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MODULI OF RING DOMAINS OBTAINED BY A CONFORMAL WELDING

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Abstract

We are concerned with ring domains which are conformally welded along a pair of opposite sides of a square. Oikawa studied moduli of these ring domains and left some problems. We shall answer one of these open problems.

1. Introduction

Welding of polygons and the type of Riemann surfaces were considered by Nevanlinna, Oikawa and others (cf. [3], [4]). We are concerned with the relation of weldings and the moduli of Riemann surfaces. Oikawa studied this subject and got some results which he didn't publish (cf. [5]). We follow him. A square in the complex plane can be conformally welded into various ring domains by a specific kind of identification of a pair of opposite sides. We consider the range of these moduli. Oikawa gave an estimate for the range of these moduli and asked whether it is the best possible or not. We shall show a certain identification which give conformally welded ring domains with arbitrary small moduli. In addition, we shall show that the moduli of ring domains conformally welded by an unnatural identification never meet to a neighborhood of the module of ring domains conformally welded by the natural identification.

Consider the square $Q=\{x+iy : 0<x<1,\ 0<y<1\}$ in the complex plane, and put

$$L_{+}=\{x+iy : 0<x<1,\ y=1\}, \qquad L_{-}=\{x+iy : 0<x<1,\ y=0\}.$$

Let $\phi_0(x+i)=x(0<x<1)$ and $\phi : L_{+}\to L_{-}$ be a homeomorphism such that $\phi\circ\phi_0^{-1}(x)$ is strictly increasing. Let G be a ring domain and C be a Jordan curve in G joining two boundary components of G. Let f be a continuous mapping from $Q\cup L_{+}\cup L_{-}$ onto G. We say the triple (G, C, f) is a conformal welding obtained by ϕ if f is conformal in Q, $f\circ\phi=f$ on L_{+}, and $f(L_{+})=f(L_{-})=C$. And we call ϕ a welding function. We say a conformal welding by ϕ is unique, if, for any two conformal weldings $(G_i, C_i, f_i)_{i=1,2}$ obtained by ϕ, $f_2\circ f_1^{-1}$ is a con-

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formal mapping from G_1 to G_2.

If a ring domain G is conformally equivalent to an annulus $\{z: R_1<|z|<R_2\}$, then the quantity $M(G)=\log(R_2/R_1)$ is called the modulus of G. The $M(G)$ is represented by means of extremal length. Let $\Lambda(G)$ be the set of Borel measurable conformal densities on G and $\Gamma(G)$ be the set of rectifiable closed curves in G separating the boundary components of G. For $\rho\in\Lambda(G)$, set

$$A(\rho, G)=\iint_G \rho(z)^2 dx dy, \ (z=x+iy)$$

$$L(\rho, \Gamma(G))=\inf_{\gamma\in\Gamma(G)}\int_\gamma \rho(z)|dz|.$$

Then

$$\lambda(\Gamma(G))=\sup_{\rho\in\Lambda(G)}\frac{L(\rho, \Gamma(G))^2}{A(\rho, G)}$$

is called the extremal length of $\Gamma(G)$ on G. We know $M(G)=2\pi/\lambda(\Gamma(G))$.

For a welding function ϕ, put

$$M_\phi=\{M(G): (G, C, f) \textit{ is a conformal welding obtained by } \phi\}.$$

If a conformal welding by ϕ is unique, then M_ϕ consists of a single point. On the other hand, if the 2-dimensional measure of C is positive, then a conformal welding by ϕ is not unique. In fact, by using a variational formula (cf. Gardinar [1]), we know that M_ϕ contains an interval. Oikawa remaked that there is a welding function ϕ which has two conformal weldings (G, C, f) and (G', C', f') such that the area $|C|$ of C is positive but the area $|C'|$ of C' is zero. So the following Oikawa's problem is interesting for us.

Is there a welding function ϕ such that a conformal welding by ϕ is not unique, but M_ϕ consists of a single point?

Now Oikawa [5] proved the following three theorems.

