \circ \beta_\gamma$ is of class C^∞ over $[0,1]$. Since $\gamma(1) \in \pi'(S_2 \cap \pi^{-1}(\lim_{t\to 1-0}\beta(t)))$ and hence $\gamma_{n-r}(1) < -1$, we see that $\gamma_{n-r} < -1$ holds over $(1-\varepsilon, 1]$ for a sufficiently small $\varepsilon > 0$. Hence we have $\lambda \circ \gamma_{n-r} = 1$ over $(1-\varepsilon, 1]$, that is,

$$(\phi \circ \beta_\gamma)(t) = \begin{cases} \left(t-1, (t-1)\left(2+\sin\frac{1}{(t-1)^2}\right)e^{-1/|(t-1)(2+\sin(1/(t-1)^2))|},\right. \\ (t-1)\left(2+\cos\frac{1}{(t-1)^2}\right)e^{-1/|(t-1)(2+\cos(1/(t-1)^2))|}, \\ \left. 0,\dots,0,\gamma_1(t),\dots,\gamma_{n-r}(t)\right) & (1-\varepsilon < t < 1) \\ (0,\dots,0,\gamma_1(1),\dots,\gamma_{n-r}(1)) & (t=1). \end{cases}$$

This implies that $\phi \circ \beta_y$ is of class C^∞ over $(1-\varepsilon, 1]$ and hence so is it over $[0, 1]$. Thus S_1 and S_2 satisfy the condition (vii). Since $\phi|_{\boldsymbol{R}^n \setminus (S_1 \cup S_2)}$ is a C^∞-diffeomorphism, $\mathfrak{F}$ becomes a foliation on $M \setminus S_2$. Let $\mathfrak{F}_1$ be a foliation on M whose leaves are the fibres of the projection $\pi_1 : M \to \boldsymbol{R}^r$ defined by $\pi_1(x_1, \ldots, x_n) = (y_1(x_1, \ldots, x_n), \ldots, y_r(x_1, \ldots, x_n))$ $((x_1, \ldots, x_n) \in M)$. Set $W := \{(x_1, \ldots, x_n) \in M \mid x_n < -1\}$. On W, $\phi(x_1, \ldots, x_n) = (x_1, \mu(2x_1 + x_2, 1), \mu(2x_1 + x_3, 1), x_4, \ldots, x_n)$ holds. This implies that $\mathfrak{F} = \mathfrak{F}_1$ on W. Therefore, $\mathfrak{F}$ becomes a foliation on $(M \setminus S_2) \cup W = M$. Thus S_1 and S_2 satisfy the condition (viii).

Next we shall construct a complementary distribution D to $\mathfrak{F}$ on M satisfying the conditions (ix)–(xii). Let $\{U_1, U_2\}$ be an open covering of M defined by

$$
\begin{aligned}
U_1 &:= \{(x_1, \ldots, x_n) \in M \mid x_n \geq -1 \quad \text{or} \\
&\qquad (-2 < x_n < -1 \text{ and } (2x_1 + x_2)(2x_1 + x_3) \neq 0)\}, \\
U_2 &:= \{(x_1, \ldots, x_n) \in M \mid x_n < -1\}
\end{aligned}
$$

and $\{\eta_1, \eta_2\}$ a partition of unity subordinating to $\{U_1, U_2\}$. Set

$$
\begin{aligned}
X_1 &:= \frac{\partial}{\partial y_1} + \eta_1 \left(\frac{\partial y_2}{\partial x_1} \frac{\partial}{\partial y_2} + \frac{\partial y_3}{\partial x_1} \frac{\partial}{\partial y_3} \right), \\
X_2 &:= \left(\eta_1 \frac{\partial y_2}{\partial x_2} + \eta_2 \right) \frac{\partial}{\partial y_2} + \eta_1 x_3 x_n \left(\frac{\partial y_2}{\partial x_n} \frac{\partial}{\partial y_2} + \frac{\partial y_3}{\partial x_n} \frac{\partial}{\partial y_3} + \frac{\partial}{\partial y_n} \right) \\
&\quad - \frac{\eta_1 \eta_2 x_3 x_n}{\eta_1 \frac{\partial y_3}{\partial x_3} + \eta_2} \frac{\partial}{\partial y_n}, \\
X_3 &:= \left(\eta_1 \frac{\partial y_3}{\partial x_3} + \eta_2 \right) \frac{\partial}{\partial y_3} - \eta_1 x_2 x_n \left(\frac{\partial y_2}{\partial x_n} \frac{\partial}{\partial y_2} + \frac{\partial y_3}{\partial x_n} \frac{\partial}{\partial y_3} + \frac{\partial}{\partial y_n} \right) \\
&\quad + \frac{\eta_1 \eta_2 x_2 x_n}{\eta_1 \frac{\partial y_2}{\partial x_2} + \eta_2} \frac{\partial}{\partial y_n}.
\end{aligned}
$$

Since $x_i, \partial y_i/\partial x_1, \partial y_i/\partial x_i, \partial y_i/\partial x_n$ $(i = 2, 3)$ are C^∞-functions on $M \setminus S_2$ and $\eta_1 = 0$ on a neighbourhood of S_2, we see that $\eta_1 x_i$, $\eta_1(\partial y_i/\partial x_1)$, $\eta_1(\partial y_i/\partial x_i)$, $\eta_1(\partial y_i/\partial x_n)$ $(i = 2, 3)$ are C^∞-functions on M. Also, it is clear that x_n is a C^∞-function on M. Furthermore, for $i = 2, 3$, we have

$$
(1.1) \qquad \frac{\partial y_i}{\partial x_i} = \begin{cases} \left(1 + \dfrac{\lambda(x_n)}{|2x_1 + x_i|} \right) \cdot e^{-\lambda(x_n)/|2x_1 + x_i|} & (2x_1 + x_i \neq 0) \\ 1 & \left(2x_1 + x_i = 0,\ x_n \geq -\dfrac{1}{2} \right) \\ 0 & \left(2x_1 + x_i = 0, x_n < -\dfrac{1}{2} \right) \end{cases}
$$

and hence $\eta_1(\partial y_i/\partial x_i)+\eta_2>0$ $(i=2,3)$ hold on M. Thus X_i $(i=1,2,3)$ are C^∞-vector fields on M and furthermore $(X_1, X_2, X_3, \partial/\partial y_4, \dots, \partial/\partial y_n)$ is a frame field on M. Define a $(C^\infty\text{-})$distribution D on M by $D := \mathrm{Span}\{X_1, X_2, X_3, \partial/\partial y_4, \dots, \partial/\partial y_r\}$. Since the tangent bundle of $\mathfrak{F}$ is given by $\mathrm{Span}\{\partial/\partial y_{r+1}, \dots, \partial/\partial y_n\}$, we see that D is a complementary distribution to $\mathfrak{F}$. First we shall show that D satisfies the condition (ix). Since $\phi|_{\boldsymbol{R}^n\setminus(S_1\cup S_2)}$ is a C^∞-diffeomorphism, $\partial/\partial x_i = \sum_{k=1}^n (\partial y_k/\partial x_i)(\partial/\partial y_k)$ $(1\le i\le n)$ hold on $M\setminus S_2$. In more detail, we can obtain