THEOREM A. *$M_\phi=\{2\pi\}$ if and only if $\phi=\phi_0$.*

THEOREM B. *Let Φ be the set of welding functions ϕ, then*

$$\bigcup_{\phi\in\Phi} M_\phi=(0, 2\pi].$$

For the sake of simplicity, we will often denote $\phi\circ\phi_0^{-1}(x)$ by $\phi(x)$.

THEOREM C.

$$M_\phi\subset\left[2\pi\int_0^1\frac{\min(\phi'(x), 1)}{1+(\phi(x)-x)^2}dx, 2\pi\right].$$

Further Oikawa [5] presented the following problem.

Find a welding function ϕ satisfying

$$M_\phi \supset \left(2\pi \int_0^1 \frac{\min(\phi'(x), 1)}{1+(\phi(x)-x)^2} dx,\ 2\pi\right).$$

In this paper, we shall show the following theorems, which answer this problem.

THEOREM 1. *There are a welding function ϕ and $\varepsilon > 0$ such that*

$$M_\phi \supset (0, \varepsilon).$$

THEOREM 2. *For any $\phi \neq \phi_0$, there is an $m < 2\pi$ such that*

$$M_\phi \subset (0, m].$$

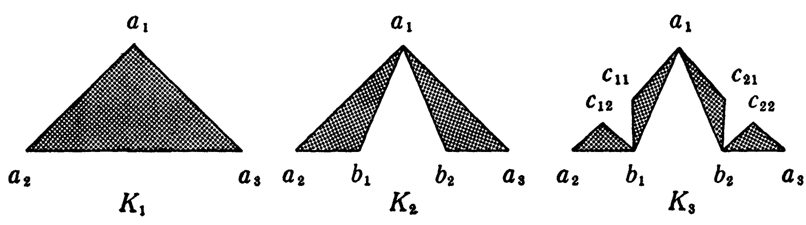
2. An example of a welding function which yields ring domains with arbitrary small moduli

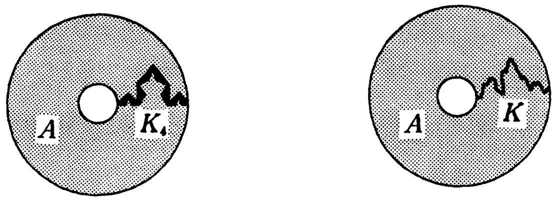
In this section we prove Theorem 1. Let A be an annulus. Then, for any $\varepsilon > 0$ we can construct a Jordan curve K in A and a quasiconformal (=q.c.) mapping h from A onto a ring domain G_1 such that

(1) K joins two boundary components of A,

(2) h is conformal on $A-K$ and $M(G_1) < \varepsilon$.

Take a closed rectilinear triangle $K_1 = \triangle a_1 a_2 a_3$ such that $K_1 - \{a_2, a_3\} \subset A$ and a_2, a_3 lie on the different boundary components of A. Choose a vertex a_1 and take a segment $[b_1, b_2]$ on the edge $[a_2, a_3]$ which contains the midpoint of $[a_2, a_3]$. Delete the open triangle $\triangle_1 = \triangle a_1 b_1 b_2$ from K_1 and put $K_2 = K_1 - \triangle_1 = K_2^1 \cup K_2^2$, where $K_2^j (j=1, 2)$ is the closed triangle $\triangle b_j a_1 a_{j+1}$. Next take an open triangle $\triangle_2^j = \triangle b_j c_{j1} c_{j2}$ on $K_2^j (j=1, 2)$, where c_{j1} and c_{j2} lie on the edge $[a_1, a_{j+1}]$ and the edge $[c_{j1}, c_{j2}]$ contains the midpoint of the edge $[a_1, a_{j+1}]$. Set $K_3 =$





$\cup_{j=1,2}(K_2^j-\triangle_2^j)$. We repeat these processes to get a decreasing sequence of connected compact sets $\{K_n\}$. Then $K=\cap K_n$ becomes a Jordan curve, which is a kind of Koch curve. To attain our result, we must choose the deleted triangles in a certain manner, which is descrived after Lemma 1.

LEMMA 1. *Let A be an annulus and K_n be constructed as above. For any $\varepsilon>0$, there exists a q.c. mapping h on A which is conformal on $A-K_n$ and satisfies $M(h(A))<\varepsilon$.*

Proof. Set $l=2^{n-1}$. The interior of K_n consists of disjoint l open triangles $\{O_n^m\}_{m=1,2,\dots,l}$. Take a set of closed triangles $\{T_m\}(m=1, 2, \dots, l)$ such that each T_m is contained in O_n^m. Let K_n^m be the closure of O_n^m. Then $K_n=\cup_{m=1}^{l}K_n^m$. Let g be a conformal mapping from $(l+2)$-connected domain $A-\cup T_m$ to an annulus $\Omega=\Omega(\{T_m\})$ such that $\Omega-g(A-\cup T_m)$ consists of concentric circular slits. The boundary components of Ω are denoted by $C_0, C_1, \dots, C_{l+1}$, and the radius of the circle on which C_i lies are denoted by R_i, where $R_0<R_1\leqq\cdots\leqq R_l<R_{l+1}$.

Suppose that there exists a positive number α such that

$$\log\frac{R_{l+1}}{R_0}\geqq\alpha$$

for $\Omega(\{T_m\})$ made by arbitrary choosen $\{T_m\}$. We can choose p so that

$$\log\frac{R_{p+1}}{R_p}\geqq\frac{\alpha}{l+1}.$$

Let $\gamma_i(i=0, 1, \dots, p-1, p+1, \dots l)$ be a set of Jordan curves such that each γ_i joins C_i to C_{i+1}, $\gamma_i(i\leqq p-1)$ is contained in $\{z: R_0<|z|<R_p\}$ and $\gamma_i(i\geqq p+1)$ is contained in $\{z: R_{p+1}<|z|<R_{l+1}\}$. We can get a ring domain $\tilde{A}=\tilde{A}(\{T_m\}, \{\gamma_i\})=g^{-1}(\Omega-\cup_{i\neq p}\gamma_i)$. It is proved in [2] that if the spherical diameters of the boundary components of $\tilde{A}$ are larger than $\tilde{d}$ and if their spherical distance is smaller than d, then

$$M(\tilde{A})\leqq\pi^2\left(\log\tan\frac{\tilde{d}}{2}-\log\tan\frac{d}{2}\right)^{-1}.$$

Hence

$$M(\tilde{A})=M(\Omega-\bigcup_{i\neq p}\gamma_i)$$

tends to zero if $\mathrm{dist}(T_p, T_{p+1})$ tends to zero. On the other hand,

$$\log\frac{R_{p+1}}{R_p}=M(\{z: R_p<|z|<R_{p+1}\})\leqq M(\Omega-\bigcup_{i\neq p}\gamma_i).$$

This is a contradiction. Hence there exists $\{T_m\}$ such that $\log(R_{l+1}/R_0)$ with respect to $\Omega(\{T_m\})$ is arbitrarily small.

Next take a set of simply connected domains $\{B_m\}$ such that each B_m contains T_m, each ∂B_m is real analytic and $B_m \cup \partial B_m$ is contained in the interior of K_n^m. Let $g_{1,m}$ and $g_{2,m}$ be conformal mappings from B_m and $g(B_m - T_m) \cup C_m$ to the unit disk D respectively. Then $f_0 = g_{2,m} \circ g \circ g_{1,m}^{-1}$ has a conformal extension to a neighborhood of ∂D. Hence

$$\frac{\partial}{\partial \theta} \log f_0(e^{i\theta}) = i \frac{\partial}{\partial \theta} \arg f_0(e^{i\theta}) \neq 0.$$