$$
(1.2)\qquad
\begin{aligned}
&\frac{\partial}{\partial x_1}=\frac{\partial}{\partial y_1}+\frac{\partial y_2}{\partial x_1}\frac{\partial}{\partial y_2}+\frac{\partial y_3}{\partial x_1}\frac{\partial}{\partial y_3},\\
&\frac{\partial}{\partial x_2}=\frac{\partial y_2}{\partial x_2}\frac{\partial}{\partial y_2},\quad \frac{\partial}{\partial x_3}=\frac{\partial y_3}{\partial x_3}\frac{\partial}{\partial y_3},\quad \frac{\partial}{\partial x_i}=\frac{\partial}{\partial y_i}\quad (4\le i\le r)
\end{aligned}
$$

and

$$
(1.3)\qquad \frac{\partial}{\partial x_j}=\frac{\partial}{\partial y_j}\quad (r+1\le j\le n-1),\quad \frac{\partial}{\partial x_n}=\frac{\partial y_2}{\partial x_n}\frac{\partial}{\partial y_2}+\frac{\partial y_3}{\partial x_n}\frac{\partial}{\partial y_3}+\frac{\partial}{\partial y_n}
$$

on $M\setminus S_2$. Since $\eta_1=1$ and $\eta_2=0$ on $M\setminus U_2$, we have $X_1=\partial/\partial x_1$, $X_2=\partial/\partial x_2 + x_3x_n(\partial/\partial x_n)$ and $X_3=\partial/\partial x_3 - x_2x_n(\partial/\partial x_n)$ on $M\setminus U_2$. Hence $D=D_1$ holds on $M\setminus U_2$. Since $M\setminus U_2$ is a neighbourhood of $\tilde{\beta}_1([0,1))\cup\{\lim_{t\to1-0}\tilde{\beta}(t)\}$, D satisfies the condition (ix). Next we shall show that D satisfies the conditions (x)–(xii). Let β_s^L (resp. $\tilde{\beta}_s$) be the D-lift (resp. the D_1-lift) of β starting from $\alpha(s)$. Fix $s\in[0,1)$. Set $t_0 := \sup\{t \mid \beta_s^L$ is defined over $[0,t]\}$. Set $t_1 := \sup\{t\in[0,t_0) \mid \beta_s^L([0,t])\subset M\setminus U_2\}$. Since $D=D_1$ on $M\setminus U_2$, we have $\beta_s^L=\tilde{\beta}_s$ on $[0,t_1)$. From $\lim_{t\to1-0} x_n(\tilde{\beta}_s(t)) = \lim_{t\to1-0}(s-1)/(1-t) = -\infty$, we have $t_1<t_0$. We can express β_s^L as

$$
\begin{aligned}
\beta_s^L(t)=\Bigg(&t-1,(t-1)\sin\frac{1}{(t-1)^2},(t-1)\cos\frac{1}{(t-1)^2},\\
&0,\dots,0,x_{r+1}(\beta_s^L(t)),\dots,x_n(\beta_s^L(t))\Bigg)\quad (t\in[0,t_0)).
\end{aligned}
$$

Then we have

$$
(1.4)\qquad
\begin{aligned}
\dot{\beta}_s^L(t)=\frac{\partial}{\partial x_1}&+\left(\sin\frac{1}{(t-1)^2}-\frac{2}{(t-1)^2}\cos\frac{1}{(t-1)^2}\right)\frac{\partial}{\partial x_2}\\
&+\left(\cos\frac{1}{(t-1)^2}+\frac{2}{(t-1)^2}\sin\frac{1}{(t-1)^2}\right)\frac{\partial}{\partial x_3}\\
&+\sum_{i=r+1}^n\frac{d(x_i\circ\beta_s^L)}{dt}\frac{\partial}{\partial x_i}\quad (t\in[0,t_0)).
\end{aligned}
$$

Set $I := \{t\in[0,t_0) \mid \beta_s^L(t)\in\bar{U}_2\}$. It follows from $y_i=\mu(2x_1+x_i,\lambda(x_n))$ $(i=2,3)$

and $(\lambda \circ x_n)|_{\bar{U}_2} = 1$ that

$$(1.5)\qquad \left(\frac{\partial y_2}{\partial x_n} \circ \beta_s^L\right)(t) = \left(\frac{\partial y_3}{\partial x_n} \circ \beta_s^L\right)(t) = 0 \quad (t \in I).$$

This together with (1.2), (1.3) and (1.4) deduces

$$(1.6)\quad \dot{\beta}_s^L(t) = \frac{\partial}{\partial y_1} + \left\{ \frac{\partial y_2}{\partial x_1}\bigg|_{\beta_s^L(t)} + \frac{\partial y_2}{\partial x_2}\bigg|_{\beta_s^L(t)} \times \left(\sin\frac{1}{(t-1)^2} - \frac{2}{(t-1)^2}\cos\frac{1}{(t-1)^2}\right)\right\}\frac{\partial}{\partial y_2}$$
$$+ \left\{ \frac{\partial y_3}{\partial x_1}\bigg|_{\beta_s^L(t)} + \frac{\partial y_3}{\partial x_3}\bigg|_{\beta_s^L(t)} \left(\cos\frac{1}{(t-1)^2} + \frac{2}{(t-1)^2}\sin\frac{1}{(t-1)^2}\right)\right\}\frac{\partial}{\partial y_3}$$
$$+ \sum_{i=r+1}^{n} \frac{d(x_i \circ \beta_s^L)}{dt}\frac{\partial}{\partial y_i} \quad (t \in I).$$

Since this vector belongs to $D_{\beta_s^L(t)}$, we can obtain $d(x_i \circ \beta_s^L)/dt = 0$ $(r+1 \le i \le n-1)$ and

$$(1.7)\quad \frac{d(x_n \circ \beta_s^L)}{dt} = (x_n \circ \beta_s^L)(t) \cdot \left(\frac{\eta_1^2(\partial y_2/\partial x_2)(\partial y_3/\partial x_3)}{(\eta_1(\partial y_2/\partial x_2) + \eta_2)(\eta_1(\partial y_3/\partial x_3) + \eta_2)}\right)(\beta_s^L(t))$$
$$\times \left\{ 2\eta_2(\beta_s^L(t)) \cdot (t-1)\left(\cos\frac{1}{(t-1)^2} - \sin\frac{1}{(t-1)^2}\right) + \frac{2}{1-t}\right\}$$

on I, where we use $\partial y_i/\partial x_1(\beta_s^L(t)) = 2(\partial y_i/\partial x_i)(\beta_s^L(t))$ $(i = 2, 3,\ t \in I)$. In particular, if $\beta_s^L(t)$ is a boundary point of U_2, then $\eta_2(\beta_s^L(t)) = 0$ and hence $(d(x_n \circ \beta_s^L)/dt)(t) = (x_n \circ \beta_s^L)(t) \times 2/(1-t) \le -2/(1-t) < 0$. This implies that $\beta_s^L(t) \in U_2$ holds for every $t \in (t_1, t_0)$. Suppose that there is $t_2 \in [t_1, t_0)$ with $(x_n \circ \beta_s^L)(t_2) \le -2$. Then, since $\beta_s^L(t_2) \in M \backslash U_1$ and hence $\eta_1(\beta_s^L(t_2)) = 0$, by (1.7), we have $d(x_n \circ \beta_s^L)/dt|_{t=t_2} = 0$. This implies that $(x_n \circ \beta_s^L)(t) \ge -2$ for every $t \in [0, t_0)$, which furthermore implies $t_0 = 1$. That is, β_s^L is defined over $[0,1)$. Also, we have $I = [t_1, 1)$. It follows from (1.1) and $(\lambda \circ x_n)|_{\bar{U}_2} = 1$ that

$$(1.8)\quad \frac{\partial y_2}{\partial x_2}(\beta_s^L(t)) = \left(1 + \frac{1}{(1-t)(2+\sin(1/(t-1)^2))}\right)e^{-1/((1-t)(2+\sin(1/(t-1)^2)))} > 0,$$

$$(1.9)\quad \frac{\partial y_3}{\partial x_3}(\beta_s^L(t)) = \left(1 + \frac{1}{(1-t)(2+\cos(1/(t-1)^2))}\right)e^{-1/((1-t)(2+\cos(1/(t-1)^2)))} > 0$$

for $t \in I = [t_1, 1)$. Therefore, from (1.7), (1.8), (1.9) and $(x_n \circ \beta_s^L)(t) \le -1$ $(t \in$