Therefore $\eta(\theta) = \arg f_0(e^{i\theta})$ is real analytic and $\eta'(\theta)$ satisfies $\eta'(\theta) \geqq \varepsilon > 0$. We define a mapping $F_0: D \to D$ by $F_0(re^{i\theta}) = re^{i\eta(\theta)}$, where $x+iy = re^{i\theta}$. Put $F = (\log) \circ F_0 \circ (\exp)$ i.e., $F(x+iy) = x + i\eta(y)$. Therefore

$$\left| \frac{F_{\bar{z}}}{F_z} \right| = \left| \frac{1-\eta'(y)}{1+\eta'(y)} \right| \leqq \left| \frac{1-\varepsilon}{1+\varepsilon} \right| < 1.$$

Consider

$$h = \begin{cases} g_{2,m}^{-1} \circ (\exp) \circ F \circ (\log) \circ g_{1,m} & \text{on } \cup B_m \\ g & \text{on } A - \cup B_m . \end{cases}$$

Then $h: A \to \Omega$ is a q.c. mapping which is conformal in $A - K_n$ and satisfies

$$m(h(A)) < \varepsilon. \qquad \text{q.e.d.}$$

We construct a Jordan curve K and a q.c. mapping h satisfying (1), (2). By Lemma 1, there exists a q.c. mapping h_1 on A which is conformal on $A - K_1$ and satisfies

$$M(h_1(A)) < \frac{M(A)}{2^4}.$$

Let $\mu_1 = (h_1)_{\bar{z}}/(h_1)_z$ be the Beltrami coefficient of h_1. Since

$$M(h_1(A)) = 2\pi \inf_{\rho \in \Lambda(h_1(A))} \frac{\iint_{h_1(A)} \rho(w)^2 du\, dv}{(\inf_{\gamma \in \Gamma(h_1(A))} \int_\gamma \rho(w) |dw|)^2}$$

$$= 2\pi \inf_{\rho \in \Lambda(h_1(A))} \frac{\iint_A \rho(h_1(z))^2 |(h_1)_z(z)|^2 |1-\mu_1(z)|^2 dx\, dy}{(\inf_{\gamma \in \Gamma(A)} \int_\gamma \rho(h_1(z)) |(h_1)_z(z)| \, |dz + \mu_1(z) d\bar{z}|)^2},$$

we can choose a conformal density $\rho_1 \in \Lambda(h_1(A))$ so that

$$2\pi \frac{A(\rho_1, h_1(A))}{L(\rho_1, \Gamma(h_1(A)))^2} < \frac{M(A)}{2^4}.$$

In our above construction, by choosing the segment $[b_1, b_2]$ sufficiently short, we can take a compact set K_2 so that

$$\iint_{K_1 - K_2} \rho_1(h_1)^2 |(h_1)_z|^2 |\mu_1|^2 dx\, dy \leqq \frac{1}{2} \iint_A \rho_1(h_1)^2 |(h_1)_z|^2 |1-\mu_1|^2 dx\, dy.$$

Again, by Lemma 1, there exists a q.c. mapping h_2 on A which is conformal on $A - K_2$ and satisfies

$$M(h_2(A))<\frac{M(A)}{2^5}.$$

Let $\mu_2=((h_2)_{\bar{z}}/(h_2)_z)$ be the Beltrami coefficient of h_2. We can take a conformal density $\rho_2\in\Lambda(h_2(A))$ such that

$$2\pi\frac{A(\rho_2, h_2(A))}{L(\rho_2, \Gamma(h_2(A)))^2}<\frac{M(A)}{2^5}.$$

Now suppose that for $n\geqq 3$, $\{K_l\}_{l\leqq n-1}$, $\{h_l\}_{l\leqq n-1}$, $\{\mu_l\}_{l\leqq n-1}$, $\{\rho_l\}_{l\leqq n-1}$ satisfy the following:

$$\iint_{K_{l-1}-K_l}\rho_{l-j}(h_{l-j})^2|(h_{l-j})_z|^2|\mu_{l-j}|^2dxdy$$
$$\leqq\frac{1}{2^j}\iint_A\rho_{l-j}(h_{l-j})^2|(h_{l-j})_z|^2|1-\mu_{l-j}|^2dxdy,$$

for $1\leqq j\leqq l-1$,

$$M(h_l(A))<\frac{M(A)}{2^{l+3}},\quad \mu_l=\frac{(h_l)_{\bar{z}}}{(h_l)_z},\quad 2\pi\frac{A(\rho_l, h_l(A))}{L(\rho_l, \Gamma(h_l(A)))^2}<\frac{M(A)}{2^{l+3}}.$$