$I = [t_1, 1))$, we see that $d(x_n \circ \beta_s^L)/dt \le 0$ holds on $[\max\{t_1, 1 - 1/(\sqrt[4]{2})\}, 1)$. This together with $x_n \circ \beta_s^L \ge -2$ implies that $\lim_{t\to 1-0}(x_n \circ \beta_s^L)(t)$ exists and $\lim_{t\to 1-0}(x_n \circ \beta_s^L)(t) < -1$, which furthermore implies $\lim_{t\to 1-0} \pi'(\beta_s^L(t)) \in \pi'(S_2 \cap \pi^{-1}(\lim_{t\to 1-0} \beta(t)))$. Since $\lim_{t\to 1-0} \beta_s^L(t) \in M \setminus U_1$, there is a sufficiently small positive number ε with $\eta_1(\beta_s^L(t)) = 0$ for $t \in [1-\varepsilon, 1)$. It follows from (1.7) that $d(x_n \circ \beta_s^L)/dt = 0$ over $[1-\varepsilon, 1)$. This together with $x_i \circ \beta_s^L \equiv 0$ $(r+1 \le i \le n-1)$ implies that a continuous curve $\widehat{\pi' \circ \beta_s^L} : [0,1] \to \boldsymbol{R}^{n-r}$ defined as in (xii) is a C^∞-curve. Thus D satisfies the conditions (x)–(xii). That is, this triple $(M, \mathfrak{F}, D)$ admits a singular non-extendable rectangle without terminal vertex. In this example, it is sufficient that $r \ge 3$ and $n \ge r+1$. Therefore, Theorem 1 has been proved. □

It is natural to consider the following problem.

PROBLEM. *Is there a triple $(M, \mathfrak{F}, D)$ admitting a singular non-extendable rectangle without terminal vertex for $r = 2$ and $n \ge 3$?*

2. Proof of Theorem 2

In this section, we shall prove Theorem 2 by investigating a property of a singular non-extendable rectangle without terminal vertex. First we prepare the following lemma.

LEMMA. *Let $\mathfrak{F}$ be a foliation on a Riemannian manifold (M, g) and take the orthogonal complementary distribution $F^\perp$ of $\mathfrak{F}$ as a complementary distribution to $\mathfrak{F}$. If δ is a singular non-extendable rectangle without terminal vertex, then $\lim_{t\to 1-0} l(\delta_{\cdot 1}|_{[0,t]}) = \infty$ holds.*

Proof. Set $p_0 := \lim_{t\to 1-0} \delta(t, 1)$. Take a foliated coordinate neighbourhood $(\tilde{U}, \tilde{\phi} = (x_1, \dots, x_n))$ around p_0 with $\tilde{\phi}(p_0) = (0, \dots, 0)$ and $\tilde{\phi}(\tilde{U}) = (-2, 2)^n$, where $n = \dim M$ and the foliatedness of $(\tilde{U}, \tilde{\phi})$ implies that fibres of the submersion $\pi := (x_1, \dots, x_r) : \tilde{U} \to \boldsymbol{R}^r$ $(r = \operatorname{codim} \mathfrak{F})$ are leaves of $\mathfrak{F}|_{\tilde{U}}$. Let D be a complementary distribution to $\mathfrak{F}$ on $\tilde{U}$ spanned by $\partial/\partial x_1, \dots, \partial/\partial x_r$. Denote by $L_{p_0}^V$ the leaf of $\mathfrak{F}|_{\tilde{U}}$ through p_0 and $L_{p_0}^D$ the maximal integral manifold of D through p_0. Let $\pi_V : \tilde{U} \to L_{p_0}^V$ (resp. $\pi_D : \tilde{U} \to L_{p_0}^D$) be the projection whose fibres are the maximal integral manifolds of D (resp. leaves of $\mathfrak{F}|_{\tilde{U}}$). Give $\tilde{U}$ a flat Riemannian metric g_0 defined by $g_0(\partial/\partial x_i, \partial/\partial x_j) = \delta_{ij}$ $(i, j = 1, \dots, n)$ and denote by d_0 the distance function induced from g_0, where δ_{ij} is the Kronecker's delta. Set $U := \tilde{\phi}^{-1}((-1, 1)^n)$. Take increasing sequences $\{t_k\}_{k=1}^\infty$ and $\{s_k\}_{k=1}^\infty$ in $[0, 1)$ satisfying $\lim_{k\to\infty} t_k = \lim_{k\to\infty} s_k = 1$, $\delta_{\cdot 1}([t_k, 1)) \cup \delta_{t_k \cdot}([s_k, 1]) \subset U$, $\max_{t\in[t_k,1)} d_0(\pi_D(\delta(t,1)), p_0) < 1/k$ and $\max_{s\in[s_k,1]} d_0(\pi_V(\delta(t_k, s)), p_0) < 1/k$ (see Figure 2.1).

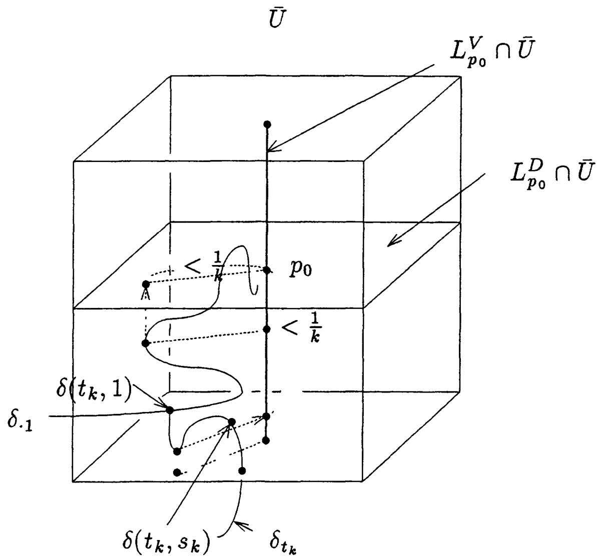


Figure 2.1.

We can show that $\delta_{\cdot s_k}([t_k, 1])$ is not contained in $\tilde{U}$ for every k. In fact, if $\delta_{\cdot s_k}([t_k, 1]) \subset \tilde{U}$ for some k, then the existence of the $F^{\perp}$-lift of $\pi \circ \delta_{\cdot s_k}|_{[t_k, 1]}$ starting from $\delta(t_k, 1)$ is assured because $\lim_{t \to 1-0} \delta_{\cdot 1}(t) = p_0 \in \tilde{U}$ and hence δ is extendable. This contradicts the fact that δ is non-extendable. Thus $\delta_{\cdot s_k}([t_k, 1]) \not\subset \tilde{U}$ for every k. That is, $\delta_{\cdot s_k}([t_k, 1]) \cap \partial U \neq \emptyset$ holds for every k, where ∂U is the boundary of U in M. Set $t'_k := \min\{t \in [t_k, 1] \mid \delta_{\cdot s_k}(t) \in \partial U\}$. Take $(t''_k, s'_k) \in (t_k, t'_k] \times [s_k, 1)$ satisfying $\delta([t_k, t''_k] \times [s'_k, 1]) \cap \partial U = \{\delta(t''_k, s'_k)\}$ (see Figure 2.2).

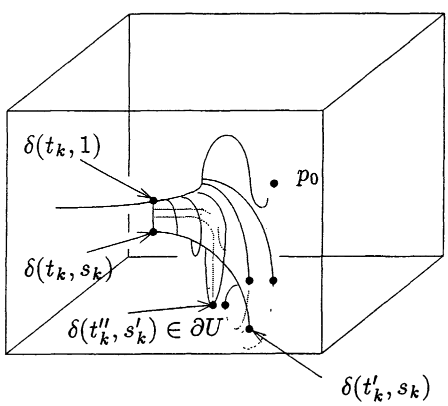


Figure 2.2.

Denote by $S(\bar{U})$ the unit tangent bundle of $\bar{U}$ with respect to g_0 and F the tangent bundle of $\mathfrak{F}$, where $\bar{U}$ is the closure of U. Define $\beta_k : [t_k, t_k''] \to M$ by $\beta_k(t) := \delta_{s_k'}(t)$ for $t \in [t_k, t_k'']$ and $X_k(t) := \dot{\beta}_k(t)/\|\dot{\beta}_k(t)\| \in S(\bar{U})$, where $\|\dot{\beta}_k(t)\| = \sqrt{g_0(\dot{\beta}_k(t), \dot{\beta}_k(t))}$. Take $t_k''' \in [t_k, t_k'']$ such that

$$\frac{\|X_k(t_k''')^V\|}{\|X_k(t_k''')^D\|} = \max_{t \in [t_k, t_k'']} \frac{\|X_k(t)^V\|}{\|X_k(t)^D\|},$$

where $X_k(t)^V$ (resp. $X_k(t)^D$) is the F-component (resp. the D-component) of $X_k(t)$. It follows from the compactness of $S(\bar{U})$ that there is a convergent subsequence $\{Y_k\}_{k=1}^{\infty}$ of $\{X_k(t_k''')\}_{k=1}^{\infty}$. Set $Y_0 := \lim_{k\to\infty} Y_k$. Since $Y_k \in F^{\perp}$ for every k, we have $Y_0 \in F^{\perp}$. Suppose $\lim_{k\to\infty}\|X_k(t_k''')^V\|/\|X_k(t_k''')^D\| = \infty$. Then, we have $\lim_{k\to\infty}\|Y_k^V\|/\|Y_k^D\| = \infty$, which implies $Y_0 \in F$. Thus $Y_0 \in F \cap F^{\perp}$, that is, $Y_0 = 0$ is deduced. This contradicts $Y_0 \in S(\bar{U})$. Therefore, $\lim_{k\to\infty}\|X_k(t_k''')^V\|/\|X_k(t_k''')^D\| = \infty$ does not hold. Hence, for a sufficiently large positive number c, there is a subsequence $\{X_{a(k)}(t_{a(k)}''')\}_{k=1}^{\infty}$ of $\{X_k(t_k''')\}_{k=1}^{\infty}$ such that $\|X_{a(k)}(t_{a(k)}''')^V\|/\|X_{a(k)}(t_{a(k)}''')^D\| < c$ for every k. Since $\|X_{a(k)}(t)^V\|/\|X_{a(k)}(t)^D\| < c$ $(t \in [t_{a(k)}, t_{a(k)}''])$ by the definition of $t_{a(k)}'''$, we have

$$l_0(\pi_D \circ \beta_{a(k)}) > \frac{1}{c}\, l_0(\pi_V \circ \beta_{a(k)}), \tag{2.1}$$

where $l_0(\cdot)$ is the length of a curve $\cdot$ with respect to g_0. Also, it follows from $\max_{s \in [s_{a(k)}, 1]} d_0(\pi_V(\delta(t_{a(k)}, s)), p_0) < 1/a(k)$ and $\delta(t_{a(k)}'', s_{a(k)}') \in \partial U$ that $d_0(\pi_V(\delta(t_{a(k)}, s_{a(k)}')), p_0) < 1/a(k)$ and $d_0(\pi_V(\delta(t_{a(k)}'', s_{a(k)}')), p_0) \geq 1$, respectively. Hence we have

$$\begin{aligned} l_0(\pi_V \circ \beta_{a(k)}) &> d_0(\pi_V(\delta(t_{a(k)}, s_{a(k)}')), \pi_V(\delta(t_{a(k)}'', s_{a(k)}'))) \\ &\geq d_0(\pi_V(\delta(t_{a(k)}'', s_{a(k)}')), p_0) - d_0(\pi_V(\delta(t_{a(k)}, s_{a(k)}')), p_0) \\ &> 1 - \frac{1}{a(k)} \\ &\geq 1 - \frac{1}{k}. \end{aligned} \tag{2.2}$$

Since $\delta_{t\cdot}|_{[s_{a(k)}', 1]}$ $(t \in [t_{a(k)}, t_{a(k)}''])$ are vertical curves in U by $\delta([t_{a(k)}, t_{a(k)}''] \times [s_{a(k)}', 1]) \cap \partial U = \{\delta(t_{a(k)}'', s_{a(k)}')\}$, we have $\pi_D \circ \delta_{\cdot 1}|_{[t_{a(k)}, t_{a(k)}'']} = \pi_D \circ \beta_{a(k)}$. Therefore, it follows from (2.1) and (2.2) that $l_0(\pi_D \circ \delta_{\cdot 1}|_{[t_{a(k)}, t_{a(k)}'']}) \geq (1/c)(1 - 1/k)$. We may assume that $t_k'' \leq t_{k+1}$ holds for every k by retaking $\{t_k\}_{k=1}^{\infty}$ if necessary. Hence we obtain

$$\begin{aligned}\lim_{t\to 1-0} l_0(\delta_{\cdot 1}|_{[t_{a(1)},t]}) &\geq \lim_{t\to 1-0} l_0(\pi_D \circ \delta_{\cdot 1}|_{[t_{a(1)},t]})\\ &\geq \sum_{k=1}^{\infty} l_0(\pi_D \circ \delta_{\cdot 1}|_{[t_{a(k)},t''_{a(k)}]})\\ &\geq \sum_{k=1}^{\infty} \frac{1}{c}\left(1-\frac{1}{k}\right)\\ &= \infty.\end{aligned}$$

Define a function ρ on the projective bundle $Pr(T\tilde{U})$ of $T\tilde{U}$ by

$$\rho(W) := \sqrt{\frac{g(X,X)}{g_0(X,X)}}$$

for $W \in Pr(T\tilde{U})$, where X is a non-zero vector belonging to W. It is clear that ρ is continuous. Since $\delta_{\cdot 1}([t_{a(1)},1)) \cup \{p_0\}$ is compact, so is also $Pr(T\tilde{U})|_{\delta_{\cdot 1}([t_{a(1)},1))\cup\{p_0\}}$. Therefore, the minimum of ρ on $Pr(T\tilde{U})|_{\delta_{\cdot 1}([t_{a(1)},1))\cup\{p_0\}}$ exists. Denote by c' this minimum. Clearly we have $c' > 0$. Then, it is easy to show that $l(\delta_{\cdot 1}|_{[t_{a(1)},t]}) \geq c' l_0(\delta_{\cdot 1}|_{[t_{a(1)},t]})$ for every $t \in [t_{a(1)},1)$. Therefore, we obtain

$$\lim_{t\to 1-0} l(\delta_{\cdot 1}|_{[t_{a(1)},t]}) = c' \lim_{t\to 1-0} l_0(\delta_{\cdot 1}|_{[t_{a(1)},t]}) = \infty,$$

that is, $\lim_{t\to 1-0} l(\delta_{\cdot 1}|_{[0,t]}) = \infty$. □

Now we shall prove Theorem 2 in terms of this lemma.