Then we can take a compact set K_n such that

$$\iint_{K_{n-1}-K_n}\rho_{n-j}(h_{n-j})^2|(h_{n-j})_z|^2|\mu_{n-j}|^2dxdy$$
$$\leqq\frac{1}{2^j}\iint_A\rho_{n-j}(h_{n-j})^2|(h_{n-j})_z|^2|1-\mu_{n-j}|^2dxdy,$$

for $1\leqq j\leqq n-1$. By Lemma 1, we can get h_n, μ_n and ρ_n such that

$$M(h_n(A))<\frac{M(A)}{2^{n+3}},\quad \mu_n=\frac{(h_n)_{\bar{z}}}{(h_n)_z},\quad 2\pi\frac{A(\rho_n, h_n(A))}{L(\rho_n, \Gamma(h_n(A)))^2}<\frac{M(A)}{2^{n+3}}.$$

Thus we can get a Jordan curve $K=\cap K_n$ which satisfies

$$\iint_{K_i-K}\rho_i(h_i)^2|(h_i)_z|^2|\mu_i|^2dxdy\leqq\iint_A\rho_i(h_i)^2|(h_i)_z|^2|1-\mu_i|^2dxdy$$

for every i. This K joins two boundary components of A and has positive 2-dimensional measure.

Let $f=f_n$ be a q.c. mapping on $\boldsymbol{C}$ such that

$$\nu=\frac{f_{\bar{z}}}{f_z}=\begin{cases}\mu_n & \text{on } K\\ 0 & \text{on } C-K.\end{cases}$$

Let

$$\tilde{\rho}_n(w)=\rho_n(h_n\circ f^{-1}(w))\left|\frac{(h_n)_z(f^{-1}(w))}{f_z(f^{-1}(w))}\right|.$$

We have

$$M(f(A))=2\pi\inf_{\rho\in\Lambda(f(A))}\frac{\iint_A\rho(f(z))^2|f_z(z)|^2(1-|\nu(z)|^2)dxdy}{(\inf_{\gamma\in\Gamma(A)}\int_\gamma\rho(f(z))|f_z(z)||dz+\nu(z)d\bar{z}|)^2}$$

$$\leqq 2\pi\frac{\iint_A\tilde{\rho}_n(f(z))^2|f_z(z)|^2(1-|\nu(z)|^2)dxdy}{(\inf_{\gamma\in\Gamma(A)}\int_\gamma\tilde{\rho}_n(f(z))|f_z(z)||dz+\nu(z)d\bar{z}|)^2}$$

$$=2\pi\frac{\iint_A\rho_n(h_n(z))^2|(h_n)_z(z)|^2(1-|\nu(z)|^2)dxdy}{(\inf_{\gamma\in\Gamma(A)}\int_\gamma\rho_n(h_n(z))|(h_n)_z(z)||dz+\nu(z)d\bar{z}|)^2}.$$