Proof of Theorem 2. Suppose that $F^{\perp}$ is not an Ehresmann connection. Then there is a non-extendable rectangle δ without terminal vertex. If δ is non-singular, then $\lim_{t\to 1-0} l(\delta_{\cdot 1}|_{[0,t]}) = \infty$ is deduced from the completeness of (M,g). Also, if δ is singular, then $\lim_{t\to 1-0} l(\delta_{\cdot 1}|_{[0,t]}) = \infty$ is deduced from Lemma. Whether δ is non-singular or not, we obtain $\lim_{t\to 1-0} l(\delta_{\cdot 1}|_{[0,t]}) = \infty$. This deduces $\lim_{t\to 1-0} G^{\perp}_{\delta_0}(\delta|_{[0,t]\times[0,1]}) = \infty$. Hence $\sup G^{\perp}_{\delta_0} < \infty$ does not hold, which contradicts the assumption. Therefore, $F^{\perp}$ is an Ehresmann connection. □

3. Examples

In this section, we shall first give examples of non-singular non-extendable rectangles without terminal vertex.

EXAMPLE 1. Let $B := \{(x_1,\dots,x_n) \in \boldsymbol{R}^n \mid x_1^2 + \cdots + x_n^2 < 1\}$ $(n \geq 2)$ and $\mathfrak{F}$ a foliation of codimension r on B whose leaves are fibres of the projection $\pi : B \to \boldsymbol{R}^r$ defined by $\pi(x_1,\dots,x_n) = (x_1,\dots,x_r)$ for $(x_1,\dots,x_n) \in B$, where $1 \leq r \leq n-1$. Let $D = \mathrm{Span}\{\partial/\partial x_1,\dots,\partial/\partial x_r\}$, where we regard $(x_1,\dots,x_n)$ as a coordinate of B. Define a rectangle δ without terminal vertex by $\delta(t,s) := (t/\sqrt{2}, 0,\dots,0,s/\sqrt{2})$. It is clear that δ is non-extendable and non-singular (see Figure 3.1).

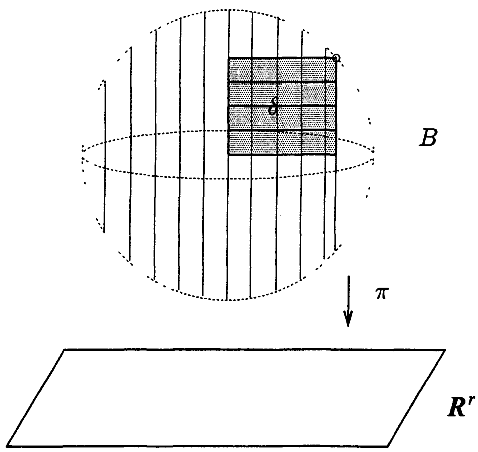


Figure 3.1.

EXAMPLE 2. Let $\mathfrak{F}$ be a foliation of codimension one on an n-dimensional affine space $\boldsymbol{R}^n$ ($n \geq 2$) whose leaves are

$$\left\{ (x_1,\dots,x_{n-1},k-\exp\left(1-\sum_{i=1}^{n-1}x_i^2\right)^{-1} \,\middle|\, \sum_{i=1}^{n-1}x_i^2<1 \right\} \quad (k\in\boldsymbol{R})$$

and

$$\left\{ (x_1,\dots,x_n) \,\middle|\, \sum_{i=1}^{n-1}x_i^2=k^2, x_n\in\boldsymbol{R} \right\} \quad (k\geq 1).$$

Let D be the orthogonal complementary distribution of $\mathfrak{F}$ with respect to the Euclidean metric g of $\boldsymbol{R}^n$ defined by $g(\partial/\partial x_i,\partial/\partial x_j)=\delta_{ij}$, where we regard $(x_1,\dots,x_n)$ as a coordinate of $\boldsymbol{R}^n$ and δ_{ij} is the Kronecker's delta. Let α be a vertical curve defined by $\alpha(s)=((1-s)/2,0,\dots,0,k-e^{4/((3-s)(1+s))})$ for $s\in[0,1]$ and β be a horizontal curve defined by $\beta(t)=((1+t)/2,0,\dots,0,(1/32)\int_0^t(((t+3)^2(t-1)^2)/(t+1))e^{4/((t+3)(t-1))}\,dt+k-e^{4/3})$ for $t\in[0,1]$. It is clear that there is a rectangle δ without terminal vertex satisfying $\delta_{0\cdot}=\alpha$ and $\delta_{\cdot 0}=\beta$ but it is non-extendable and non-singular (see Figure 3.2).

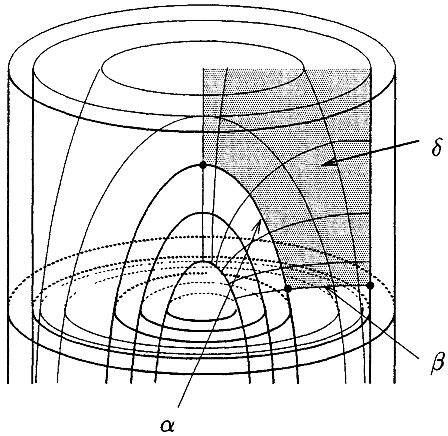


Figure 3.2.

Next we shall give examples of a foliated Riemannian manifold which satisfies the assumption of Theorem 2 but does not satisfy the condition (IV) in Introduction.

EXAMPLE 3. Let M be a hypersurface of an $(n+1)$-dimensional Euclidean space $\boldsymbol{R}^{n+1}$ $(n \geq 2)$ defined by the equation $x_1^2 + \cdots + x_{r+1}^2 - x_{r+2}^2 - \cdots - x_{n+1}^2 = 1$ $(1 \leq r \leq n-1)$ and give M the Riemannian metric g induced from the Euclidean metric of $\boldsymbol{R}^{n+1}$, where $(x_1, \ldots, x_{n+1})$ is a Euclidean coordinate system of $\boldsymbol{R}^{n+1}$. It is clear that (M, g) is complete. Let $\mathfrak{F}$ be a foliation on (M, g) whose leaves are the intersections of M and $(n-r+1)$-dimensional halfplanes

$$\{(tc_1, \ldots, tc_{r+1}, x_{r+2}, \ldots, x_{n+1}) | (x_{r+2}, \ldots, x_{n+1}) \in \boldsymbol{R}^{n-r},\ t \geq 0\}$$

$$((c_1, \ldots, c_{r+1}) \in S^r(1) = \{(x_1, \ldots, x_{r+1}) | x_1^2 + \cdots + x_{r+1}^2 = 1\}).$$

Then the orthogonal complementary distribution $F^{\perp}$ of $\mathfrak{F}$ is an integrable distribution whose maximal integral manifolds are the intersections of M and $(r+1)$-dimensional planes

$$\{(x_1, \ldots, x_{r+1}, c_{r+2}, \ldots, c_{n+1}) | (x_1, \ldots, x_{r+1}) \in \boldsymbol{R}^{r+1}\}$$

$$((c_{r+2}, \ldots, c_{n+1}) \in \boldsymbol{R}^{n-r}).$$

It is shown that $G_{\alpha}^{\perp} \equiv \sqrt{\alpha_1(1)^2 + \cdots + \alpha_{r+1}(1)^2} / \sqrt{\alpha_1(0)^2 + \cdots + \alpha_{r+1}(0)^2}$ holds for each vertical curve α (see Figure 3.3), where $\alpha = (\alpha_1, \ldots, \alpha_{n+1})$.