On one hand,

$$\iint_A\rho_n(h_n(z))^2|(h_n)_z(z)|^2(1-|\nu(z)|^2)dxdy$$

$$=\iint_{A-(K_n-K)}\rho_n(h_n(z))^2|(h_n)_z(z)|^2(1-|\mu_n(z)|^2)dxdy$$

$$+\iint_{(K_n-K)}\rho_n(h_n(z))^2|(h_n)_z(z)|^2dxdy$$

$$=\iint_A\rho_n(h_n(z))^2|(h_n)_z(z)|^2(1-|\mu_n(z)|^2)dxdy$$

$$+\iint_{(K_n-K)}\rho_n(h_n(z))^2|(h_n)_z(z)|^2|\mu_n(z)|^2dxdy$$

$$\leqq\iint_A\rho_n(h_n(z))^2|(h_n)_z(z)|^2(1-|\mu_n(z)|^2)dxdy$$

$$+\iint_A\rho_n(h_n(z))^2|(h_n)_z(z)|^2(1-|\mu_n(z)|^2)dxdy$$

$$=2\iint_A\rho_n(h_n(z))^2|(h_n)_z(z)|^2(1-|\mu_n(z)|^2)dxdy.$$

On the other hand,

$$\int_\gamma\rho_n(h_n(z))|(h_n)_z(z)||dz+\nu(z)d\bar{z}|$$

$$=\int_{\gamma\cap(C-(K_n-K))}\rho_n(h_n(z))|(h_n)_z(z)||dz+\mu_n(z)d\bar{z}|$$

$$+\int_{\gamma\cap(K_n-K)}\rho_n(h_n(z))|(h_n)_z(z)||dz|$$

$$\geqq\int_{\gamma\cap(C-(K_n-K))}\rho_n(h_n(z))|(h_n)_z(z)||dz+\mu_n(z)d\bar{z}|$$

$$+\frac{1}{2}\int_{\gamma\cap(K_n-K)}\rho_n(h_n(z))|(h_n)_z(z)||dz+\mu_n(z)d\bar{z}|$$

$$\geqq \frac{1}{2}\int_{\gamma}\rho_n(h_n(z))|(h_n)_z(z)||dz+\mu_n(z)d\bar{z}|.$$

Therefore

$$M(f(A))\leqq 2\pi\frac{2\iint_A\rho_n(h_n(z))^2|(h_n)_z(z)|^2(1-|\mu_n(z)|^2)dxdy}{(\inf_{\gamma\in\Gamma(A)}(1/2)\int_\gamma\rho_n(h_n(z))|(h_n)_z(z)||dz+\mu_n(z)d\bar{z}|)^2}$$

$$=16\pi\frac{A(\rho_n,\ h_n(A))}{L(\rho_n,\ \Gamma(h_n(A)))^2}<\frac{M(A)}{2^n},$$

which shows (2).

Proof of Theorem 1. Let f_t be a q.c. mapping on A which satisfies

$$\frac{(f_t)_{\bar{z}}}{(f_t)_z}=t\nu(0\leqq t\leqq 1).$$

By using the variational formula [1], we know

$$\{M(f_t(A));\ 0\leqq t\leqq 1\}\supset\left[\frac{M(A)}{2^n},\ M(A)\right].$$

Let g_3 be a conformal mapping from $A-K$ to a rectangle Q_0 whose horizontal sides correspond to K, g_4 be a q.c. mapping from Q to Q_0 which preserve the vertical and horizontal sides. There exists a q.c. mapping g_5 on A into $\boldsymbol{C}$ such that $g_5\circ g_3^{-1}\circ g_4$ is conformal on Q. Let $k=\text{esssup}|(g_5)_{\bar{z}}/(g_5)_z|<1$. Set $g=g_5\circ g_3^{-1}\circ g_4$, $g_5(A)=G$, $g_5(K)=C$. Since g has a continuous, injective extension $\tilde{g}$ to the boundary, we can define $\phi=\tilde{g}^{-1}\circ\tilde{g}:\ L_+\to L_-$. Then $(G,\ C,\ \tilde{g})$ becomes a conformal welding obtained from Q by ϕ. It follows that for sufficiently large n,

$$M_\phi\supset\left[\frac{(1+k)M(A)}{(1-k)2^n},\ \frac{(1-k)M(A)}{(1+k)}\right].$$

Hence M_ϕ contains $(0,\ (1-k)M(A)/(1+k)]$.

Remark. Let a welding function ϕ satisfy Theorem 1. Then by Theorem C, we have

$$\int_0^1\frac{\min(\phi'(x),\ 1)}{1+(\phi(x)-x)^2}dx=0.$$

Hence $\phi'(x)$ vanishes almost everywhere on (0, 1). Contrary, we don't know whether the vanishing of ϕ' a.e. on (0, 1) implies that $M_\phi\supset(0,\ \varepsilon)$.