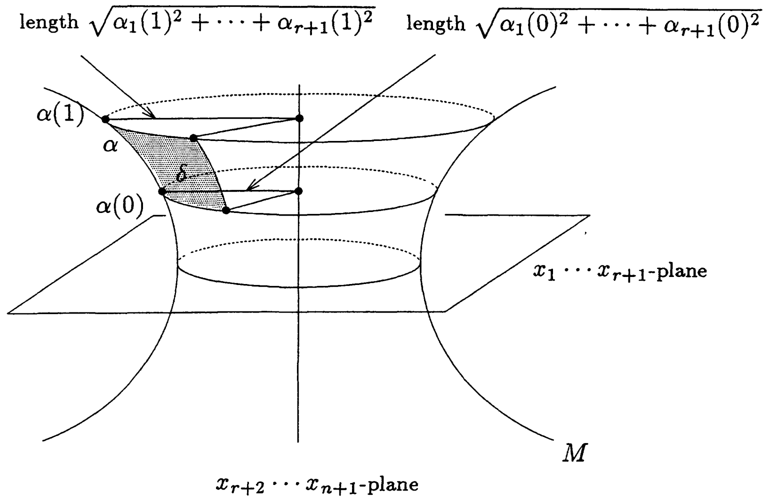


Figure 3.3.

Thus $\mathfrak{F}$ satisfies the assumption of Theorem 2. Let α_0 be a vertical curve without terminal point defined by $\alpha_0(s) := (1/(1-s), 0, \ldots, 0, \sqrt{2s-s^2}/(1-s))$ for $s \in [0,1)$. From $G^{\perp}_{\alpha_0|_{[0,s]}} \equiv \sqrt{(\alpha_0)_1(s)^2+\cdots+(\alpha_0)_{r+1}(s)^2}/\sqrt{(\alpha_0)_1(0)^2+\cdots+(\alpha_0)_{r+1}(0)^2} = 1/(1-s)$, we have $\lim_{s\to 1-0}\sup G^{\perp}_{\alpha_0|_{[0,s]}} = \infty$ and hence $\sup_{s\in[0,1)}\sup G^{\perp}_{\alpha_0|_{[0,s]}} < \infty$ does not hold. Thus $\mathfrak{F}$ does not satisfy the condition (IV).

In this example, the base manifold M is not compact. Next we shall give an example such that the base manifold is compact.

EXAMPLE 4. Let $\tilde{\mathfrak{F}}$ be a foliation on a 2-dimensional Euclidean space $\boldsymbol{R}^2$ whose leaves are

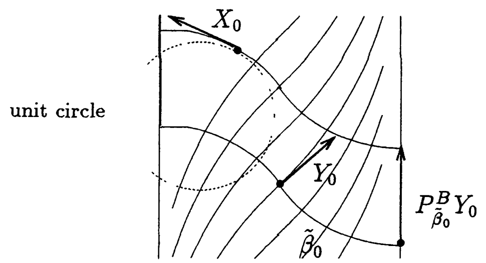
$$\left\{(x_1, \tan x_1 + c)\,\middle|\, \frac{(2k-1)\pi}{2} < x_1 < \frac{(2k+1)\pi}{2}\right\} \quad (c \in \boldsymbol{R}, k \in \boldsymbol{N})$$

and

$$\left\{\left(\frac{(2k+1)\pi}{2}, x_2\right)\,\middle|\, x_2 \in \boldsymbol{R}\right\} \quad (k \in \boldsymbol{N}).$$

Let ϕ_1 be a translation of $\boldsymbol{R}^2$ defined by $\phi_1(x_1,x_2) = (x_1+\pi, x_2)$ for $(x_1,x_2) \in \boldsymbol{R}^2$ and ϕ_2 a translation of $\boldsymbol{R}^2$ defined by $\phi_2(x_1,x_2) = (x_1, x_2+1)$ for $(x_1,x_2) \in \boldsymbol{R}^2$. Denote by G the transformation group of $\boldsymbol{R}^2$ generated by ϕ_1 and ϕ_2. Denote by M the orbit space $\boldsymbol{R}^2/G$ of G. Since G is an isometry group of $\boldsymbol{R}^2$, the Euclidean metric $\tilde{g}$ of $\boldsymbol{R}^2$ induces a Riemannian metric on M, which we denote by g. Also, since G preserves $\tilde{\mathfrak{F}}$, $\tilde{\mathfrak{F}}$ induces a foliation on M, which we denote by $\mathfrak{F}$. Denote by $\tilde{F}^{\perp}$ (resp. $F^{\perp}$) the orthogonal complementary distribution of $\tilde{\mathfrak{F}}$

(resp. $\mathfrak{F}$) and $\tilde{\mathfrak{F}}^{\perp}$ (resp. $\mathfrak{F}^{\perp}$) the foliation whose leaves are the maximal integral manifolds of $\tilde{F}^{\perp}$ (resp. $F^{\perp}$). Denote by $\Gamma(\cdot)$ the space of all cross sections of a vector bundle $\cdot$. We define $h \in \Gamma((\tilde{F}^{\perp})^* \otimes (\tilde{F}^{\perp})^* \otimes \tilde{F})$ by $h(X, Y) := (\tilde{\nabla}_X Y)_{\tilde{F}}$ for $X, Y \in \Gamma(\tilde{F}^{\perp})$, where $\tilde{\nabla}$ is the Levi-Civita connection of $\tilde{g}$ and $(\tilde{\nabla}_X Y)_{\tilde{F}}$ is the $\tilde{F}$-component of $\tilde{\nabla}_X Y$. Let $S\tilde{F}^{\perp}$ (resp. $S\tilde{F}$) be a sphere bundle consisting of all unit vectors belonging to $\tilde{F}^{\perp}$ (resp. the tangent bundle $\tilde{F}$ of $\tilde{\mathfrak{F}}$). Then, it is easy to show that $\|h(X, X)\| < 1$ holds for every $X \in S\tilde{F}^{\perp}$, where $\|h(X, X)\|$ is the norm of $h(X, X)$ (see Fig. 3.4). Also, it is shown that $\|P^B_{\tilde{\beta}} Y\| \le \sqrt{2}$ holds for every horizontal curve $\tilde{\beta}$ in $\boldsymbol{R}^2$ and every $Y \in S\tilde{F}_{\tilde{\beta}(0)}$, where $P^B_{\tilde{\beta}}$ is the parallel translation along $\tilde{\beta}$ with respect to the Bott connection on the orthogonal complementary distribution F of $\tilde{\mathfrak{F}}^{\perp}$ (see Figure 3.4).



$$\max_{X \in S\tilde{F}^{\perp}} \|h(X_0, X_0)\| < 1$$

$$\max_{\tilde{\beta}} \max_{Y \in S\tilde{F}_{\tilde{\beta}(0)}} \|P^B_{\tilde{\beta}} Y\| = \|P^B_{\tilde{\beta}_0} Y_0\| \le \sqrt{2}$$

Figure 3.4.