3. Upper bound of the moduli for a welding function

In this section we prove Theorem 2. Suppose that there exists a welding function ϕ such that the closure of M_ϕ contains 2π. There exist conformal weldings (G_n, C_n, f_n) such that $r_n=M(G_n)$ increases monotonously to 2π. Here we may assume $G_n=\{z: 1<|z|<e^{r_n}\}$. Let $G=\{z: 1<|z|<e^{2\pi}\}$, $f=e^{2\pi}z$ and C be an open interval $(1, e^{2\pi})$. Then (G, C, f) is the conformal welding by the welding function $\phi_0(x+i)=x$. Each f_n and f have continuous extensions to the boundary of Q by Carathéodory's theorem, which are denoted by the same symbols. Set

$$w_n=\frac{1}{r_n}\log|f_n|, \quad w=\frac{1}{2\pi}\log|f|.$$

Since each f_n is univalent and uniformly bounded, we can take a subsequence of $\{f_n\}$ which converges to a conformal mapping f_0 uniformly on each compact set in Q. We may assume that f_n converges to f_0 uniformly on each compact set in Q. We can show the following.

LEMMA 2. *$f_0=e^{i\alpha}f$, where α is a real constant.*

Proof. Set $L_a=\{x+ia: 0<x<1\}$, where $0<a<1$. We have

$$2\pi\leqq\int_{L_a}|d\log f_0|=\int_0^1\frac{|f_0'|}{|f_0|}dx.$$

Hence

$$4\pi^2\leqq\left(\int_0^1\frac{|f_0'|}{|f_0|}dx\right)^2\leqq\int_0^1 dx\int_0^1\frac{|f_0'|^2}{|f_0|^2}dx=\int_0^1\frac{|f_0'|^2}{|f_0|^2}dx.$$

Furthermore,

$$4\pi^2\leqq\int_0^1 dy\int_0^1\frac{|f_0'|^2}{|f_0|^2}dx=\iint_Q\frac{|f_0'|^2}{|f_0|^2}dxdy$$

$$=\iint_{f_0(Q)}\frac{dudv}{|w|^2}\leqq\iint_G\frac{dudv}{|w|^2}=4\pi^2 \quad (w=u+iv).$$

It follows that for almost all a in $(0, 1)$

$$2\pi=\int_{L_a}|d\log f_0|,$$

and $f_0(L_a)$ lies on a line through 0. Since

$$\iint_{f_0(Q)}\frac{dudv}{|w|^2}=\iint_G\frac{dudv}{|w|^2},$$

$f_0(L_+)=f_0(L_-)$ lies on a line through 0 with an argument α and

$$f_0(Q)=\{z: 1<|z|<e^{2\pi}\}-\{z: \arg z=\alpha, 1<|z|<e^{2\pi}\}.$$

Let g_6 denote the inverse mapping f_0^{-1} from $f_0(L_+)$ to L_+ and g_7 from $f_0(L_-)$ to L_-. Set $\phi_1=g_7\circ g_6^{-1}$. Since $M(f_0(Q))=2\pi$, Theorem A implies $\phi_1=\phi_0$. Thus $f_0\circ f^{-1}$ is a conformal mapping on G. Hence $f_0=e^{i\alpha}f$. q.e.d.

LEMMA 3. *The Dirichlet norm* $\|dw_n\|_Q=\sqrt{\iint_Q dw_n\wedge {}^*dw_n}$ *converges to* $\|dw\|_Q$.