Therefore, we can obtain

$$\sup_{\tilde{\beta}:\text{horizontal curve}} \ \sup_{X \in SF^{\perp}_{\tilde{\beta}(1)},\, Y \in SF_{\tilde{\beta}(0)}} |\tilde{g}(P^B_{\tilde{\beta}} Y, h(X, X))| \le \sqrt{2} < \infty.$$

Set

$$A := \sup_{\tilde{\beta}.\text{horizontal curve}} \ \sup_{X \in SF^{\perp}_{\tilde{\beta}(1)},\, Y \in SF_{\tilde{\beta}(0)}} |\tilde{g}(P^B_{\tilde{\beta}} Y, h(X, X))|.$$

Then we can show that $\sup G^{\perp}_{\tilde{\alpha}} \le \exp(A \cdot l(\tilde{\alpha}))$ holds for every vertical curve $\tilde{\alpha}$ in $\boldsymbol{R}^2$, where $l(\tilde{\alpha})$ is the length of $\tilde{\alpha}$ with respect to $\tilde{g}$ (see the proof of Corollary 3.10 in [11]). Take an arbitrary vertical curve α in M and an arbitrary rectangle δ with $\delta_{0\cdot} = \alpha$. Let α^L be one of lifts of α to $\boldsymbol{R}^2$ and δ^L the lift of δ to $\boldsymbol{R}^2$ with $\delta^L_{0\cdot} = \alpha^L$. Clearly we have $G^{\perp}_{\alpha}(\delta) = G^{\perp}_{\alpha^L}(\delta^L)$, which implies $\sup G^{\perp}_{\alpha} \le \sup G^{\perp}_{\alpha^L}$ by the arbitrariness of δ. Therefore, we can obtain

$$\sup G_{\alpha}^{\perp} \leq \sup G_{\alpha^L}^{\perp} \leq \exp(A \cdot l(\alpha^L)) = \exp(A \cdot l(\alpha)),$$

where $l(\alpha)$ is the length of α with respect to g. Thus $\mathfrak{F}$ satisfies the assumption of Theorem 2. Define a vertical curve $\tilde{\alpha}_0$ in $\boldsymbol{R}^2$ without terminal point by $\tilde{\alpha}_0(s) := (-\pi/2, 1/(1-s))$ for $s \in [0,1)$ and let $\tilde{\beta}_0$ be the horizontal curve in $\boldsymbol{R}^2$ satisfying $\tilde{\beta}_0(0) = \tilde{\alpha}_0(0)$ and $(\tilde{\beta}_0)_1(t) = t - (\pi/2) (t \in [0,1])$, where $\tilde{\beta}_0 = ((\tilde{\beta}_0)_1, (\tilde{\beta}_0)_2)$. Take a sequence $\{t_k\}_{k=1}^{\infty}$ in $(0,1]$ satisfying $l(\tilde{\beta}_0|_{[0,t_k]}) < 1/k$ $(k \geq 1)$. For each t_k, there is $s_k \in [0,1)$ satisfying $l((\delta_{\tilde{\alpha}_0|_{[0,s_k]}\tilde{\beta}_0|_{[0,t_k]}})_{\cdot 1}) > \pi/2$ (see Figure 3.5), where $\delta_{\tilde{\alpha}_0|_{[0,s_k]}\tilde{\beta}_0|_{[0,t_k]}}$ is the rectangle with $(\delta_{\tilde{\alpha}_0|_{[0,s_k]}\tilde{\beta}_0|_{[0,t_k]}})_{0\cdot} = \tilde{\alpha}_0|_{[0,s_k]}$ and $(\delta_{\tilde{\alpha}_0|_{[0,s_k]}\tilde{\beta}_0|_{[0,t_k]}})_{\cdot 0} = \tilde{\beta}_0|_{[0,t_k]}$. Then we have

$$G^{\perp}_{\tilde{\alpha}_0|_{[0,s_k]}}(\delta_{\tilde{\alpha}_0|_{[0,s_k]}\tilde{\beta}_0|_{[0,t_k]}}) = \frac{l((\delta_{\tilde{\alpha}_0|_{[0,s_k]}\tilde{\beta}_0|_{[0,t_k]}})_{\cdot 1})}{l(\tilde{\beta}_0|_{[0,t_k]})} > \frac{\pi/2}{1/k} = \frac{k\pi}{2}.$$

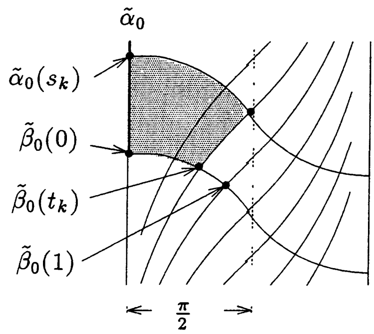


Figure 3.5.

Set $\alpha_0 := \pi \circ \tilde{\alpha}_0$ and $\beta_0 := \pi \circ \tilde{\beta}_0$, where π is the projection of $\boldsymbol{R}^2$ onto M. From $G^{\perp}_{\alpha_0|_{[0,s_k]}}(\delta_{\alpha_0|_{[0,s_k]}\beta_0|_{[0,t_k]}}) = G^{\perp}_{\tilde{\alpha}_0|_{[0,s_k]}}(\delta_{\tilde{\alpha}_0|_{[0,s_k]}\tilde{\beta}_0|_{[0,t_k]}})$, we have $\lim_{k\to\infty} G^{\perp}_{\alpha_0|_{[0,s_k]}}(\delta_{\alpha_0|_{[0,s_k]}\beta_0|_{[0,t_k]}}) = \lim_{k\to\infty} k\pi/2 = \infty$ and hence $\sup_{s\in[0,1)} \sup G^{\perp}_{\alpha_0|_{[0,s]}} < \infty$ does not hold. Thus $\mathfrak{F}$ does not satisfy the condition (IV).

From these examples, it is guessed that there are a lots of examples of a foliated Riemannian manifold which satisfies the assumption of Theorem 2 but does not satisfy the condition (IV). Thus we can recognize the essential gap between the assumption of Theorem 2 and the condition (IV).

Next we shall give examples showing the topological gap between Riemannian foliations (i.e., foliations admitting a bundlelike metric) and foliations admitting a Riemannian metric satisfying the assumption of Theorem 2.

EXAMPLE 5. Let $(M, \mathfrak{F})$ be a foliated manifold in Example 4. The above Riemannian metric g satisfies the assumption of Theorem 2. However, $\mathfrak{F}$ is not a Riemannian foliation. In fact, for an arbitrary Riemannian metric on M, $G^{\perp}_{\alpha_0}(\delta_0) > 1$ holds, where α_0 and δ_0 are as in Figure 3.6.

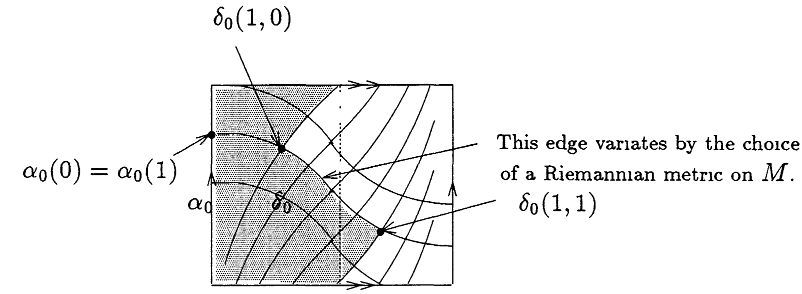


Figure 3.6.