Proof. Set $L^-=\{iy: 0<y<1\}$ and $L^+=\{1+iy: 0<y<1\}$. Each conformal mapping f_n on Q has a conformal extension to a neighborhood of L^+, L^-. We denote the extension by the same symbol. For any positive number ε there exists a rectangle $Q_\varepsilon=\{x+iy: 0<x<1, 0<b_\varepsilon\leqq y\leqq b'_\varepsilon<1\}$ such that

$$\mathrm{Area}(f_0(Q))-\mathrm{Area}(f_0(Q_\varepsilon))\leqq\varepsilon.$$

Since $\{f_n\}$ converges to f_0 uniformly on Q_ε, there exists an integer N such that for all $n\geqq N$

$$\mathrm{Area}(f_0(Q_\varepsilon))-\mathrm{Area}(f_n(Q_\varepsilon))\leqq\varepsilon.$$

Then

$$\begin{aligned}\mathrm{Area}(f_n(Q-Q_\varepsilon))&=\mathrm{Area}(f_n(Q))-\mathrm{Area}(f_n(Q_\varepsilon))\\&\leqq\mathrm{Area}(f_0(Q))-\mathrm{Area}(f_n(Q_\varepsilon))\\&\leqq\mathrm{Area}(f_0(Q))-\mathrm{Area}(f_0(Q_\varepsilon))+\varepsilon\leqq2\varepsilon.\end{aligned}$$

Since $w_n=r_n^{-1}\log|f_n|$, $dw_n=r_n^{-1}\Re(df_n/f_n)$. We have

$$\begin{aligned}\|dw_n\|_Q^2&=\|dw_n\|_{Q_\varepsilon}^2+\|dw_n\|_{Q-Q_\varepsilon}^2\leqq\|dw_n\|_{Q_\varepsilon}^2+\frac{1}{r_n^2}\|df_n\|_{Q-Q_\varepsilon}^2\\&=\|dw_n\|_{Q_\varepsilon}^2+\frac{1}{r_n^2}\mathrm{Area}(f_n(Q-Q_\varepsilon))\leqq\|dw_n\|_{Q_\varepsilon}^2+2\varepsilon.\end{aligned}$$

Therefore

$$\begin{aligned}\|dw\|_Q^2&\leqq\varliminf_{n\to\infty}\|dw_n\|_Q^2\leqq\varlimsup_{n\to\infty}(\|dw_n\|_{Q_\varepsilon}^2+2\varepsilon)\\&=\|dw\|_{Q_\varepsilon}^2+2\varepsilon\leqq\|dw_n\|_Q^2+2\varepsilon.\end{aligned}$$

q.e.d.

Proof of Theorem 2. By the parallelogram law

$$\begin{aligned}\|dw_n-dw_m\|_Q^2&=2(\|dw_n\|_Q^2+\|dw_m\|_Q^2)-\|dw_n+dw_m\|_Q^2\\&\leqq2\{\|dw_n\|_Q^2+\|dw_m\|_Q^2-\|dw\|_Q^2\}\longrightarrow0.\end{aligned}$$

Hence $\{dw_n\}$ is a Cauchy sequence and converges to a differential dw_*. Since $\{w_n\}$ converges to w uniformly on every compact set in Q, $dw_*=dw$.

Let g be a conformal mapping from the unit disk D to Q. Then g has a continuous extension to the boundary of D by Carathéodory's theorem, which is denoted by the same symbol. Set

$$W_n=w_n\circ g-w_n\circ g(0),\qquad W=w\circ g-w\circ g(0).$$

Then $\{dW_n\}$ converges to dW with respect to Dirichlet norm on D. Since W_n-W is harmonic on D, we can write it as

$$W_n - W = \sum_{k=1}^{\infty} (a_k(n) z^k + \overline{a_k(n)} \bar{z}^k).$$

We see that $\{W_n\}$ converges to W on ∂D in L_2-norm, because

$$\|W_n - W\|_{\partial D}^2 = 4\pi \sum_{k=1}^{\infty} |a_k(n)|^2 \leqq 4\pi \sum_{k=1}^{\infty} k |a_k(n)|^2 = \|dW_n - dW\|_D^2.$$

Now for every n, $W_n \circ g^{-1} \circ \phi = W_n \circ g^{-1}$ on L_+. The left hand side converges to $W \circ g^{-1} \circ \phi$ and the right hand side converges to $W \circ g^{-1}$ almost everywhere on L_+. Hence $\phi = \phi_0$ if the closure of M_ϕ contains 2π. This shows Theorem 2.

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