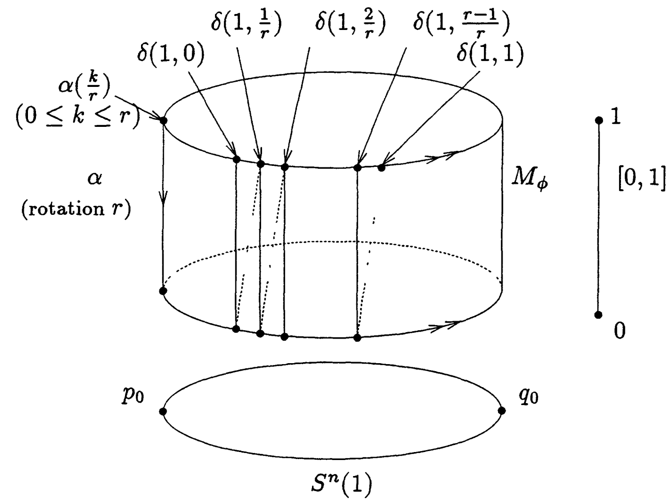
EXAMPLE 6. Let p_0 be a point of the n-dimensional unit sphere $S^n(1)$ and q_0 the antipodal point of p_0, where we give $S^n(1)$ the standard Riemannian metric. Denote by g_1 the standard Riemannian metric. Define a map ϕ of $S^n(1)$ into itself by

$$\phi(p) := \begin{cases} \exp_{p_0}(f(\|X\|)X) & (p \neq q_0) \\ q_0 & (p = q_0) \end{cases}$$

for $p \in S^n(1)$, where $\exp_{p_0}$ is the exponential map of $S^n(1)$ at p_0, $\|\cdot\|$ is the norm of $\cdot$ with respect to g_1, X is the tangent vector of $S^n(1)$ at p_0 satisfying $\exp_{p_0} X = p$ and $\|X\| < \pi$ and f is a C^∞-function over $[0,\pi)$ satisfying $1 \leq f \leq 4/3$ over $[0,\pi)$, $f^{-1}(4/3) = [0,\pi/4]$, $f^{-1}(1) = [3\pi/4,\pi)$ and $f' > -1/\pi$ over $[0,\pi)$. Then, $(f(t)t)' = f(t) + f'(t)t \geq 1 - (t/\pi) > 0$ holds over $[0,\pi)$, that is, $f(t)t$ is an increasing function over $[0,\pi)$. Also, we have $\lim_{t\to\pi-0} f(t)t = \pi$. These facts imply that ϕ is a diffeomorphism. Let M_ϕ be the mapping torus of ϕ and $\mathfrak{F}$ a foliation on M_ϕ induced naturally from the foliation $\mathfrak{F}$ on $S^n(1) \times [0,1]$ whose leaves are the fibres of the projection of $S^n(1) \times [0,1]$ onto $S^n(1)$. Denote by π the quotient map of $S^n(1) \times [0,1]$ onto M_ϕ and P (resp. Q) the projection of $S^n(1) \times [0,1]$ onto $S^n(1)$ (resp. $[0,1]$). Also, denote by g_2 the standard Riemannian metric of $[0,1]$. Define a Riemannian metric g_0 on M_ϕ by

$$g_0(X,Y) := ug_1(P_*\tilde{X}, P_*\tilde{Y}) + (1-u)(\phi^*g_1)(P_*\tilde{X}, P_*\tilde{Y}) + g_2(Q_*\tilde{X}, Q_*\tilde{Y})$$

for $X, Y \in T_{\pi(p,u)}M_\phi$, where $\tilde{X}$ (resp. $\tilde{Y}$) is the tangent vector of $S^n(1) \times [0,1]$ at (p,u) with $\pi_*\tilde{X} = X$ (resp. $\pi_*\tilde{Y} = Y$) and ϕ^*g_1 is the Riemannian metric induced from g_1 by ϕ. It is clear that g_0 is well-defined. Take an arbitrary vertical curve α in M_ϕ. Since $g_1(\phi_*X;\phi_*X) \leq (4/3)^2 g_1(X,X)$ for every $X \in TS^n(1)$, we see that $G^{\perp}_\alpha(\delta) \leq (4/3)^{[l(\alpha)]+1}$ for every $\delta \in \mathrm{Rec}(\alpha,\cdot)$, that is, $\sup G^{\perp}_\alpha \leq (4/3)^{[l(\alpha)]+1} < \infty$, where $l(\alpha)$ is the length of α with respect to g_0 and $[\cdot]$ is the Gauss's symbol of $\cdot$ (see Figure 3.7).



$$l(\delta_{\cdot 1}) \leq \tfrac{4}{3} l(\delta_{\cdot \frac{r-1}{r}}) \leq \cdots \leq (\tfrac{4}{3})^r l(\delta_{\cdot 0})$$

Figure 3.7.

Thus $\mathfrak{F}$ satisfies the assumption of Theorem 2 with respect to g_0.

Take an arbitrary Riemannian metric g on M_ϕ. Let α_0 be a vertical curve defined by $\alpha_0(s) = \pi(p_0, 1-s)$ and β_0 be a horizontal curve (with respect to g) satisfying $\beta_0(0) = \alpha_0(0)$, $\beta_0(1) \in \pi(P^{-1}(\exp_{p_0}(\pi/4)X_0))$ and $\beta_0([0,1]) \subset$

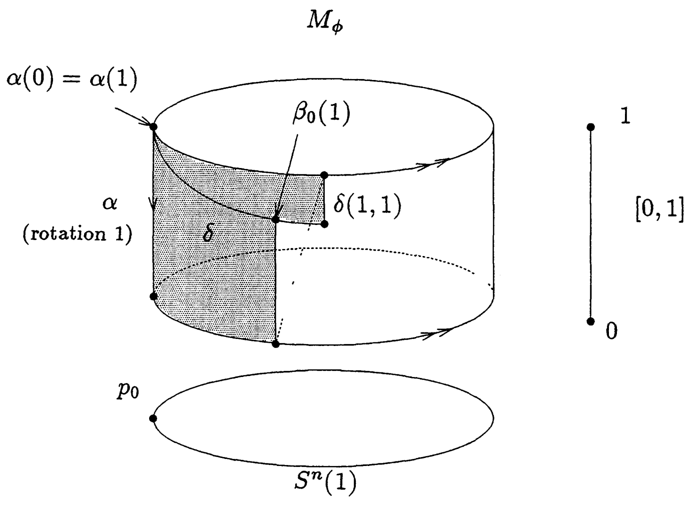


Figure 3.8.

$\pi(P^{-1}(\{\exp_{p_0}((\pi t/4)X_0)|t\in[0,1]\}))$, where X_0 is some unit tangent vector of $S^n(1)$ at p_0. Let δ be the rectangle with $\delta_{0\cdot}=\alpha_0$ and $\delta_{\cdot 0}=\beta_0$. Then we have $G^{\perp}_{\alpha_0}(\delta)>1$ (see Figure 3.8). Thus any Riemannian metric g on M_ϕ is not bundle-like for $\mathfrak{F}$, that is, $\mathfrak{F}$ is not a Riemannian foliation.

Similarly, we can give examples showing the topological gap between totally geodesicable foliations and foliations admitting a Riemannian metric satisfying the condition (V) in Introduction.

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